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# Real and Complex Singularities

Edited By  
M. Manoel, M. C. Romero Fuster  
and C. T. C Wall

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This volume is dedicated to  
*Maria Aparecida Ruas*  
and  
*Terence Gaffney*  
on their 60<sup>th</sup> birthdays



*Maria Aparecida Ruas*



*Terence Gaffney*



# Contents

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	<i>Preface</i>	<i>page xiii</i>
<b>1</b>	<b>On a conjecture by A. Durfee    <i>E. Artal Bartolo, J. Carmona Ruber &amp; A. Melle-Hernández</i></b>	<b>1</b>
1	Introduction	1
2	Invariants of singularities	2
3	Superisolated surface singularities	5
4	Zariski pairs	11
	References	15
<b>2</b>	<b>On normal embedding of complex algebraic surfaces</b>	
	<b><i>L. Birbrair, A. Fernandes &amp; W. D. Neumann</i></b>	<b>17</b>
1	Introduction	17
2	Proof	18
	References	21
<b>3</b>	<b>Local Euler obstruction, old and new, II    <i>Jean-Paul Brasselet &amp; Nivaldo G. Grulha Jr.</i></b>	<b>23</b>
1	Introduction	23
2	Definition of the Euler obstruction	26
3	Local Euler obstruction for a frame	29
4	Euler obstruction and hyperplane sections	30
5	The local Euler obstruction of a function	32
6	The Euler obstruction and the Euler defect	33
7	The Euler defect at general points	34
8	The Euler obstruction via Morse theory	34
9	Generalizations of Milnor number	35
10	Bruce and Roberts' Milnor Number	36
11	Local Euler obstruction of an analytic map	38



12	Local Euler obstruction of 1-forms	39
13	Local Chern obstruction of collections of 1-forms and special points	41
	References	43
<b>4</b>	<b>Branching of periodic orbits in reversible Hamiltonian systems</b> <i>C. A. Buzzi, L. A. Roberto &amp; M. A. Teixeira</i>	<b>46</b>
1	Introduction	46
2	Preliminaries	49
3	Belitskii normal form	55
4	Liapunov–Schmidt reduction	56
5	Birkhoff normal form	59
6	Two degrees of freedom	60
7	Three degrees of freedom	65
8	Examples	68
	References	69
<b>5</b>	<b>Topological invariance of the index of a binary differential equation</b> <i>L. S. Challapa</i>	<b>71</b>
1	Introduction	71
2	Index of a binary differential equation	72
3	Topological invariance of $I(E, 0)$	75
4	Examples	79
	References	80
<b>6</b>	<b>About the existence of Milnor fibrations</b> <i>J. L. Cisneros-Molina &amp; R. N. Araújo Dos Santos</i>	<b>82</b>
1	Introduction	82
2	Milnor fibrations for complex maps	86
3	Milnor fibrations for real maps	91
	References	101
<b>7</b>	<b>Counting hypersurfaces invariant by one-dimensional complex foliations</b> <i>Maurício Corrêa Jr. &amp; Márcio G. Soares</i>	<b>104</b>
1	Introduction	104
2	Statement of results	106
3	Examples	109
4	Proofs	110
	References	113
<b>8</b>	<b>A note on topological contact equivalence</b> <i>J. C. F. Costa</i>	<b>114</b>
1	Introduction	114
2	Topological contact equivalence	115

3	Properties and invariants	115
	References	123
<b>9</b>	<b>Bi-Lipschitz equivalence, integral closure and invariants</b> <i>Terence Gaffney</i>	<b>125</b>
1	Introduction	125
2	The Lipschitz saturation of an ideal or module	126
3	Bi-Lipschitz equisingularity of families of curves	132
	References	136
<b>10</b>	<b>Solutions to PDEs and stratification conditions</b> <i>Terence Gaffney</i>	<b>138</b>
1	Introduction	138
2	The method of characteristics and W stratifications	138
	References	145
<b>11</b>	<b>Real integral closure and Milnor fibrations</b> <i>Terence T. Gaffney &amp; Raimundo Araújo Dos Santos</i>	<b>146</b>
1	Introduction	146
2	Notations and setup	149
3	Some results	150
	References	156
<b>12</b>	<b>Surfaces around closed principal curvature lines, an inverse problem</b> <i>R. Garcia, L. F. Mello &amp; J. Sotomayor</i>	<b>158</b>
1	Introduction	158
2	Preliminary results	160
3	Hyperbolic principal cycles	162
4	Concluding remarks	165
	References	166
<b>13</b>	<b>Euler characteristics and atypical values</b> <i>Helmut A. Hamm</i>	<b>167</b>
1	Cohomological results	168
2	Absence of vanishing cycles	174
3	Topological triviality	178
	References	183
<b>14</b>	<b>Answer to a question of Zariski</b> <i>A. Hefez &amp; M. E. Hernandez</i>	<b>185</b>
1	Introduction	185
2	Classification of the equisingularity class $\langle 7, 8 \rangle$	185
3	The question of Zariski	186
	References	189

<b>15</b>	<b>Projections of timelike surfaces in the de Sitter space</b>	<b>190</b>
	<i>Shyuichi Izumiya &amp; Farid Tari</i>	
1	Introduction	190
2	Preliminaries	191
3	The family of projections along geodesics in $S_1^n$	195
4	Projections of timelike surfaces in $S_1^3$	198
5	Duality	205
6	Appendix	208
	References	209
<b>16</b>	<b>Spacelike submanifolds of codimension at most two in de Sitter space</b>	<b>211</b>
	<i>M. Kasedou</i>	
1	Introduction	211
2	Lightcone Gauss images of spacelike hypersurfaces	212
3	Lightcone height functions	215
4	Generic properties and de Sitter hyperhorospheres	217
5	Spacelike surfaces in de Sitter 3-space	220
6	Spacelike submanifolds of codimension 2	221
7	Lightlike hypersurfaces and contact with lightcones	223
8	Lightcone Gauss maps and lightcone height functions	224
9	Classification in de Sitter 4-space	226
	References	227
<b>17</b>	<b>The geometry of Hopf and saddle-node bifurcations for waves of Hodgkin-Huxley type</b>	<b>229</b>
	<i>Isabel S. Labouriau &amp; Paulo R. F. Pinto</i>	
1	Introduction – equations of Hodgkin-Huxley type	230
2	Linearisation	233
3	The exchange of stability set	235
4	The exchange of stability set in (HHW)	238
5	Example: single channel Hodgkin-Huxley equations	241
	References	244
<b>18</b>	<b>Global classifications and graphs</b>	<b>246</b>
	<i>J. Martínez-Alfaro, C. Mendes de Jesus &amp; M. C. Romero-Fuster</i>	
1	Introduction	246
2	Surfaces and graphs	247
3	3-Manifolds and graphs	262
	References	264
<b>19</b>	<b>Real analytic Milnor fibrations and a strong Łojasiewicz inequality</b>	<b>268</b>
	<i>David B. Massey</i>	
1	Introduction	268
2	Singular values of matrices	271

