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EXISTENCE OF GLOBAL SOLUTIONS TO SOME NONLINEAR DISSIPATIVE WAVE EQUATIONS

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ABSTRACT. – Let Ω be a smooth bounded domain. We prove existence of global solutions, *i.e.*, solutions defined for all $t \in \mathbb{R}$, for dissipative wave equations of the form:

$$u'' - \Delta u + |u'|^{p-1} u' = 0 \quad \text{in } \Omega \times (-\infty, \infty),\ p > 1,$$

with Dirichlet boundary conditions. When Ω is unbounded the same existence result holds for $p \geqq 2$.

Introduction

In this paper we are concerned with proving the existence of global solutions to damped wave equations of the form:

$$(\mathcal{D}) \quad \begin{cases} u'' - \Delta u + g(u') = 0 & \text{in } \mathbb{R} \times \Omega, \\ u = 0 & \text{on } \mathbb{R} \times \partial\Omega, \end{cases}$$

where $g(s)$ is a locally Lipschitz continuous nondecreasing function satisfying $g(0) = 0$ and Ω stands for a bounded domain in $\mathbb{R}^n$. When $g(s)$ is sublinear at infinity global existence for the backward and forward initial value problems is assured. Therefore, we shall only deal here with the superlinear case.

Let us state our main results in the model case:

$$(\mathcal{H}) \quad \begin{cases} u'' - \Delta u + |u'|^{p-1} u' = 0 & \text{in } \mathbb{R} \times \Omega, \\ u = 0 & \text{on } \mathbb{R} \times \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ and $p > 1$.

THEOREM 1. – *Let us assume that Ω is a smooth (at least of class C^2) bounded domain and that $1 < p < \infty$ when $n \leqq 2$ or $1 < p \leqq n + 2/n - 2 = 2^* - 1$ when $n > 2$. Then,*

1) *if $p \geqq 2$, for each solution ψ of the elliptic problem*

$$(\mathcal{E}) \quad \begin{cases} -\Delta\psi = \left(\dfrac{p}{p-1}\right)^p |\psi|^{p-1} \psi & \text{in } \Omega, \\ \psi|_{\partial\Omega} = 0, \end{cases}$$

there exists at least one solution

$$u \in C(\mathbb{R}, H_0^1(\Omega)) \cap C^1(\mathbb{R}, L^2(\Omega))$$

behaving like $\Phi = |t|^{\frac{p}{p-1}} \psi$ *at* $-\infty$.

2) *if* $1 < p < 2$, *there exists at least one solution*

$$u \in C(\mathbb{R}, H_0^1(\Omega)) \cap C^1(\mathbb{R}, L^2(\Omega))$$

behaving like $\Phi = |t|^{p/p-1} \psi$ *at* $-\infty$ *for each solution* ψ *of* $(\mathcal{E})$ *with constant sign.*

Remark 1. – Let us define $2E(v(t)) = \|\nabla v(t)\|_{L^2(\Omega)}^2 + \|v'(t)\|_{L^2(\Omega)}^2$. By "behave like Φ" we mean that the energy $E(u-\Phi)(t) = o(|t|^{\frac{2p}{p-1}})$ when t tends to $-\infty$. In fact, for any fixed $t_0 < 0$ and any ψ as above we construct a solution u defined when $t \leqq t_0$ that can be decomposed as $u = v + w$ where:

– $v \in C((-\infty, t_0], L^\infty(\Omega))$ is a sum of functions of the form $|t|^r f(x)$ in which the highest order term (that is, the one growing the fastest as $|t|$ increases) is Φ.

– $w \in C((-\infty, t_0], H_0^1(\Omega)) \cap C^1((-\infty, t_0], L^2(\Omega))$ is such that $E(w(t))$ tends to zero when t goes to $-\infty$.

Then, we extend this u to a global solution defined for all $t \in \mathbb{R}$ by solving the dissipative initial value problem with initial data $(u(t_0), u'(t_0))$.

Once a solution $u(t)$ is known, its translates $u(t+k)$, $k \in \mathbb{R}$ furnish solutions of (H) with the same asymptotic behavior as $u(t)$. We ignore whether we have uniqueness up to translations, *i.e.*, given two solutions u_1, u_2 behaving like $|t|^{\frac{p}{p-1}} \psi$ at $-\infty$ we don't know if there exists $k \in \mathbb{R}$ such that $u(t+k) = u(t)$, $\forall t \in \mathbb{R}$.

Remark 2. – Some comments concerning the existence of solutions to $(\mathcal{E})$ are in order. Let us first consider the cases where $n \leqq 2$ or $1 < p < 2^* - 1$ and $n > 2$ so that injection $H_0^1(\Omega) \to L^{p+1}(\Omega)$ is compact. Since the nonlinear term is odd, infinitely many solutions to $(\mathcal{E})$ are found (*see* [AR]) by looking for critical points in $H_0^1(\Omega)$ of the functional

$$F(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \left(\frac{p}{p-1}\right)^p \frac{1}{p+1} \int_\Omega |u|^{p+1}.$$

On the other hand, the existence of (at least) a positive solution is easily proved by using a minimization approach (for more detail *see* [LI]).

In this case, a bootstrap argument yields that any solution $\psi \in H_0^1(\Omega)$ of $(\mathcal{E})$ belongs to $C^{1,\alpha}(\bar{\Omega})$ (*see* [GT]).

In the critical case $p = 2^* - 1$ further hypothesis on the topology and geometry of the domain Ω are needed in order to ensure the existence of solutions (*see* [BR] and the references therein). For instance, positive solutions are known to exist for smooth domains with holes. Positive solutions in $H_0^1(\Omega)$ of $(\mathcal{E})$ in this limit case are known [BK] to be in $L^p(\Omega)$ for $p \geqq 1$.

In both cases, the solutions $\psi \in C^\infty(\Omega)$ and, up to the boundary, they are as smooth as $\partial\Omega$ and p allow.

Note that $2^* - 1 < 2$ when $n \geqq 6$. Therefore, since we assume $p \leqq 2^* - 1$, in this range of dimensions we are necessarily in case ii).

Remark 3. – The boudedness of Ω is only necessary when $1 < p < 2$.

In order to understand the meaning of this theorem it is worth recalling some facts about dissipative problems like $(\mathcal{D})$. It is well known that, due to the monotonicity of the nonlinear term, the forward initial value problem

$$\begin{array}{ll} u'' - \Delta u + g(u') = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ u = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ u(x, 0) = u_0,\ u'(x, 0) = u_1 & \text{in } \Omega, \end{array}$$

can always be solved for $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ yielding a unique global bounded solution in $H_0^1(\Omega) \times L^2(\Omega)$. Moreover, when $g(s)\, s > 0$ for $s \neq 0$ the trajectories (u, u') are shown to converge in $H_0^1(\Omega) \times L^2(\Omega)$ to the single stationary state of the system, *i.e.* $(0, 0)$. (*See* [H1], for instance, and the references therein.)