3	$\mathbb{L}$ -maps and $\mathbb{L}$ -weights	274
4	Milnor's conditions (a) and (b)	282
5	The main theorem	286
6	Comments and questions	290
	References	291
<b>20</b>	<b>An estimate of the degree of <math>\mathcal{L}</math>-determinacy by the degree of <math>\mathcal{A}</math>-determinacy for curve germs</b> <i>T. Nishimura</i>	<b>293</b>
1	Introduction	293
2	Proof of Theorem 1.1	294
	References	297
<b>21</b>	<b>Regularity of the transverse intersection of two regular stratifications</b> <i>Patrice Orro &amp; David Trotman</i>	<b>298</b>
1	Introduction	298
2	Regular stratifications	299
	References	303
<b>22</b>	<b>Pairs of foliations on surfaces</b> <i>Farid Tari</i>	<b>305</b>
1	Introduction	305
2	Examples from differential geometry	307
3	Classification	311
4	Invariants	325
5	Bifurcations	330
6	Appendix	331
	References	334
<b>23</b>	<b>Bi-Lipschitz equisingularity</b> <i>David Trotman</i>	<b>338</b>
1	An oft-heard slogan	338
2	Existence of Whitney and Verdier stratifications	339
3	O-minimal structures	340
4	Local triviality along strata	341
5	Mostowski's Lipschitz stratifications	342
6	Consequences of Valette's triangulation theorem	344
7	Appendix. Definition of Lipschitz stratifications	345
	References	347
<b>24</b>	<b>Gaffney's work on equisingularity</b> <i>C. T. C. Wall</i>	<b>350</b>
	Introduction	350
1	Early results on equisingularity	351
2	Notations	356
3	Integral closures	357
4	Invariants	362
5	Criteria for equisingularity	366

6	Recent work	369
7	Bibliography of Terry Gaffney's papers	372
	References	374
<b>25</b>	<b>Singularities in algebraic data acquisition</b> <i>Y. Yomdin</i>	<b>378</b>
1	Introduction	378
2	Examples of moment inversion	381
3	Singularities in reconstruction of linear combinations of $\delta$ -functions	389
4	Noisy measurements	393
	References	394

## Preface

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The *Workshops on Real and Complex Singularities* form a series of biennial meetings organized by the Singularities group at Instituto de Ciências Matemáticas e de Computação of São Paulo University (ICMC-USP), Brazil. Their main purpose is to bring together world experts and young researchers in singularity theory, applications and related fields to report recent achievements and exchange ideas, addressing trends of research in a stimulating environment.

These meetings started in 1990 following two pioneer symposia on Singularity Theory (in fact the very first talks on Singularities held in Brazil) organized respectively by G. Loibel and L. Favaro in 1982 and 1988 at ICMC-USP. Since then, with Maria Aparecida Ruas as a driving force, meetings have taken place every two years between singularists from around the world who find in São Carlos a centre to interact and develop new ideas.

The meeting held from the 27th of July to the 2nd of August 2008 was the tenth of these workshops. This was a special occasion, for it was also dedicated to Maria Aparecida Ruas (Cidinha) and Terence Gaffney on their 60th birthdays.

Cidinha and Terry started their scientific connection in 1976 when she was a Ph.D. student at Brown University in the U.S.A. At that time Terry held a position as instructor at that university. Their common interest in singularity theory brought them together and he became her (very young) thesis supervisor. Cidinha returned to Brazil in 1980 and joined the Singularities group created by G. Loibel at the ICMC-USP. Her great capacity and notable enthusiasm has brought the group to a leading position in the Brazilian mathematical community.

The mathematical interaction between Cidinha and Terry has had an important influence on the development of Singularity Theory at the ICMC-USP. From the beginning Terry has attended almost all workshops organized by the

members of Cidinha's group, contributing to their research with stimulating discussions and seminars.

This workshop had a total of about 170 participants from about 15 different countries. The formal proceedings consisted of 27 plenary talks, 27 ordinary sessions and 3 poster sessions, with a total of 19 posters. The topics were divided into six categories: real singularities, classification of singularities, topology of singularities, global theory of singularities, singularities in geometry, and dynamical systems.

The Scientific Committee was composed of Lev Birbrair (Universidade Federal do Ceará, Brazil), Jean-Paul Brasselet (Institut de Mathématiques de Luminy, France), Goo Ishikawa (Hokkaido University, Japan), Shyuichi Izumiya (Hokkaido University, Japan), Steven Kleiman (Massachusetts Institute of Technology, USA), David Massey (Northeastern University, USA), David Mond (University of Warwick, UK), Maria del Carmen Romero Fuster (Universitat de València, Spain), Marcio Gomes Soares (Universidade Federal de Minas Gerais, Brazil), Marco Antonio Teixeira (Universidade Estadual de Campinas, Brazil), David Trotman (Université de Provence, France) and Terry Wall (University of Liverpool, UK).

Thanks are due to many people and institutions crucial in the realization of the workshop. We start by thanking the Organizing Committee: Roberta Wik Atique, Abramo Hefez, Isabel Labouriau, Miriam Manoel, Ana Claudia Nabarro, Regilene Oliveira and Marcelo José Saia. We also thank the members of the Scientific Committee for their support. The workshop was funded by FAPESP, CNPq, CAPES, USP and SBM, whose support we gratefully acknowledge. Finally, it is a pleasure to thank the speakers and the other participants whose presence was the real success of the tenth Workshop.

The editors

## Introduction

This book is a selection of papers submitted for the proceedings of the *10th Workshop on Real and Complex Singularities*. They are grouped into three categories: singularity theory (7 papers), singular varieties (8 papers) and applications to dynamical systems, generic geometry, singular foliations, etc. (10 papers). Among them, four are survey papers: *Local Euler obstruction, old and new, II*, by N. G. Grulha Jr. and J.-P. Brasselet, *Global classifications and graphs*, by J. Martínez-Alfaro, C. Mendes de Jesus and M. C. Romero-Fuster, *Pairs of foliations on surfaces*, by F. Tari, and *Gaffney's work on equisingularity*, by C. T. C. Wall.

We thank the staff members of the London Mathematical Society involved with the preparation of this book. All papers presented here have been refereed.





# 1

## On a conjecture by A. Durfee

E. ARTAL BARTOLO, J. CARMONA RUBER  
AND A. MELLE-HERNÁNDEZ

### Abstract

We show how *superisolated surface singularities* can be used to find a counterexample to the following conjecture by A. Durfee [8]: for a complex polynomial  $f(x, y, z)$  in three variables vanishing at 0 with an isolated singularity there, “the local complex algebraic monodromy is of finite order if and only if a resolution of the germ  $(\{f = 0\}, 0)$  has no cycles”. A Zariski pair is given whose corresponding superisolated surface singularities, one has complex algebraic monodromy of finite order and the other not (answering a question by J. Stevens).

### 1. Introduction

In this paper we give an example of a *superisolated surface singularity*  $(V, 0) \subset (\mathbb{C}^3, 0)$  such that a resolution of the germ  $(V, 0)$  has no cycles and the local complex algebraic monodromy of the germ  $(V, 0)$  is not of finite order, contradicting a conjecture proposed by Durfee [8].

For completeness in the second section we recall well known results about monodromy of the Milnor fibration, about normal surface singularities and state the question by Durfee.

In the third section we recall results on superisolated surface singularities and with them we study in detail the counterexample.

In the last section we show a Zariski pair  $(C_1, C_2)$  of curves of degree  $d$  given by homogeneous polynomials  $f_1(x, y, z)$  and  $f_2(x, y, z)$  whose

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corresponding superisolated surface singularities  $(V_1, 0) = (\{f_1(x, y, z) + l^{d+1} = 0\}, 0) \subset (\mathbb{C}^3, 0)$  and  $(V_2, 0) = (\{f_2(x, y, z) + l^{d+1} = 0\}, 0) \subset (\mathbb{C}^3, 0)$  ( $l$  is a generic hyperplane) satisfy: 1)  $(V_1, 0)$  has complex algebraic monodromy of finite order and 2)  $(V_2, 0)$  has complex algebraic monodromy of infinite order (answering a question proposed to us by J. Stevens).

## 2. Invariants of singularities

### 2.1. Monodromy of the Milnor fibration

Let  $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$  be an analytic function defining a germ  $(V, 0) := (f^{-1}\{0\}, 0) \subset (\mathbb{C}^{n+1}, 0)$  of a hypersurface singularity. The *Milnor fibration* of the holomorphic function  $f$  at 0 is the  $C^\infty$  locally trivial fibration  $f| : B_\varepsilon(0) \cap f^{-1}(\mathbb{D}_\eta^*) \rightarrow \mathbb{D}_\eta^*$ , where  $B_\varepsilon(0)$  is the open ball of radius  $\varepsilon$  centered at 0,  $\mathbb{D}_\eta = \{z \in \mathbb{C} : |z| < \eta\}$  and  $\mathbb{D}_\eta^*$  is the open punctured disk ( $0 < \eta \ll \varepsilon$  and  $\varepsilon$  small enough). Milnor's classical result also shows that the topology of the germ  $(V, 0)$  in  $(\mathbb{C}^{n+1}, 0)$  is determined by the pair  $(S_\varepsilon^{2n+1}, L_V^{2n-1})$ , where  $S_\varepsilon^{2n+1} = \partial B_\varepsilon(0)$  and  $L_V^{2n-1} := S_\varepsilon^{2n+1} \cap V$  is the *link* of the singularity.

Any fiber  $F_{f,0}$  of the Milnor fibration is called the *Milnor fiber* of  $f$  at 0. The *monodromy transformation*  $h : F_{f,0} \rightarrow F_{f,0}$  is the well-defined (up to isotopy) diffeomorphism of  $F_{f,0}$  induced by a small loop around  $0 \in \mathbb{D}_\eta$ . The *complex algebraic monodromy of  $f$  at 0* is the corresponding linear transformation  $h_* : H_*(F_{f,0}, \mathbb{C}) \rightarrow H_*(F_{f,0}, \mathbb{C})$  on the homology groups.

If  $(V, 0)$  defines a germ of isolated hypersurface singularity then  $\tilde{H}_j(F_{f,0}, \mathbb{C}) = 0$  but for  $j = 0, n$ . In particular the non-trivial complex algebraic monodromy will be denoted by  $h : H_n(F_{f,0}, \mathbb{C}) \rightarrow H_n(F_{f,0}, \mathbb{C})$  and  $\Delta_V(t)$  will denote its characteristic polynomial.

### 2.2. Monodromy Theorem and its supplements

The following are the **main properties of the monodromy operator**, see e.g. [11]:

- (a)  $\Delta_V(t)$  is a product of cyclotomic polynomials.
- (b) Let  $N$  be the maximal size of the Jordan blocks of  $h$ , then  $N \leq n + 1$ .
- (c) Let  $N_1$  be the maximal size of the Jordan blocks of  $h$  for the eigenvalue 1, then  $N_1 \leq n$ .
- (d) The monodromy  $h$  is called of *finite order* if there exists  $N > 0$  such that  $h^N = Id$ . Lê D.T. [12] proved that the monodromy of an irreducible plane curve singularity is of finite order.
- (e) This result was extended by van Doorn and Steenbrink [7] who showed that if  $h$  has a Jordan block of maximal size  $n + 1$  then

$N_1 = n$ , i.e. there exists a Jordan block of  $h$  of maximal size  $n$  for the eigenvalue 1.

Milnor proved that the link  $L_V^{2n-1}$  is  $(n-2)$ -connected. Thus the link is an *integer (resp. rational) homology  $(2n-1)$ -sphere* if  $H_{n-1}(L_V^{2n-1}, \mathbb{Z}) = 0$  (resp.  $H_{n-1}(L_V^{2n-1}, \mathbb{Q}) = 0$ ). These can be characterized considering the natural map  $h - id : H_n(F_{f,0}, \mathbb{Z}) \rightarrow H_n(F_{f,0}, \mathbb{Z})$  and using Wang's exact sequence which reads as (see e.g. [19, 21]):

$$0 \rightarrow H_n(L_V^{2n-1}, \mathbb{Z}) \rightarrow H_n(F_{f,0}, \mathbb{Z}) \rightarrow H_n(F_{f,0}, \mathbb{Z}) \rightarrow H_{n-1}(L_V^{2n-1}, \mathbb{Z}) \rightarrow 0.$$

Thus  $\text{rank } H_n(L_V^{2n-1}) = \text{rank } H_{n-1}(L_V^{2n-1}) = \dim \ker(h - id)$  and:

- $L_V^{2n-1}$  is a rational homology  $(2n-1)$ -sphere  $\iff \Delta_V(1) \neq 0$ ,
- $L_V^{2n-1}$  is an integer homology  $(2n-1)$ -sphere  $\iff \Delta_V(1) = \pm 1$ .

### 2.3. Normal surface singularities

Let  $(V, 0) = (\{f_1 = \dots = f_m = 0\}, 0) \subset (\mathbb{C}^N, 0)$  be a normal surface singularity with link  $L_V := V \cap S_\varepsilon^{2N-1}$ ,  $L_V$  is a connected compact oriented 3-manifold. Since  $V \cap B_\varepsilon$  is a cone over the link  $L_V$  then  $L_V$  characterizes the topological type of  $(V, 0)$ . The link  $L_V$  is called a *rational homology sphere* ( $\mathbb{Q}$ HS) if  $H_1(L_V, \mathbb{Q}) = 0$ , and  $L_V$  is called an *integer homology sphere* ( $\mathbb{Z}$ HS) if  $H_1(L_V, \mathbb{Z}) = 0$ . One of the main problems in the study of normal surfaces is to determine which analytical properties of  $(V, 0)$  can be read from the topology of the singularity, see the very nice survey paper by Nemethi [20].