However, almost nothing is known regarding the backward problem even as far as local existence is concerned. Let us recall several facts pointed out in [H2] in connection with this question. By reversing the time we deduce that $w(t) = u(-t)$ must satisfy:

$$\begin{array}{ll} w'' - \Delta w + g(-w') = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ w = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega, \\ w(x, 0) = u_0,\ w'(x, 0) = -u_1 & \text{in } \Omega. \end{array}$$

This time the nonlinear term has "the bad sign" so that monotonicity theory does not apply. On the other hand, since the map $g : v \to g(v)$ is not Lipschitz in $L^2(\Omega)$ we cannot consider the nonlinear term as a Lipschitz perturbation of the linear wave equation unless some restrictions on the smoothness of the initial data and the dimension are made. For instance, when $n = 1$ the map $g : v \to g(v)$ turns out to be locally Lipschitz from $H_0^1(\Omega)$ to $H_0^1(\Omega)$ so that we can construct local solutions to initial value problems with data $(u_0, u_1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$.

Nevertheless, global existence can not be guaranteed even for small initial data. In fact, when $g(s) = |s|^{p-1} s$, $p > 1$ we can construct solutions blowing up in a finite time for arbitrary small initial data. Indeed, let $B_r = B(x_0, r)$ be any ball contained in Ω. We take initial data $(u_0, v_0) \in H_0^1(\Omega) \times L^2(\Omega)$ such that $u_0 = 0$ and $v_0 = A$ on B_r, A being a constant. Since in the light cone with basis B_r

$$C_r = \{(x, t) : 0 \leqq t \leqq r,\ |x - x_0| \leqq r - t\}$$

the solution of the wave equation depends only on the values of the initial data in B_r, it agrees there with the solution of the ordinary differential equation:

$$\begin{array}{l} u'' - |u'|^{p-1} u' = 0 \quad t \geqq 0, \\ u(0) = 0,\ u'(0) = A, \end{array}$$

whose energy blows up like $(|t| - T_0)^{\frac{-2}{p-1}}$ at time $T_0(A) = 1/(p-1)\, A^{p-1}$. In order to guarantee that the solution blows up in the cone C_r it suffices to choose A large enough to have $T_0(A) < r$. By taking $v(t) = u(-t)$ and then extending it to $t > 0$ by the solution of the dissipative problem with initial data $(u_0, -v_0)$ we obtain a solution v defined on $(-T_0(A), \infty)$ whose energy tends to zero when t goes to ∞. Thus, $(v(t), v'(t))$ are initial values for the backward problem whose norm can be made arbitrarily small by taking t large enough and that generate blowing-up solutions.

On the other hand, it follows from the energy decay to zero at infinity that any global solution of $(\mathcal{H})$ must be unbounded at $-\infty$. Indeed, the energy of such a solution must satisfy (*see* [H1]) the inequality:

$$E(u(t+\tau)) \leqq F(E(t))\,\tau^{\frac{-2}{p-1}} \qquad t \in \mathbb{R},\ \tau > 0,$$

where $F(E(t))$ is an increasing bounded fonction of the energy. Supposing the energy to be bounded by a constant K and taking $t = -\tau$ it follows that:

$$E(u(0)) \leqq F(K)\,\tau^{\frac{-2}{p-1}} \qquad \tau > 0.$$

Thus, $E(u(0)) = 0$ and $u = 0$.

In the particular case $p = 2$ we can easily construct infinitely many (up to translations) global solutions of $(\mathcal{H})$, under restrictions on n, p and Ω analogous to these made in theorem 1. It suffices to take:

$$u(x,\, t) = t^2\,\varphi(x) + \eta(x) \qquad t \leqq 0,$$

where φ and η solve:

$$(P_\varphi) \quad \begin{cases} -\Delta\varphi = 4\,|\varphi|\,\varphi & \text{in } \Omega \\ \varphi\,|_{\partial\Omega} = 0, \end{cases}$$

$$(P_\eta) \quad \begin{cases} -\Delta\eta = -2\,\varphi & \text{in } \Omega \\ \eta\,|_{\partial\Omega} = 0, \end{cases}$$

and then extend them to $t > 0$ by solving the dissipative initial value problem with initial data $(\eta(x),\, 0)$.

It follows from standard Min-Max theory that infinitely many solutions to (P_φ) exist provided that $2 < (n+2)/(n-2)$ that is, $n < 6$. The existence of at least one positive solution can be proved by solving a minimization problem. Owing to the maximum principle, the η corresponding to a positive φ turns out to be negative. Nevertheless, for smooth Ω, the resulting u will be positive for t large enough. Indeed, it suffices to remark that, by the strong maximum principle, both the normal derivatives of φ and $-\eta$ are strictly negative on $\partial\Omega$.

In the case $n = 6$, (P_φ) correspond to a problem with critical exponent for the Sobolev's embedding so that, as we pointed out before, we need additional hypotheses on the geometry and topology of Ω to conclude.

Theorem 1 is a natural extension of this result to any $p > 1$, but the solutions we obtain are no more explicit for large negative t when $p \neq 2$.

Since forward dissipative initial value problems can always be solved, we must manage to find solutions of

$$\begin{aligned} &u'' - \Delta u + |\,u'\,|^{p-1}\,u' = 0 && \text{in } (-\infty,\, t_0] \times \Omega, \\ &u = 0 && \text{on } (-\infty,\, \mathrm{t}_0] \times \partial\Omega, \end{aligned}$$

for some fixed t_0. When $p = 2$ this is done explicitely but we ignore if solutions different from those exhibited above exist. In order to find solutions for other values of p we remark that the functions $v(t, x) = |t|^{\frac{p}{p-1}} \varphi(x)$ where

$$\begin{cases} -\Delta\varphi = \left(\dfrac{p}{p-1}\right)^p |\varphi|^{p-1} \varphi & \text{in } \Omega, \\ \varphi|_{\partial\Omega} = 0, \end{cases}$$

are still solutions of the parabolic problem:

$$(\mathcal{P}) \quad \begin{cases} -\Delta v + |v'|^{p-1} v' = 0 & \text{in } (-\infty, 0] \times \Omega, \\ v = 0 & \text{on } (-\infty, 0] \times \partial\Omega. \end{cases}$$

Heuristically, for an unbounded solution u of the wave equation behaving as a power of $|t|$ when $t \to -\infty$ the term u'' should be much smaller than the others so that it might approach the solutions of this "parabolic" problem. On the other hand, any v of this form satisfies:

$$v'' - \Delta v + |v'|^{p-1} v' = \left(\frac{p}{p-1}\right)\left(\frac{1}{p-1}\right) |t|^{\frac{p}{p-1}-2} \varphi(x) \quad \text{in } (-\infty, 0] \times \Omega,$$
$$v = 0 \quad \text{on } (-\infty, 0] \times \partial\Omega,$$

where the second term tends to zero as $t \to -\infty$ so that v is "almost" a solution of the wave equation. This is the starting point in our existence proof.

Theorem 1 will follow by putting together the following two results:

LEMMA 1. – *Let us assume that there exists a function*

$$v \in C((-\infty, t_0], H_0^1(\Omega)) \cap C^1((-\infty, t_0], L^2(\Omega))$$

satisfying:

$$(\mathcal{A}) \quad \begin{cases} v'' - \Delta v + |v'|^{p-1} v' = f(t, x) & \textit{in } (-\infty, t_0] \times \Omega, \\ v = 0 & \textit{on } (-\infty, t_0] \times \partial\Omega, \end{cases}$$

where $p > 1$, $t_0 < 0$ *and* $f \in C((-\infty, t_0]; L^2(\Omega))$ *with*

$$\|f(t)\|_{L^2(\Omega)} \leqq C|t|^\alpha,$$

for some positive C *and* $\alpha < -1$. *Suppose further that* $E(v(t)) \geqq K > 0$ *for every* $t < t_0$ *and some* $K > 0$.