The resolution graph  $\Gamma(\pi)$  of a resolution  $\pi : \tilde{V} \rightarrow V$  allows to relate analytical and topological properties of  $V$ . W. Neumann [22] proved that the information carried in any resolution graph is the same as the information carried by the link  $L_V$ . Let  $\pi : \tilde{V} \rightarrow V$  be a *good* resolution of the singular point  $0 \in V$ . Good means that  $E = \pi^{-1}\{0\}$  is a normal crossing divisor. Let  $\Gamma(\pi)$  be the dual graph of the resolution (each vertex decorated with the genus  $g(E_i)$  and the self-intersection  $E_i^2$  of  $E_i$  in  $\tilde{V}$ ). Mumford proved that the intersection matrix  $I = (E_i \cdot E_j)$  is negative definite and Grauert proved the converse, i.e., any such graph comes from the resolution of a normal surface singularity.

Considering the exact sequence of the pair  $(\tilde{V}, E)$  and using  $I$  is non-degenerated then

$$0 \longrightarrow \text{coker } I \longrightarrow H_1(L_V, \mathbb{Z}) \longrightarrow H_1(E, \mathbb{Z}) \longrightarrow 0$$

and  $\text{rank } H_1(E) = \text{rank } H_1(L_V)$ . In fact  $L_V$  is a  $\mathbb{Q}$ HS if and only if  $\Gamma(\pi)$  is a tree and every  $E_i$  is a rational curve. If additionally  $I$  has determinant  $\pm 1$  then  $L_V$  is an  $\mathbb{Z}$ HS.

## 2.4. Number of cycles in the exceptional set $E$ and Durfee's conjecture

In general one gets

$$\text{rank } H_1(L_V) = \text{rank } H_1(\Gamma(\pi)) + 2 \sum_i g(E_i),$$

where  $\text{rank } H_1(\Gamma(\pi))$  is the number of independent cycles of the graph  $\Gamma(\pi)$ . Let  $n : \tilde{E} \rightarrow E$  be the normalization of  $E$ . Durfee showed in [8] that the *number of cycles*  $c(E)$  in  $E$ , i.e.  $c(E) = \text{rank } H_1(E) - \text{rank } H_1(\tilde{E})$ , does not depend on the resolution and in fact it is equal to  $c(E) = \text{rank } H_1(\Gamma(\pi))$ . Therefore,  $E$  contains cycles only when the dual graph of the intersections of the components contains a cycle. Durfee in [8] proposed the following

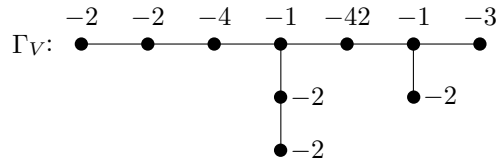
**Conjecture.** *For a complex polynomial  $f(x, y, z)$  in three variables vanishing at 0 with an isolated singularity there, “the local complex algebraic monodromy  $h$  is of finite order if and only if a resolution of the germ  $(\{f = 0\}, 0)$  has no cycles”.*

He showed that the conjecture is true in the following two cases:

- (1) if  $f$  is weighted homogeneous (the resolution graph is star-shaped and therefore its monodromy is finite)
- (2) if  $f = g(x, y) + z^n$ . Using Thom-Sebastiani [27], the monodromy of  $f$  is finite if and only if the monodromy of  $g$  is finite. Theorem 3 in [8] proves that the monodromy of  $f$  is of finite order if and only if a resolution of  $f$  has no cycles.

## 2.5. Example (main result)

In this paper we show that the conjecture is not true in general, and for that we use superisolated surface singularities. Let  $(V, 0) \subset (\mathbb{C}^3, 0)$  be the germ of normal surface singularity defined by  $f := (xz - y^2)^3 - ((y - x)x^2)^2 + z^7 = 0$ . Then the minimal good resolution graph  $\Gamma_V$  of (the superisolated singularity)  $(V, 0)$  is



where every dot denotes a rational non-singular curve with the corresponding self-intersection. Thus the link  $L_V$  is a rational homology sphere and in particular this graph is a tree, i.e. it has no cycles. But the complex algebraic monodromy of  $f$  at 0 does not have finite order because there exists a Jordan block of size  $2 \times 2$  for an eigenvalue  $\neq 1$ .

### 3. Superisolated surface singularities

**Definition 3.1.** A hypersurface surface singularity  $(V, 0) \subset (\mathbb{C}^3, 0)$  defined as the zero locus of  $f = f_d + f_{d+1} + \dots \in \mathbb{C}\{x, y, z\}$  (where  $f_j$  is homogeneous of degree  $j$ ) is *superisolated*, SIS for short, if the singular points of the complex projective plane curve  $C := \{f_d = 0\} \subset \mathbb{P}^2$  are not situated on the projective curve  $\{f_{d+1} = 0\}$ , that is  $\text{Sing}(C) \cap \{f_{d+1} = 0\} = \emptyset$ . Note that  $C$  must be reduced.

The SIS were introduced by I. Luengo in [17] to study the  $\mu$ -constant stratum. The main idea is that for a SIS the embedded topological type (and the equisingular type) of  $(V, 0)$  does not depend on the choice of  $f_j$ 's (for  $j > d$ , as long as  $f_{d+1}$  satisfies the above requirement), e.g. one can take  $f_j = 0$  for any  $j > d + 1$  and  $f_{d+1} = l^{d+1}$  where  $l$  is a linear form not vanishing at the singular points [18].

#### 3.1. The minimal resolution of a SIS

Let  $\pi : \tilde{V} \rightarrow V$  be the monoidal transformation centered at the maximal ideal  $\mathfrak{m} \subset \mathcal{O}_V$  of the local ring of  $V$  at 0. Then  $(V, 0)$  is a SIS if and only if  $\tilde{V}$  is a non-singular surface. Thus  $\pi$  is the *minimal resolution* of  $(V, 0)$ . To construct the resolution graph  $\Gamma(\pi)$  consider  $C = C_1 + \dots + C_r$  the decomposition in irreducible components of the reduced curve  $C$  in  $\mathbb{P}^2$ . Let  $d_i$  (resp.  $g_i$ ) be the degree (resp. genus) of the curve  $C_i$  in  $\mathbb{P}^2$ . Then  $\pi^{-1}\{0\} \cong C = C_1 + \dots + C_r$  and the self-intersection of  $C_i$  in  $\tilde{V}$  is  $C_i \cdot C_i = -d_i(d - d_i + 1)$ , [17, Lemma 3]. Since the link  $L_V$  can be identified with the boundary of a regular neighbourhood of  $\pi^{-1}\{0\}$  in  $\tilde{V}$  then the topology of the tangent cone determines the topology of the abstract link  $L_V$  [17].