Then, there exists a global solution

$$u \in C((-\infty, t_0], H_0^1(\Omega)) \cap C^1((-\infty, t_0], L^2(\Omega))$$

of:

$$(\mathcal{H}) \quad \begin{cases} u'' - \Delta u + |u'|^{p-1} u' = 0 & \textit{in } \mathbb{R} \times \Omega, \\ u = 0 & \textit{on } \mathbb{R} \times \partial\Omega, \end{cases}$$

which behaves like v *at* $-\infty$ *in the sense that* $E(u-v)(t)$ *tends to zero when t goes to* $-\infty$.

Remark 4. – As a matter of fact, we shall prove that the solutions u_T on $[-T, t_0]$ of initial value problems with initial data $(v(T), v'(-T))$ at $-T$ are "atracted" by v in the sense that

$$E(v - u_T)(t) \leqq C\,|\,t\,|^{2(\alpha+1)}.$$

This estimate will allow to prove the convergence of u_T to a solution u such that $E(v-u)(t) \to 0$ as $t \to -\infty$ by using the monotonicity of the nonlinear term.

Remark 5. – In this lemma Ω can either bounded or unbounded and no smoothness hypothesis is needed.

LEMMA 2. – *We set $t_0 < 0$ and assume that either $n \leqq 2$ or $n > 2$ and $1 < p \leqq \dfrac{n+2}{n-2}$. Let ψ be*

– any solution of (E) if $p \geqq 2$

– a solution with constant sign of (E) if $1 < p < 2$.

Then, there exists

$$v \in C((-\infty, t_0], H_0^1(\Omega)) \cap C^1((-\infty, t_0], L^2(\Omega))$$

behaving like $\Phi = |\,t\,|^{p/(p-1)}\,\psi$ at $-\infty$ and satisfying:

$$(\mathcal{A}) \quad \begin{cases} v'' - \Delta v + |\,v'\,|^{p-1}\,v' = f(t, x) & in\ (-\infty, t_0] \times \Omega, \\ v = 0 & on\ (-\infty, t_0] \times \partial\Omega, \end{cases}$$

with f depending on ψ such that $\|\,f(t)\,\|_{L^2} \leqq C\,|\,t\,|^{\alpha}$ for some $\alpha < -1$ depending on p.

Remark 6. – We already know that $v_0(t, x) = \psi(x)\,|\,t\,|^{p/(p-1)}$ satisfies $(\mathcal{A})$ with $\alpha = \dfrac{p}{p-1} - 2 < 0$. The functions v are constructed by adding to v_0 several terms of the form $v_k = \psi_k(x)\,|\,t\,|^{p/(p-1)-a_k}$, with $a_k > 0$ in such a way that the sum satisfies $(\mathcal{A})$ with a right hand side decreasing fast enough.

Remark 7. – An analogous of Lemma 2 can be proved for lower order perturbations of $|\,s\,|^{p-1}\,s$ such as $g(x, s) = a(x)\,|\,s\,|^{p-1}\,s + b(x)\,|\,s\,|^{q-1}\,s$ with $q < p$, $p > 1$, a, b, being nonnegative bounded functions. Lemma 1 remains valid if we replace $|\,s\,|^{p-1}\,s$ by any continuous nondecreasing function g such that $g(0) = 0$. So, the same existence result applies in this case.

Remark 8. – The boudedness of Ω is only needed when $1 < p < 2$.

It is interesting to compare these results to those known for the analogous (in a certain sence) ordinary differential equation

$$u'' + u + g(u') = 0$$

where g is any superlinear and locally lipschitz continuous non decreasing function such that $g(0) = 0$. By reversing the sign we obtain $v'' + v = -g(-v')$, where $v(t) = u(-t)$. We recall some facts established in [S2]:

i) if $g(s)\,s \geqq K\,s^2$ for $|\,s\,|$ large enough, then for every $t_0 \in \mathbb{R}$ and for every $v_0 \in \mathbb{R}$ there exists at least a global solution $v(t)$ such that $v(t_0) = v_0$.

ii) if $m\,|\,s\,|^p \leqq |\,g(s)\,|$ for some $m > 0$ and $p > 1$ then, either

$$v > 0, \quad v' > 0 \text{ when } t \to \infty, \lim_{t\to\infty} v = \infty, \lim_{t\to\infty} v' = \infty$$

or

$$v < 0,\ v' < 0 \text{ when } t \to \infty, \lim_{t\to\infty} v = -\infty, \lim_{t\to\infty} v' = -\infty.$$

iii) if $m\,|\,s\,|^p \leqq |\,g(s)\,| \leqq M\,|\,s\,|^p$ for some $m,\ M > 0$ and $p > 1$ for $|\,x\,|$ large enough then every nontrivial global solution v satisfies:

$$C_1\,|\,t\,|^{\frac{p}{p-1}} \leqq |\,v(t)\,| \leqq C_2\,|\,t\,|^{\frac{p}{p-1}},$$

$$C_1'\,|\,t\,|^{\frac{1}{p-1}} \leqq |\,v'(t)\,| \leqq C_2'\,|\,t\,|^{\frac{1}{p-1}},$$

for certain positive $C_1,\ C_2,\ C_1',\ C_2'$ when $t \to \infty$.

iv) if g satisfies the growing condition in iii) and is differentiable on some neighbourhoods of $-\infty$ and ∞ with

$$\lim_{s\to\pm\infty} g'(s) = \infty,$$

then there exist exactly two non trivial global solutions (up to translations), one tending to ∞ and the other to $-\infty$ when $t \to \infty$. Moreover,

$$v'' = o(v) \quad t \to \infty.$$

Let us assume $g(s) = |\,s\,|^{p-1}\,s$, so that we are concerned with solutions of

$$(H) \quad u'' + u + |\,u'\,|^{p-1}\,u' = 0 \quad t \in \mathbb{R}.$$

By eliminating the second derivative we get the equation

$$(P) \quad u + |\,u'\,|^{p-1}\,u' = 0 \quad t \in \mathbb{R}.$$

Since we are mainly interested in solving the backward problem we reverse the time and get that $v(t) = u(-t)$ must satisfy

$$\begin{aligned} &(H') \quad v'' + v = |\,v'\,|^{p-1}\,v' \quad t \in \mathbb{R},\\ &(P') \quad v \qquad\;\; = |\,v'\,|^{p-1}\,v' \quad t \in \mathbb{R}. \end{aligned}$$

This equation can be solved for $t \geqq t_0$ for any initial data $v(t_0) = v_0$. Indeed,

$$v = \left\{\frac{p-1}{p}\,(t - t_0) + v_0^{\frac{p-1}{p}}\right\}^{\frac{p}{p-1}}$$

when $v_0 > 0$. The same formula holds for negative v_0 by changing the sing of v and v_0. If $v_0 = 0$ we get three different solutions (the nonlinearity obtained when writing the equation as $v' = g^{-1}(v)$ is not Lipschitz at zero), namely: $0,\ \pm\left(\frac{p-1}{p}\,(t - t_0)\right)^{\frac{p}{p-1}}$.

We observe that, up to translations, there are only three solutions: $0,\ \pm\left(\frac{p-1}{p}\,t\right)^{\frac{p}{p-1}}$.

They can be classified by their behavior at ∞ : either $v = 0$ or $\dfrac{v}{t^{\frac{p}{p-1}}} \to \left(\dfrac{p-1}{p}\right)^{\frac{p}{p-1}}$ or $\dfrac{v}{t^{\frac{p}{p-1}}} \to -\left(\dfrac{p-1}{p}\right)^{\frac{p}{p-1}}$.

If we make the changes $v = wt^{\frac{p}{p-1}}$ and $\tau = \ln t$ equations (H') and (P') become:

$$
\begin{aligned}
(H'')\quad & (\lambda(\lambda-1)w + (2\lambda-1)w_\tau + w_{\tau\tau})e^{-2\tau} + w = |\lambda w + w_\tau|^{p-1}(\lambda w + w_\tau),\\
(P'')\quad & w = |\lambda w + w_\tau|^{p-1}(\lambda w + w_\tau).
\end{aligned}
$$

The second one has now two stationary solutions $\pm l = \pm\left(\dfrac{p-1}{p}\right)^{\frac{p}{p-1}}$ which satisfy:

$$(E)\quad (\psi) = \left(\frac{p}{p-1}\right)^p |\psi|^{p-1}\psi.$$

By using the fact that $\dfrac{d}{dt}\dfrac{|w|^2}{2} > 0$ when $|w| < l$ it follows that any non trivial bounded solution of this equation converges to one of the stationary points $\pm l$ when $t \to \infty$. In the same way, we deduce that the function w obtained from any global solution u of (H) also converges to $\pm l$ by using the information about its asymptotic behavior.

Therefore, all nontrivial global solutions u of (H) and (P) are "attracted" by the functions $\Phi(t) = \pm l\,|t|^{\frac{p}{p-1}}$ at $-\infty$ in the sense that $|(\Phi - u)(t)|\,|t|^{\frac{-p}{p-1}} - 0$ when $t \to -\infty$.

We can also prove analogous to Lemma 1 and Lemma 2 which give the existence of two solutions behaving like $t^{\frac{p}{p-1}}\psi$ for any $1 < p < \infty$. Here, the solutions $\pm\psi = \pm l$ of (E) play the role of the solutions of the elliptic problems:

$$
(\mathcal{E})\quad \begin{cases} -\Delta\psi = \left(\dfrac{p}{p-1}\right)^p |\psi|^{p-1}\psi & \text{in } \Omega,\\ \psi|_{\partial\Omega} = 0. \end{cases}
$$

In this case, Lemma 1 says that any v satisfying

$$(A)\quad v'' + v + |v'|^{p-1}v' = f \quad t \leqq t_0$$

with $|f(t)| \leqq C|t|^\alpha$, $\alpha < 1$ and $|v(t)| \geqq K > 0$ for $t \leqq t_0$ generates a global solution u such that $|(v-u)(t)|\,|t|^{\frac{-p}{p-1}} \to 0$ when $t \to -\infty$. Lemma 2 will provide two functions v, each of them associated to one of the ψ. By the above considerations on the ordinary differential equation we know that the only possible global solutions of (H) behave at $-\infty$ like $t^{\frac{p}{p-1}}\psi$, so that all v satisfying (A) must also behave at $-\infty$ like $|t|^{\frac{p}{p-1}}\psi$.

If we take u to be a global nontrivial solution of $(\mathcal{H})$ or $(\mathcal{P})$ and make the changes $u(-t) = v(t) = wt^{\frac{p}{p-1}}$ and $\tau = \ln t$ [BNP] we shall obtain for w the equations:

$$
\begin{aligned}
(\mathcal{H}'')\quad & (\lambda(\lambda-1)w + (2\lambda-1)w_\tau + w_{\tau\tau})e^{-2\tau} - \Delta w = |\lambda w + w_\tau|^{p-1}(\lambda w + w_\tau),\\
(\mathcal{P}'')\quad & -\Delta w = |\lambda w + w_\tau|^{p-1}(\lambda w + w_\tau).
\end{aligned}
$$

The asymptotic behavior of the global solutions of $(\mathcal{P}'')$ is now unknown in general. With regard to $(\mathcal{H}'')$ we lack of estimates analogous to iii) and iv) so that we cannot prove the convergence of w to solutions of $(\mathcal{E})$ as we did before. The complete structure of the set of global solutions in both cases remains an open question.

In view of the results existing for ordinary differential equations, one should expect a similar existence result to hold for the wave equation with more general superlinear g that those considered here. Unfortunately, the proof of Lemma 2 relies heavily of the fact that g only involves powers of s, which allows us to control the behavior of the solutions of certain auxiliary problems and its derivatives. As a consequence, we are note able to extend our existence result to a larger class of g.

To conclude, let us remark that the existence of global solutions to $(\mathcal{H})$ established in theorem 1 proves the optimality of the energy decay estimates for forward dissipative initial value problems obtained in [C1]. In this article we proved that the energy of any solution of

$$\begin{aligned} & u'' - \Delta u + g(u') = 0 && \text{in } \mathbb{R}^+ \times \Omega, \\ & u = 0 && \text{on } \mathbb{R}^+ \times \partial\Omega, \\ & u(x, 0) = u_0,\ u'(x, 0) = u_1 && \text{in } \Omega, \end{aligned}$$

with $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ satisfies:

$$(I) \quad E(u(t+\tau)) \leqq F(E(u(t))\tau^{\frac{-2}{p-1}} \quad \tau \geqq 0,$$

where $F(X)$ is a bounded function of the initial energy growing like $C(1 + X^{\frac{p+1}{p}})$ for some positive constant C. If we take u to be a global solution whose energy grows like $|t|^{\frac{2p}{p-1}}$ when $t \to -\infty$ and assume the inequality (I) to hold for some $F(X)$ we deduce that $F(X) \geqq CX^{\frac{p+1}{p}}$ for large X.

1. Proof of Lemma 1

We are going to take advantage of the existence of unbounded "approximated" solutions, *i.e.* solutions v of:

$$\begin{aligned} & v'' - \Delta v + |v'|^{p-1} v' = f(t, x) && \text{in } (-\infty, t_0] \times \Omega, \\ & v = 0 && \text{on } (-\infty, t_0] \times \partial\Omega, \end{aligned}$$

where f decays fast enough at infinity and $E(v(t)) > K > 0$ when t tends to $-\infty$ to construct global non trivial solutions of the backward problem. We shall give here a simplified version of our original proof (that can be found in [C2]) suggested to us by A. Haraux.

Let us asssume that:

i) $\| f(x, t)\|_{L^2(\Omega)} \leqq C|t|^\alpha$ for some $\alpha < -1$ and $t \leqq t_0$,

ii) $v \in C((-\infty, t_0], H_0^1(\Omega)) \cap C^1((-\infty, t_0], L^2(\Omega))$,

where $t_0 < 0$ is fixed.