#### 3.2. The minimal good resolution of a SIS

The minimal good resolution of a SIS  $(V, 0)$  is obtained after  $\pi$  by doing the minimal embedded resolution of each plane curve singularity  $(C, P) \subset (\mathbb{P}^2, P)$ ,  $P \in \text{Sing}(C)$ . This means that the support of the minimal good resolution graph

$\Gamma_V$  is the same as the minimal embedded resolution graph  $\Gamma_C$  of the projective plane curve  $C$  in  $\mathbb{P}^2$ . The decorations of the minimal good resolution graph  $\Gamma_V$  are as follows:

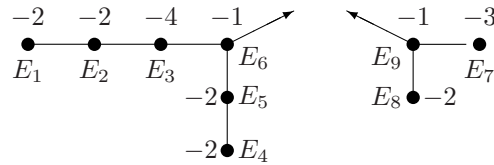
1) the genus of (the strict transform of) each irreducible component  $C_i$  of  $C$  is a birational invariant and then one can compute it as an embedded curve in  $\mathbb{P}^2$ . All the other curves are non-singular rational curves.

2) Let  $C_j$  be an irreducible component of  $C$  such that  $P \in C_j$  and with multiplicity  $n \geq 1$  at  $P$ . After blowing-up at  $P$ , the new self-intersection of the (strict transform of the) curve  $C_j$  in the (strict transform of the) surface  $\tilde{V}$  is  $C_j^2 - n^2$ . In this way one constructs the minimal good resolution graph  $\Gamma$  of  $(V, 0)$ .

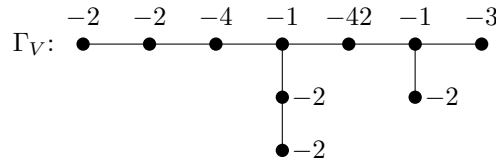
In particular the theory of hypersurface superisolated surface singularities “contains” in a canonical way the theory of complex projective plane curves.

**Example 3.2.** If  $(V, 0) \subset (\mathbb{C}^3, 0)$  is a SIS with an irreducible tangent cone  $C \subset \mathbb{P}^2$  then  $L_V$  is a rational homology sphere if and only if  $C$  is a rational curve and each of its singularities  $(C, p)$  is locally irreducible, i.e a cusp.

**Example 3.3.** For instance, if  $f = f_6 + z^7$  is given by the equation  $f_6 = (xz - y^2)^3 - ((y - x)x^2)^2$ . The plane projective curve  $C$  defined by  $f_6 = 0$  is irreducible with two singular points:  $P_1 = [0 : 0 : 1]$  (with a singularity of local singularity type  $u^3 - v^{10}$ ) and  $P_2 = [1 : 1 : 1]$  (with a singularity of local singularity type  $\mathbb{A}_2$ ) which are locally irreducible. Let  $\pi : X \rightarrow \mathbb{P}^2$  be the minimal embedded resolution of  $C$  at its singular points  $P_1, P_2$ . Let  $E_i, i \in I$ , be the irreducible components of the divisor  $\pi^{-1}(f^{-1}\{0\})$ .



The minimal good resolution graph  $\Gamma_V$  of the superisolated singularity  $(V, 0)$  is given by



### 3.3. The embedded resolution of a SIS

In [2], the first author has studied, for SIS, the Mixed Hodge Structure of the cohomology of the Milnor fibre introduced by Steenbrink and Varchenko, [28], [29]. For that he constructed in an effective way an embedded resolution of a SIS and described the MHS in geometric terms depending on invariants of the pair  $(\mathbb{P}^2, C)$ .

The first author determined the Jordan form of the complex monodromy on  $H_2(F_{f,0}, \mathbb{C})$  of a SIS. Let  $\Delta_V(t)$  be the corresponding characteristic polynomial of the complex monodromy on  $H_2(F_{f,0}, \mathbb{C})$ . Denote by  $\mu(V, 0) = \deg(\Delta_V(t))$  the Milnor number of  $(V, 0) \subset (\mathbb{C}^3, 0)$ .

Let  $\Delta^P(t)$  be the characteristic polynomial (or Alexander polynomial) of the action of the complex monodromy of the germ  $(C, P)$  on  $H_1(F_{g^P}, \mathbb{C})$ , (where  $g^P$  is a local equation of  $C$  at  $P$  and  $F_{g^P}$  denotes the corresponding Milnor fiber). Let  $\mu^P$  be the Milnor number of  $C$  at  $P$ . Recall that if  $n^P : \tilde{C}^P \rightarrow (C, P)$  is the normalization map then  $\mu^P = 2\delta^P - (r^P - 1)$ , where  $\delta^P := \dim_{\mathbb{C}} n_*^P(\mathcal{O}_{\tilde{C}^P})/\mathcal{O}_{C,P}$  and  $r^P$  is the number of local irreducible components of  $C$  at  $P$ .

Let  $H$  be a  $\mathbb{C}$ -vector space and  $\varphi : H \rightarrow H$  an endomorphism of  $H$ . The  $i$ -th Jordan polynomial of  $\varphi$ , denoted by  $\Delta_i(t)$ , is the monic polynomial such that for each  $\zeta \in \mathbb{C}$ , the multiplicity of  $\zeta$  as a root of  $\Delta_i(t)$  is equal to the number of Jordan blocks of size  $i + 1$  with eigenvalue equal to  $\zeta$ .

Let  $\Delta_1$  and  $\Delta_2$  be the first and the second Jordan polynomials of the complex monodromy on  $H_2(F_{f,0}, \mathbb{C})$  of  $V$  and let  $\Delta_1^P$  be the first Jordan polynomial of the complex monodromy of the local plane singularity  $(C, P)$ . After the Monodromy Theorem these polynomials joint with  $\Delta_V(t)$  and  $\Delta^P$ ,  $P \in \text{Sing}(C)$ , determine the corresponding Jordan form of the complex monodromy. Let us denote the Alexander polynomial of the plane curve  $C$  in  $\mathbb{P}^2$  by  $\Delta_C(t)$ , it was introduced by A. Libgober [13, 14] and F. Loeser and Vaquié [16].

**Theorem 3.4 [2].** *Let  $(V, 0)$  be a SIS whose tangent cone  $C = C_1 \cup \dots \cup C_r$  has  $r$  irreducible components and degree  $d$ . Then the Jordan form of the complex monodromy on  $H_2(F_{f,0}, \mathbb{C})$  is determined by the following polynomials*

(i) *The characteristic polynomial  $\Delta_V(t)$  is equal to*

$$\Delta_V(t) = \frac{(t^d - 1)^{\chi(\mathbb{P}^2 \setminus C)}}{(t - 1)} \prod_{P \in \text{Sing}(C)} \Delta^P(t^{d+1}).$$



(ii) The first Jordan polynomial is equal to

$$\Delta_1(t) = \frac{1}{\Delta_C(t)(t-1)^{r-1}} \prod_{P \in \text{Sing}(C)} \frac{\Delta_1^P(t^{d+1})\Delta_{(d)}^P(t)}{\Delta_{1,(d)}^P(t)^3},$$

where  $\Delta_{(d)}^P(t) := \gcd(\Delta^P(t), (t^d - 1)^{\mu^P})$  and  $\Delta_{1,(d)}^P(t) := \gcd(\Delta_1^P(t), (t^d - 1)^{\mu^P})$ .