For every $T \geqq |t_0|$ we solve:

$$\begin{aligned} & u_T'' - \Delta u_T + |u_T'|^{p-1} u_T' = 0 && \text{in } (-T, t_0] \times \Omega, \\ & u_T = 0 && \text{on } (-T, t_0] \times \partial\Omega, \\ & u_T(-T) = v(-T),\ u_T'(-T) = v'(-T) && \text{in } \Omega. \end{aligned}$$

Since $(v(-T), v'(-T)) \in H_0^1(\Omega) \cap L^2(\Omega)$ we know (*see* [H1], [L]) that there exists a unique solution

$$u_T \in C((-T, t_0], H_0^1(\Omega)) \cap C^1((-T, t_0], L^2(\Omega)).$$

The next step consists in proving that u_T converges strongly in $C_{\text{loc}}((-\infty, t_0], H_0^1(\Omega)) \cap C_{\text{loc}}^1((-\infty, t_0], L^2(\Omega))$ to a solution u. The proof relies on the following estimate ([B2] Lemma 3.1, p. 64):

Let A be a monotone operator on a Hilbert space H, endowed with a norm $\| \ \|$. For all weak solutions U and V of $U' + AU = F$ and $V' + AV = G$ with F and $G \in L^1(\alpha, \beta; H)$, $[\alpha, \beta] \subset \mathbb{R}$ the following inequality holds:

$$(\star) \qquad \| U(t) - V(t) \| \leqq \| U(s) - V(s) \| + \int_s^t \| F(\sigma) - G(\sigma) \| \, d\sigma \; \alpha \leqq s \leqq t \leqq \beta.$$

Let us fit our problem in this framework. We set $W = (w, z)$ with $z = w'$ and $H = H_0^1(\Omega) \times L^2(\Omega)$. Then, the equation

$$w'' - \Delta w + |w'|^{p-1} w' = h$$

can be rewritten as $W' + AW = (0, h)$ where A defined by:

$$D(A) = \{(w, z) \in H, \ -\Delta w + |z|^{p-1} z \in L^2(\Omega)\},$$

$$AW = (-w, \ -\Delta \omega + |z|^{p-1} z) \quad \forall W \in D(A)$$

is maximal monotone (*cf.* [B3]).

We set $U_T = (u_T, u_T')$, $V = (v, v')$ and $G = (0, f)$. Then:

$$U_T' + AU_T = 0 \quad \text{in } [-T, t_0],$$

$$V' + AV = G \quad \text{in } [-\infty, t_0],$$

with $G \in L^1(-\infty, t_0; L^2(\Omega))$.

For any t belonging to a time interval $[\alpha, \beta]$ we get as a consequence of $(\star)$:

$$\| U_T(t) - U_{T'}(t) \| \leqq \| U_T(\alpha) - U_{T'}(\alpha) \|$$

if $-T' \leqq -T \leqq \alpha \leqq t_0$. Let us take $\alpha = -T$ and $\beta = t_0$. For $-T' < -T < t_0$ we have:

$$\begin{aligned} \| U_T(-T) - U_{T'}(-T) \| &= \| V(-T) - U_{T'}(-T) \| \\ &\leqq \| V(-T') - U_{T'}(-T') \| + \int_{-T'}^{-T} \| f(\sigma) \| \, d\sigma, \end{aligned}$$

so that

$$\| U_T(-T) - U_{T'}(-T) \| \leqq \int_{-T'}^{-T} \| f(\sigma) \| \, d\sigma$$

and

$$\| U_T(t) - U_{T'}(t) \| \leqq \int_{-T'}^{-T} \| f(\sigma) \| \, d\sigma$$

if $-T' \leqq -T \leqq t \leqq t_0$. Letting T', T tend to infinity we conclude that U_T converges strongly in $C_{\text{loc}}(-\infty, t_0; H)$ to some limit $U = (u, u')$. It follows from the general theory on dissipative initial value problems (*see* [H], ch. 2) that $|u'|^{p-1} u' \in L^1_{\text{loc}}(-\infty, t_0; L^1(\Omega))$ and also, since the nonlinear term is odd, the weak solution u turns out to be a solution in the sense of distributions (*see* Remark II.2.3.7. in [H]), that is:

$$u'' - \Delta u + |u'|^{p-1} u' = 0 \quad \text{in } \mathcal{D}'((-\infty, t_0] \times \Omega).$$

Finally, we must make sure that $u \neq 0$. Set $w = u - v$. Applying inequality $(\star)$ to u_T and v and then letting $T \to \infty$ we deduce that:

$$E(w(t)) \leqq C |t|^{2(\alpha+1)} \; \forall t \in [-\infty, t_0).$$

Therefore, $E(w(t))$ tends to zero as $t \to \infty$. Taking into account that $E(v(t)) > K$ we get:

$$2E(v(t)) \geqq E(u(t)) \geqq \frac{1}{2} E(v(t)) \geqq \frac{K}{2}$$

for $|t|$ large. Now, it suffices to extend u by the solution of the dissipative initial value problem with initial data $(u(t_0), u'(t_0))$ for $t > t_0$ to obtain a global solution.

Remark 1.1. – More generally, if a solution

$$v \in C((-\infty, t_0], H_0^1(\Omega)) \cap C^1((-\infty, t_0], L^2(\Omega))$$

of a dissipative problem of the form:

$$v'' - \Delta v + g(v') = f(t, x) \quad \text{in } (-\infty, t_0] \times \Omega$$

is known, such that $E(v(t))$ does not tend to zero as $t \to -\infty$ and $f \in L^1(-\infty, t_0; L^2(\Omega))$ then, we can construct a global solution of:

$$u'' - \Delta u + g(u') = 0 \quad \text{in } \mathbb{R} \times \Omega$$

for any continuous nondecreasing $g : \mathbb{R} \to \mathbb{R}$ such that $g(0) = 0$.

Remark 1.2. – An analogous result holds if we replace the wave operator with homogeneous Dirichlet conditions with another one, provided that the resulting operator A is monotone and a solution $V \in C((-\infty, t_0], H)$ of $V_t + AV = G$ with $G \in L^1(-\infty, t_0; H)$ is known. For instance, we could consider plate operators with some adequate boundary conditions.

2. Proof of Lemma 2

We seek for a function v of the form $v = v_0 + \cdots + v_k$ where the v_i satisfy some adequate partial differential equations.

Let us note $L\,v = v'' - \Delta v + g(v')$ where $g(v') = |\,v'\,|^{p-1}\,v'$. By expanding $g(v')$ we obtain:

$$\begin{aligned} g(v') &= g(v_0') + g'(v_0')(v_1' + \cdots + v_k') + \frac{g''(v_0')}{2}(v_1' + \cdots + v_k')^2 \\ &\quad + \cdots + \frac{g^{k)}(v_0')}{k!}(v_1' + \cdots + v_k')^k \\ &\quad + \frac{g^{k+1)}(\xi)}{(k+1)\,!}(v_1' + \cdots + v_k')^{k+1}, \end{aligned}$$

for some $\xi \in (v_0',\, v_0' + \cdots + v_k')$, where $g^{k)}$ denotes the k-th order derivative of g.