(iii) The second Jordan polynomial is equal to

$$\Delta_2(t) = \prod_{P \in \text{Sing}(C)} \Delta_{1,(d)}^P(t).$$

**Corollary 3.5 [2, Corollaire 5.5.4].** The number of Jordan blocks of size 2 for the eigenvalue 1 of the complex monodromy  $h$  is equal to

$$\sum_{P \in \text{Sing}(C)} (r^P - 1) - (r - 1). \quad (3.1)$$

Let  $\tilde{D}_i$  be the normalization of  $D_i$  and  $\tilde{C}$  the disjoint union of the  $\tilde{D}_i$  and  $n : \tilde{C} \rightarrow C$  be the projection map. Thus the first Betti number of  $\tilde{C}$  is  $2g := 2 \sum_i g(D_i)$  and the first Betti number of  $C$  is  $2g + \sum_{P \in \text{Sing}(C)} (r^P - 1) - r + 1$ . Then  $\sum_{P \in \text{Sing}(C)} (r^P - 1) - (r - 1)$  is exactly the difference between the first Betti numbers of  $C$  and  $\tilde{C}$ . In fact this non-negative integer is equal to the first Betti number of the minimal embedded resolution graph  $\Gamma_C$  of the projective plane curve  $C$  in  $\mathbb{P}^2$ , which is nothing but  $\text{rank } H_1(\Gamma_V)$ .

**Corollary 3.6.** Let  $(V, 0)$  be a SIS whose tangent cone  $C = C_1 \cup \dots \cup C_r$  has  $r$  irreducible components. Then the number of independent cycles  $c(E) = \text{rank } H_1(\Gamma_V) = \sum_{P \in \text{Sing}(C)} (r^P - 1) - (r - 1)$ .

In particular  $E$  has no cycles if and only if  $\sum_{P \in \text{Sing}(C)} (r^P - 1) = (r - 1)$  if and only if the complex monodromy  $h$  has no Jordan blocks of size 2 for the eigenvalue 1.

**Corollary 3.7 [2, Corollaire 4.3.2].** If for every  $P \in \text{Sing}(C)$ , the local monodromy of the local plane curve equation  $g^P$  at  $P$  acting on the homology  $H_1(F_{g^P}, \mathbb{C})$  of the Milnor fibre  $F_{g^P}$  has no Jordan blocks of maximal size 2 then the corresponding SIS has no Jordan blocks of size 3.

**Corollary 3.8.** Let  $(V, 0) \subset (\mathbb{C}^3, 0)$  be a SIS with a rational irreducible tangent cone  $C \subset \mathbb{P}^2$  of degree  $d$  whose singularities are locally irreducible. Then:

- (i) the link  $L_V$  is a  $\mathbb{Q}HS$  link and  $E$  has no cycles,
- (ii) the complex monodromy on  $H_2(F_{f,0}, \mathbb{C})$  has no Jordan blocks of size 2 for the eigenvalue 1,

- (iii) the complex monodromy on  $H_2(F_{f,0}, \mathbb{C})$  has no Jordan blocks of size 3.
- (iv) The first Jordan polynomial is equal to

$$\Delta_1(t) = \frac{1}{\Delta_C(t)} \prod_{P \in \text{Sing}(C)} \gcd(\Delta^P(t), (t^d - 1)^{\mu^P}).$$

The proof follows from the previous description and the fact that if every  $P \in \text{Sing}(C)$  is locally irreducible then by Lê D.T. result (see 2.2) the plane curve singularity has finite order and  $\Delta_1^P(t) = 1$ .

**Corollary 3.9.** *Let  $(V, 0) \subset (\mathbb{C}^3, 0)$  be a SIS whose tangent cone  $C = C_1 \cup \dots \cup C_r$  has  $r$  irreducible components. Assume that  $\sum_{P \in \text{Sing}(C)} (r^P - 1) = (r - 1)$ , then:*

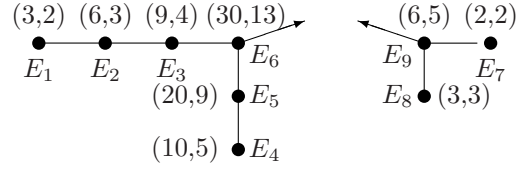
- (i)  $E$  has no cycles,
- (ii) the complex monodromy on  $H_2(F_{f,0}, \mathbb{C})$  has no Jordan blocks of size 2 for the eigenvalue 1,
- (iii) the complex monodromy on  $H_2(F_{f,0}, \mathbb{C})$  has no Jordan blocks of size 3.
- (iv) The first Jordan polynomial is equal to

$$\Delta_1(t) = \frac{1}{\Delta_C(t)(t - 1)^{r-1}} \prod_{P \in \text{Sing}(C)} \gcd(\Delta^P(t), (t^d - 1)^{\mu^P}).$$

The proof follows from Corollary 3.6 and the part (e) Monodromy Theorem 2.2.

### 3.4. The first Jordan polynomial in Example 3.3

As we described above, the plane projective curve  $C$  defined by  $f_6 = (xz - y^2)^3 - ((y - x)x^2)^3 = 0$  is irreducible, rational and with two singular points:  $P_1 = [0 : 0 : 1]$  (with a singularity of local singularity type  $u^3 - v^{10}$ ) and  $P_2 = [1 : 1 : 1]$  (with a singularity of local singularity type  $\mathbb{A}_2$ ) which are unbranched. Let  $\pi : X \rightarrow \mathbb{P}^2$  be the minimal embedded resolution of  $C$  at its singular points  $P_1, P_2$ . Let  $E_i, i \in I$ , be the irreducible components of the divisor  $\pi^{-1}(f^{-1}\{0\})$ . For each  $j \in I$ , we denote by  $N_j$  the multiplicity of  $E_j$  in the divisor of the function  $f \circ \pi$  and we denote by  $v_j - 1$  the multiplicity of  $E_j$  in the divisor of  $\pi^*(\omega)$  where  $\omega$  is the non-vanishing holomorphic 2-form  $dx \wedge dy$  in  $\mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty$ . Then the divisor  $\pi^*(C)$  is a normal crossing divisor. We attach to each exceptional divisor  $E_i$  its numerical data  $(N_i, v_i)$ .