Thus, we can write $L\,v$ in the following way:

$$\begin{aligned} L\,v &= [-\Delta v_0 + g(v_0')] + [-\Delta v_1 + g'(v_0')\,v_1' + v_0''] \\ &\quad + \left[-\Delta v_2 + g'(v_0')\,v_2' + v_1'' + \sum_{2\leqq\alpha\leqq k} \frac{g^{\alpha)}(v_0')}{\alpha!}\, v_1'^{\alpha}\right] \\ &\quad + \left[-\Delta v_3 + g'(v_0')\,v_3' + v_2'' + \sum_{2\leqq\alpha\leqq k} \frac{g^{\alpha)}(v_0')}{\alpha!}\,(v_2'^{\alpha} + \sum_{\alpha\geqq s>0} C_s\, v_1'^{\alpha-s}\, v_2'^{s}\right] \\ &\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\quad + \Bigg[-\Delta v_k + g'(v_0')\,v_k' + v_{k-1}'' + \sum_{2\leqq\alpha\leqq k} \frac{g^{\alpha)}(v_0')}{\alpha!}\, v_{k-1}'^{\alpha} \\ &\qquad + \sum_{2\leqq\alpha\leqq k} \frac{g^{\alpha)}(v_0')}{\alpha!}\,(\sum_{\alpha\geqq s>0} v_{k-1}'^{\alpha-s} \sum_{s_1+\cdots+s_s=s,\, i_j<k} C_{i_1\ldots i_s}\, v_{i_1}'^{s_1}\ldots v_{i_s}'^{s_s})\Bigg] \\ &\quad + \Bigg[v_k'' + \sum_{2\leqq\alpha\leqq k} \frac{g^{\alpha)}(v_0')}{\alpha!}\,(v_k'^{\alpha} + \sum_{\alpha\geqq s>0} v_k'^{\alpha-s} \sum_{s_1+\cdots+s_s=s,\, i_j<k} C_{i_1\ldots i_s}\, v_{i_1}'^{s_1}\ldots v_{i_\alpha}'^{s_s}) \\ &\qquad + \frac{g^{k+1)}(\xi)}{(k+1)!}\,(v_1' + \cdots + v_k')^{k+1}\Bigg]. \end{aligned}$$

We are going to choose the $v_i \in H_0^1$ in such a way that they satisfy:

$$(E_0) \quad -\Delta v_0 + g(v_0') = 0,$$

$$(E_1) \quad -\Delta v_1 + g'(v_0')\,v_1' = -v_0'',$$

$$(E_2) \qquad -\Delta v_2 + g'(v_0')\, v_2' = -v_1'' - \sum_{2 \leqq \alpha \leqq k} \frac{g^{\alpha)}(v_0')}{\alpha!}\, v_1'^{\alpha},$$

$$(E_3) \qquad -\Delta v_3 + g'(v_0')\, v_3' = -v_2'' - \sum_{2 \leqq \alpha \leqq k} \frac{g^{\alpha)}(v_0')}{\alpha!}\, (v_2'^{\alpha} + \sum_{\alpha \geqq s > 0} C_s\, v_1'^{\alpha - s}\, v_2'^{s})$$

. . . .

$$(E_k) \qquad -\Delta v_k + g'(v_0')\, v_k' = -v_{k-1}'' - \sum_{2 \leqq \alpha \leqq k} \frac{g^{\alpha)}(v_0')}{\alpha!}\, v_{k-1}'^{\alpha}$$
$$- \sum_{2 \leqq \alpha \leqq k} \frac{g^{\alpha)}(v_0')}{\alpha!} \Big(\sum_{\alpha \geqq s > 0} v_{k-1}'^{\alpha - s} \sum_{s_1 + \ldots + s_s = s,\, i_j < k} c_{i_1 \ldots i_s}\, v_{i_1}'^{s_1} \cdots v_{i_s}'^{s_s} \Big).$$

Then,

$$L v = \Bigg[v_k'' + \sum_{2 \leqq \alpha \leqq k} \frac{g^{\alpha)}(v_0')}{\alpha!} \Big(v_k'^{\alpha} + \sum_{\alpha \geqq s > 0} v_k'^{\alpha - s} \sum_{s_1 + \ldots + s_s = s,\, i_j < k} C_{i_1 \ldots i_s}\, v_{i_1}'^{s_1} \cdots v_{i_s}'^{s_s} \Big)$$
$$+ \frac{g^{k+1)}(\xi)}{(k+1)!}\, (v_0' + \cdots + v_k')^{k+1} \Bigg]$$

and we will prove that for a certain k (depending on p) and a suitable choice of the v_i the resulting v satisfies the conditions of Lemma 2.

To find v_0 we must be able to solve:

$$(P\, v_0) \qquad \begin{cases} -\Delta v_0 + |\, v_0'\,|^{p-1}\, v_0' = 0 & \text{in } (-\infty,\ 0] \times \Omega, \\ v_0 = 0 & \text{on } (-\infty,\ 0) \times \partial\Omega. \end{cases}$$

If we search for a solution v_0 of the form $f(t)\, g(x)$ we find that $v_0 = |\, t\,|^{\frac{p}{p-1}}\, \psi_0(x)$ where ψ_0 is a solution to:

$$(P\, \psi_0) \qquad \begin{cases} -\Delta \psi_0 = \left(\dfrac{p}{p-1} \right)^p |\psi_0\,|^{p-1}\, \psi_0 & \text{in } \Omega, \\ \psi_0 = 0 & \text{on } \partial\Omega, \end{cases}$$

satisfies $(P\, v_0)$. It follows from standard Min-Max theory that there exist infinitely many solutions ψ_0 provided that $p < \dfrac{n+2}{n-2}$ when $n > 2$ or $p < \infty$ if $n \leqq 2$. A bootstrap argument yields $u \in C^{1,\alpha}(\bar{\Omega})$ for a certain α if Ω is C^2. For $p = \dfrac{n+2}{n-2}$ the existence of solutions to this problem depends on the domain Ω. Solutions, if they exist, are also known to be in $u \in C^{1,\alpha}(\bar{\Omega})$.

Set $\lambda = \frac{p}{p-1}$. If we take $v = v_0$ then

$$L\,v = v_0'' \sim |\,t\,|^{\lambda-2}$$

so that the condition $(\lambda - 2) + 1 < 0$ *i.e.* $\frac{p}{p-1} < 1$ is never satisfied.