Thus  $\Delta^{P_1}(t) = \frac{(t-1)(t^{30}-1)}{(t^3-1)(t^{10}-1)} = \phi_{30}\phi_{15}\phi_6$  and  $\Delta^{P_2}(t) = \frac{(t-1)(t^6-1)}{(t^3-1)(t^2-1)} = \phi_6$ , where  $\phi_k$  is the  $k$ -th cyclotomic polynomial. Thus, by Corollary 3.8, the only possible eigenvalues of with Jordan blocks of size 2 are the roots of the polynomial  $\Delta_1(t) = \frac{\phi_6^2}{\Delta_C(t)}$ .

The proof of our main result will be finished if we show that the Alexander polynomial  $\Delta_C(t) = \phi_6$ . The Alexander polynomial, in particular of sextics, has been investigated in detail by Artal [1], Artal and Carmona [3], Degtyarev [6], Oka [24], Pho [25], Zariski [30] among others. In [23] Corollary 18, I.2, it is proved that  $\Delta_C(t) = \phi_6$ .

Consider a generic line  $L_\infty$  in  $\mathbb{P}^2$ , in our example the line  $z = 0$  is generic, and define  $f(x, y) = f_6(x, y, 1)$ . Consider the (global) Milnor fibration given by the homogeneous polynomial  $f_6 : \mathbb{C}^3 \rightarrow \mathbb{C}$  with Milnor fibre  $F$ . Randell [26] proved that  $\Delta_C(t)(t-1)^{r-1}$  is the characteristic polynomial of the algebraic monodromy acting on  $F : T_1 : H_1(F, \mathbb{C}) \rightarrow H_1(F, \mathbb{C})$ .

**Lemma 3.10 (Divisibility properties) [13].** *The Alexander polynomial  $\Delta_C(t)(t-1)^{r-1}$  divides  $\prod_{P \in \text{Sing}(C)} \Delta^P(t)$  and the Alexander polynomial at infinity  $(t^d - 1)^{d-2}(t-1)$ . In particular the roots of the Alexander polynomial are  $d$ -roots of unity.*

To compute the Alexander polynomial  $\Delta_C(t)$  we combined the method described in [1] with the methods given in [13], [16] and [9].

Consider for  $k = 1, \dots, d-1$  the ideal sheaf  $\mathcal{I}^k$  on  $\mathbb{P}^2$  defined as follows:

- If  $Q \in \mathbb{P}^2 \setminus \text{Sing}(C)$  then  $\mathcal{I}_Q^k = \mathcal{O}_{\mathbb{P}^2, Q}$ .
- If  $P \in \text{Sing}(C)$  then  $\mathcal{I}_P^k$  is the following ideal of  $\mathcal{O}_{\mathbb{P}^2, P}$ : if  $h \in \mathcal{O}_{\mathbb{P}^2, P}$  then  $h \in \mathcal{I}_P^k$  if and only if the vanishing order of  $\pi^*(h)$  along each  $E_i$  is, at least,  $-(v_i - 1) + \lceil \frac{kN_i}{d} \rceil$  (where  $\lceil \cdot \rceil$  stands for the integer part of a real number).

For  $k \geq 0$  the following map

$$\sigma_k : H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k-3)) \rightarrow \bigoplus_{P \in \text{Sing}(C)} \mathcal{O}_{\mathbb{P}^2, P} / \mathcal{I}_P^k : h \mapsto (h_P + \mathcal{I}_P^k)_{P \in \text{Sing}(C)}$$

is well defined (up to scalars) and the result of [13] and [16] reinterpreted in this language as [1] and [9] reads as follows:

**Theorem 3.11 (Libgober, Loeser-Vaqu  ).**

$$\Delta_C(t) = \prod_{k=1}^{d-1} (\Delta^k(t))^{l_k}, \quad (3.2)$$

where  $\Delta^k(t) := (t - \exp(\frac{2k\pi i}{d}))(t - \exp(\frac{-2k\pi i}{d}))$  and  $l_k = \dim \operatorname{coker} \sigma_k$

In our case, by the Divisibility properties (Lemma 3.10),  $\Delta_C(t)$  divides  $\Delta^{P_1}(t)\Delta^{P_2}(t) = \phi_{30}\phi_{15}\phi_6^2$ . Thus, by Theorem 3.11, we are only interested in the case  $k = 1$  and  $5$ ,  $\Delta^1(t) = \Delta^5(t) = \phi_6 = (t^2 - t - 1)$ . In case  $k = 1$ , we have  $l_1 = 0$ .

In case  $k = 5$ , the ideal  $\mathcal{I}_{P_1}^5$  is the following ideal of  $\mathcal{O}_{\mathbb{P}^2, P_1}$ :

$$\mathcal{I}_{P_1}^5 = \{h \in \mathcal{O}_{\mathbb{P}^2, P_1} : (\pi^*h) \geq E_1 + 3E_2 + 4E_3 + 4E_4 + 8E_5 + 13E_6\}$$

and with the local change of coordinates  $u = x - y^2$ ,  $w = y$ , the generators of the ideal are  $\mathcal{I}_{P_1}^5 = \langle uw, u^2, w^5 \rangle$  and the dimension of the quotient vector space  $\mathcal{O}_{\mathbb{P}^2, P_1}/\mathcal{I}_{P_1}^5$  is 6. A basis is given by  $1, u, w, w^2, w^3, w^4$ . The ideal

$$\mathcal{I}_{P_2}^5 = \{h \in \mathcal{O}_{\mathbb{P}^2, P_2} : (\pi^*h) \geq 0E_7 + 0E_8 + E_9\} = \mathfrak{m}_{\mathbb{P}^2, P_2}$$

and the dimension of the quotient vector space  $\mathcal{O}_{\mathbb{P}^2, P_2}/\mathcal{I}_{P_2}^5$  is 1. A basis is given by 1.

If we take as a basis for the space of conics  $1, x, y, x^2, y^2, xy$ , the map  $\sigma_5$

$$\begin{aligned} \sigma_5 : H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) &\rightarrow \mathcal{O}_{\mathbb{P}^2, P_1}/\mathcal{I}_{P_1}^5 \times \mathcal{O}_{\mathbb{P}^2, P_2}/\mathcal{I}_{P_2}^5 \\ &= \mathbb{C}^6 \times \mathbb{C} : h \mapsto (h + \mathcal{I}_{P_1}^5, h + \mathcal{I}_{P_2}^5) \end{aligned}$$

is given in such coordinates by (using  $u = x - y^2$ ):  $\sigma_5(1) = (1, 0, 0, 0, 0, 1)$ ,  $\sigma_5(x) = (0, 1, 0, 1, 0, 0, 1)$ ,  $\sigma_5(y) = (0, 0, 1, 0, 0, 0, 1)$ ,  $\sigma_5(x^2) = (0, 0, 0, 0, 0, 1, 1)$ ,  $\sigma_5(y^2) = (0, 0, 0, 1, 0, 0, 1)$  and  $\sigma_5(xy) = (0, 0, 0, 0, 1, 0, 1)$ .

Therefore  $\sigma_5$  is injective and  $\dim \operatorname{coker} \sigma_5 = 7 - 6 + 0 = 1$ . The key point is that  $u \notin \mathcal{I}_{P_1}^5$ .