Now, we must solve

$$(P\,v_1) \quad \begin{cases} -\Delta v_1 + p\,\lambda^{p-1}\,|\,t\,|\,|\,\psi_0\,|^{p-1}\,v_1' = -\lambda\,(\lambda-1)\,\psi_0\,|\,t\,|^{\lambda-2} & \text{in } (-\infty,\,0]\times\Omega, \\ v_1 = 0 & \text{on } (-\infty,\,0)\times\partial\Omega. \end{cases}$$

We remark that the operator $D = -\Delta + g'(v_0')\frac{\partial}{\partial t} = -\Delta + p\lambda^{p-1}\,|\,t\,|\,|\,\psi_0\,|^{p-1}\frac{\partial}{\partial t}$ has the property $D(|\,t\,|^k\,f(x)) = |\,t\,|^k\,D_k\,f(x)$ where $D_k = -\Delta - p\,\lambda^{p-1}\,k\,|\,\psi_0\,|^{p-1}$. In view of this, if the right hand side has the form $\sum f_i(x)\,|\,t\,|^i$ we can always find a solution $v = \sum v_i(x)\,|\,t\,|^i$ with $D_i(v_i) = f_i$. Thus, it suffices to take $v_1 = |\,t\,|^{\lambda-2}\,\psi_1$ where ψ_1 solves:

$$(P\,\psi_1) \quad \begin{cases} -\Delta\psi_1 - p\,\lambda^{p-1}\,(\lambda-2)\,|\,\psi_0\,|^{p-1}\,\psi_1 = -\lambda\,(\lambda-1)\,\psi_0 & \text{in } \Omega, \\ \psi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Since the right hand side $f = -\lambda\,(\lambda-1)\,\psi_0$ and the potential $c(x) = p\,\lambda^{p-1}(\lambda-2)\,|\,\psi_0\,|^{p-1}$ belong to $L^\infty(\Omega)$ we know by elliptic regularity theory (Th 4.2 of [ST]) that any solution of this equation must be in $L^\infty(\Omega)$ ($c \in L^{\frac{q}{2}}$ and $f \in L^q$, $q > n$ would be enough for this). In order to study the existence of solutions, we must distinguish two cases.

Case a: $p \geqq 2$.

When $\lambda - 2 \leqq 0$, *i.e.* $p \geqq 2$, the existence of a unique solution ψ_1 is guaranteed by the coercivity of the operator $D_{\lambda-2}$ in $H_0^1(\Omega)$.

If $\lambda = 2$, *i.e.* $p = 2$, this procedure yields an exact solution $v = v_0 + v_1 = \psi_0\,|\,t\,|^2 + \psi_1$ where ψ_0, ψ_1 solve:

$$(P_{\psi_0}) \quad \begin{cases} -\Delta\psi_0 = 4\,|\,\psi_0\,|\,\psi_0 & \text{in } \Omega, \\ \psi_0|_{\partial\Omega} = 0. \end{cases}$$

$$(P_{\psi_1}) \quad \begin{cases} -\Delta\psi_1 = -2\,\psi_0 & \text{in } \Omega, \\ \psi_1|_{\partial\Omega} = 0. \end{cases}$$

Indeed, in this case $L\,v_1 = v_1'' = 0$.

In general, taking $v = v_0 + v_1$ we shall have:

$$L\,v = v_1'' + \frac{p\,(p-1)}{2}\,|\,\xi\,|^{p-2}\,v_1'^2$$

for some $\xi \in (v_0',\, v_0' + v_1')$

In the sequel, we shall use the notation $f(x,\,t) \sim |\,t\,|^\alpha$ to mean that

$$m(x)\,|\,t\,|^\alpha \leqq f(x,\,t) \leqq M(x)\,|\,t\,|^\alpha,$$

with $m,\, M \in L^\infty(\Omega)$. In the same way, by $f(x,\,t) \approx \sum_s |\,t\,|^{\alpha - s}$ we mean that

$$f = \sum_s a_s(x)\,|\,t\,|^{\alpha - s}$$

for some $a_s(x) \in L^\infty$.

We always have $v_1'' \sim |\,t\,|^{\lambda-4}$. When $p \geqq 2$ it is clear that $|\,\xi\,| \sim |\,t\,|^{\lambda-1}$ so that $|\,\xi\,|^{p-2}\,v_2'^2 \sim |\,t\,|^{\lambda-4}$. Hence, if $p \geqq 2$ we have $L\,v \sim |\,t\,|^{\lambda-4}$ and the condition $\lambda - 4 < -1$ is verified since $p \geqq 2 > \dfrac{3}{2}$. Therefore, in case $p \geqq 2$ we get infinitely many global solutions of the form $v_0 + v_1$ behaving like $|\,t\,|^{\frac{p}{p-1}}$ when t goes to $-\infty$, one for each solution ψ_0 of (P_{ψ_0}).

Case b : $1 < p < 2$.

When $\lambda > 2$, *i.e.* $1 < p < 2$, the existence of a solution ψ_1 of (P_{ψ_1}) is guaranteed provided that $\beta = p\,(\lambda - 2)\,\lambda^{p-1}$ is not an eigenvalue of

$$\begin{cases} -\Delta\psi = \beta\,|\,\psi_0\,|^{p-1}\,\psi & \text{in } \Omega, \\ \psi|_{\partial\Omega} = 0. \end{cases}$$

We already know that $\beta' = \lambda^p$ is an eigenvalue with eigenfunction ψ_0. Thus, if the sign of ψ_0 remains constant in Ω, necessarily $\beta' = \lambda_1$, *i.e.* β' is the first eigenvalue for this problem. Taking into account have $\beta < \beta'$ we conclude that β is not an eigenvalue. Let us take $v = v_0 + v_1$. Again

$$L\,v = v_1'' + \frac{p\,(p-1)}{2}\,|\,\xi\,|^{p-2}\,v_1'^2$$

for some $\xi \in (v_0',\, v_0' + v_1')$

When $1 < p < 2$, in order to avoid singularities in $|\,\xi\,|^{p-2}\,v_1'^2$ we consider only the solutions ψ_0 whose sign remains constant all over Ω. This restriction on the sign also guarantees the existence of ψ_1, as we pointed out before. Since $v_0'|_{\partial\Omega} = (v_0' + v_1')\,|_{\partial\Omega} = 0$ we still could get singularities on the boundary. Now, assuming Ω to be smooth, the strong maximum principle implies that $\dfrac{\partial\psi_0}{\partial n}$ is either strictly positive or strictly negative on $\partial\Omega$ so that for t large and some $\varepsilon > 0$

$$\frac{|\,\xi - v_0'\,|}{|\,t\,|^{\lambda-1}} \leqq \varepsilon\,|\,\psi_0\,|\; i.e.\; \xi \sim v_0' + \varepsilon\,|\,\psi_0\,|\,|\,t\,|^{\lambda-1} \qquad \text{and} \qquad \left|\frac{v_1'}{\xi}\right|^2 \leqq \left|\frac{\psi_1}{\psi_0}\right|^2 |\,t\,|^{-4}.$$

It follows that $|\,\xi\,|^{p-2}\,v_1'^2 = |\,\xi\,|^p\,\dfrac{v_1'^2}{|\,\xi\,|^2}$ is bounded and $|\,\xi\,|^{p-2}\,v_1'^2 \sim |\,t\,|^{\lambda-4}$. When $2 > p > \dfrac{3}{2}$, the condition $\lambda - 4 < -1$ is fulfilled and we get a global solution for each ψ_0 of constant sign.

If $p \leqq \dfrac{3}{2}$ we must add more terms v_i to obtain the necessary rate of decay in the right hand term. Let us take $v = v_0 + v_1 + v_2$. Then, we must be able to solve $(P\,v_0)$, $(P\,v_1)$ and

$$(P\,v_2) \quad \begin{cases} -\Delta v_2 + g'(v_0')\,v_2' & = -v_1'' - \dfrac{g''(v_0')}{2}\,v_1^2 \quad \text{in } (-\infty,\, 0] \times \Omega, \\ v_2 & = 0 \quad \text{on } (-\infty,\, 0] \times \partial\Omega. \end{cases}$$

If we search for v_i of the form $f(t)\, g(x)$ we find that $v_i = |\,t\,|^{\lambda - 2i}\, \psi_i(x)$ where ψ_i satisfy:

$$(P\,\psi_0)\quad \begin{cases} -\Delta\psi_0 = \left(\dfrac{p}{p-1}\right)^p |\,\psi_0\,|^{p-1}\,\psi_0 & \text{in } \Omega,\\ \psi_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

$$(P\,\psi_1)\quad \begin{cases} -\Delta\psi_1 - p\,\lambda^{p-1}(\lambda - 2)\,|\,\psi_0\,|^{p-1}\,\psi_1 = -\lambda\,(\lambda - 1)\,\psi_0 & \text{in } \Omega,\\ \psi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

$$(P\,\psi_2)\quad \begin{cases} -\Delta\psi_2 - p\,\lambda^{p-1}(\lambda - 4)\,|\,\psi_0\,|^{p-1}\,\psi_2 = f_2(x) & \text{in } \Omega,\\ \psi_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

with $f_2 = -(\lambda - 2)(\lambda - 3)\,\psi_1 - p\,(p-1)(\lambda - 2)^2(\lambda - 1)^{p-2}\,\psi_1^2\,|\,\psi_0\,|^{p-3}\,\psi_0 \in L^\infty(\Omega)$. The existence of a unique solution for each of these problems is assured by the fact that the $\beta_s = p\,\lambda^{p-1}(\lambda - 2\,s)$ are not eigenvalues for $s = 1,\ 2$. Moreover, the solutions belong to L^∞. In this case $L\,v$ is given by:

$$L\,v = v_2'' + \frac{g''(v_0')}{2}\,(v_2'^2 + 2\,v_1'\,v_2') + \frac{g'''(\xi)}{6}\,(v_2'^3 + 3\,v_1'\,v_2'^2 + v_1'^3 + 3\,v_1'^2\,v_2').$$

For $|\,t\,|$ large, v_2'', $\dfrac{g''(v_0')}{2}\,v_1'\,v_2'$, $\dfrac{g'''(\xi)}{6}\,v_3'^3 \sim |\,t\,|^{\lambda - 6}$, the remaining terms being of lower order. Hence, $L\,v \sim |\,t\,|^{\lambda - 6}$ and the condition $\lambda - 6 < -1$ is verified for $p > \dfrac{5}{4}$.

When $p \leqq \dfrac{5}{4}$ we must add v_3. The equations for v_0, v_1 are the same, but now v_2, v_3 must solve:

$$(P\,v_2)\quad \begin{cases} -\Delta v_2 + g'(v_0')\,v_2' = f_1(t,\,x) & \text{in } (-\infty,\,0] \times \Omega,\\ v_2 = 0 & \text{on } (-\infty,\,0) \times \partial\Omega, \end{cases}$$

$$(P\,v_3)\quad \begin{cases} -\Delta v_3 + g'(v_0')\,v_3' = f_2(t,\,x) & \text{in } (-\infty,\,0] \times \Omega,\\ v_2 = 0 & \text{on } (-\infty,\,0) \times \partial\Omega, \end{cases}$$

where:

$$f_1(t,\,x) = -v_1'' - \frac{g''(v_0')}{2}\,v_1^2 - \frac{g'''(v_0')}{6}\,v_1^3,$$

$$f_2(t,\,x) = -v_2'' - \frac{g''(v_0')}{2}\,(v_2^2 + 2\,v_1'\,v_2') - \frac{g'''(v_0')}{6}\,(v_2^3 + a v_1^2\,v_2 + b v_2^2\,v_1).$$

This time $-v_1'' - \dfrac{g''(v_0')}{2}\,v_1^2 \sim |\,t\,|^{\lambda - 4}$ but $-\dfrac{g'''(v_0')}{6}\,v_1^3 \sim |\,t\,|^{\lambda - 6}$ so that $v_2 \sim |\,t\,|^{\lambda - 4} + |\,t\ |^{\lambda - 6}$. In the same way we deduce that $f_2 \sim |\,t\,|^{\lambda - 6} + \sum_{k \geq 4} |\,t\,|^{\lambda - 2k}$ and from the expression for $L\,v$ it follows that $L\,v \sim |\,t\,|^{\lambda - 8}$. We can stop here for $p > \dfrac{7}{8}$.

Following this procedure, for $v_0 = v_0 + \cdots + v_k$ we get that $L\,v \sim |\,t\,|^{\lambda - 2\,(k+1)}$. When $p > \dfrac{2\,k + 1}{2\,k}$, we have $\lambda - 2\,(k + 1) < -1$ and we can take $v_0 + \cdots + v_k$ for the needed approximation. Since $\dfrac{2k+1}{2k}$ converges to 1 as k tends to infinity, one can cover in this way all the range $1 < p < 2$. The coefficients β_i appearing in these elliptic problems are not eigenvalues, so that the problems $(P\,\psi_i)$ have a (bounded) solution. Indeed, the $v = v_0 + \cdots + v_k$ constructed by this procedure is such that equations (E_i) are verified. By proceeding as before it is easy to prove that:

$$f_0,\ v_0 \approx |\,t\,|^{\lambda},$$

$$f_1,\ v_1 \approx |\,t\,|^{\lambda-2},$$

$$f_2,\ v_2 \approx |\,t\,|^{\lambda-4} + \sum_{s>2} |\,t\,|^{\lambda-2s}.$$

Let us assume that:

$$f_{k-1},\ v_{k-1} \approx |\,t\,|^{\lambda-2\,(k-1)} + \sum_{s>k-1} |\,t\,|^{\lambda-2s}.$$

Then, the right hand side in (E_k) has the form:

$$f_k = a_k\,(x)\,|\,t\,|^{\lambda-2k} + \sum_{s>k} a_s\,(x)\,|\,t\,|^{\lambda-2s},$$

so that we can find a solution

$$v_k = \sum_{s\geq k} c_s\,(x)\,|\,t\,|^{\lambda-2s},$$

where the $c_s\,(x)$ solve elliptic problems of the form:

$$(P_s)\quad \begin{cases} -\Delta c_s - p\,\lambda^{p-1}\,(\lambda - 2\,s)\,|\,\psi_0\,|^{p-1}\,c_s = a_s\,(x) & \text{in } \Omega, \\ c_s = 0 & \text{on } \partial\Omega. \end{cases}$$

From the expression for $L\,v$ and the behavior of v_i we conclude that $L\,v \sim |\,t\,|^{\lambda-2\,(k+1)}$.

Remark 2.1. – A slight modification allows to handle the case $a\,(x)\,|\,u'\,|^{p-1}\,u' + b\,(x)\,|\,u'\,|^{q-1}\,u'$ where $p > q > 1$, $p > 1$ and $a\,(x)$, $b\,(x)$ are non negative bounded functions. We could also replace the operator $-\Delta$ with homogeneous boundary conditions by a plate operator with adequate boundary conditions (in order the maximum principle to hold) or $-\Delta u + u$ with homogeneous Neumann conditions.

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