## 4. Zariski pairs

Let us consider  $C \subset \mathbb{P}^2$  a reduced projective curve of degree  $d$  defined by an equation  $f_d(x, y, z) = 0$ . If  $(V, 0) \subset (\mathbb{C}^3, 0)$  is a SIS with tangent cone  $C$ , then the link  $L_V$  of the singularity is completely determined by  $C$ . Let us recall, that  $L_V$  is a Waldhausen manifold and its plumbing graph is the dual graph of the good minimal resolution. In order to determine  $L_V$  we do not need the

embedding of  $C$  in  $\mathbb{P}^2$ , but only its embedding in a regular neighborhood. The needed data can be encoded in a combinatorial way.

**Definition 4.1.** Let  $\text{Irr}(C)$  be the set of irreducible components of  $C$ . For  $P \in \text{Sing}(C)$ , let  $B(P)$  be the set of local irreducible components of  $C$ . The *combinatorial type* of  $C$  is given by:

- A mapping  $\deg : \text{Irr}(C) \rightarrow \mathbb{Z}$ , given by the degrees of the irreducible components of  $C$ .
- A mapping  $\text{top} : \text{Sing}(C) \rightarrow \text{Top}$ , where  $\text{Top}$  is the set of topological types of singular points. The image of a singular point is its topological type.
- For each  $P \in \text{Sing}(C)$ , a mapping  $\beta_P : T(P) \rightarrow \text{Irr}(C)$  such that if  $\gamma$  is a branch of  $C$  at  $P$ , then  $\beta_P(\gamma)$  is the global irreducible component containing  $\gamma$ .

**Remark 1.** There is a natural notion of isomorphism of combinatorial types. It is easily seen that combinatorial type determines and is determined by any of the following graphs (with vertices decorated with self-intersections):

- The dual graph of the preimage of  $C$  by the minimal resolution of  $\text{Sing}'(C)$ . The set  $\text{Sing}'(C)$  is obtained from  $\text{Sing}(C)$  by forgetting ordinary double points whose branches belong to distinct global irreducible components. We need to mark in the graph the  $r$  vertices corresponding to  $\text{Irr}(C)$ .
- The dual graph of the minimal good resolution of  $V$ . Since the minimal resolution is unique, it is not necessary to mark vertices.

Note also that the combinatorial type determine the characteristic polynomial  $\Delta_V(t)$  of  $V$  (see Theorem 3.4).

**Definition 4.2.** A *Zariski pair* is a set of two curves  $C_1, C_2 \subset \mathbb{P}^2$  with the same combinatorial type but such that  $(\mathbb{P}^2, C_1)$  is not homeomorphic to  $(\mathbb{P}^2, C_2)$ . An *Alexander-Zariski pair*  $\{C_1, C_2\}$  is a Zariski pair such that the Alexander polynomials of  $C_1$  and  $C_2$  do not coincide.

In [2], (see here Theorem 3.4) it is shown that the Jordan form of complex monodromy of a SIS is determined by the combinatorial type and the Alexander polynomial of its tangent cone. The first example of Zariski pair was given by Zariski, [30, 31]; there exist sextic curves with six ordinary cusps. If these cusps are (resp. not) in a conic then the Alexander polynomial equals  $t^2 - t + 1$  (resp. 1). Then, it gives an Alexander-Zariski pair. Many other examples of Alexander-Zariski pairs have been constructed (Artal [1], Degtyarev [6]).

We state the main result in [2].

**Theorem 4.3.** *Let  $V_1, V_2$  be two SIS such that their tangent cones form an Alexander-Zariski pair. Then  $V_1$  and  $V_2$  have the same abstract topology and characteristic polynomial of the monodromy but not the same embedded topology.*

Recall that the Jordan form of the monodromy is an invariant of the embedded topology of a SIS (see Theorem 3.4); since it depends on the Alexander polynomial  $\Delta_C(t)$  of the tangent cone.

#### 4.1. Zariski pair of reduced sextics with only one singular point of type $\mathbb{A}_{17}$

Our next Zariski-pair example  $(C_1, C_2)$  can be found in [1, Théorème 4.4]. The curves  $C_i, i = 1, 2$  are reduced sextics with only one singular point  $P$  of type  $\mathbb{A}_{17}$ , locally given by  $u^2 - v^{18}$ .

(I) the irreducible components of  $C_1$  are two non-singular cubics. These cubics meet at only one point  $P$  which moreover is an inflection point of each of the cubics, i.e. the tangent line to the singular point  $P$  goes through the infinitely near points  $P, P_1$  and  $P_2$  of  $C_1$ . The equations of  $C_1$  are given for instance by  $\{f_1(x, y, z) := (zx^2 - y^3 - ayz^2 - bz^3)(zx^2 - y^3 - ayz^2 - cz^3) = 0\}$ , with  $a, b, c \in \mathbb{C}$  generic.

(II) the irreducible components of  $C_2$  are two non-singular cubics. These cubics meet at only one point  $P$  which is not an inflection point of any of the cubics, i.e. the tangent line to the singular point  $P$  goes through the infinitely near points  $P, P_1$  of  $C_1$  but it is not going through  $P_2$ . The equations of  $C_2$  are given for instance by  $\{f_2(x, y, z) := (zx^2 - y^2x - yz^2 - a_1(z^3 - y(xz - y^2)))(zx^2 - y^2x - yz^2 - a_2(z^3 - y(xz - y^2))) = 0\}$  with  $a_1, a_2 \in \mathbb{C}$  generic.

Consider the superisolated surface singularities  $(V_1, 0) = (\{f_1(x, y, z) + l^7 = 0\}, 0) \subset (\mathbb{C}^3, 0)$  and  $(V_2, 0) = (\{f_2(x, y, z) + l^7 = 0\}, 0) \subset (\mathbb{C}^3, 0)$  ( $l$  is a generic hyperplane). In both cases the tangent cone has two irreducible components and it has only one singular point  $P$  of local type  $u^2 - v^{18}$  and therefore  $\Delta^P(t) = (t^{18} - 1)(t - 1)/(t^2 - 1) = \phi_{18}\phi_9\phi_6\phi_3\phi_1$ , where  $\phi_k$  is the  $k$ -th cyclotomic polynomial. Thus the number of local branches is 2 and  $\sum_{P \in \text{Sing}(C)}(r^P - 1) = (r - 1)$ . By Corollary 3.9, for  $(V_i, 0), i = 1, 2$ , the complex monodromy has no Jordan blocks of size 2 for the eigenvalue 1, and it has no Jordan blocks of size 3. Moreover the first Jordan polynomial is equal to

$$\Delta_1(t) = \frac{\gcd(\Delta^P(t), (t^6 - 1)^{\mu^P})}{\Delta_{C_i}(t)(t - 1)} = \frac{\phi_6\phi_3}{\Delta_{C_i}(t)}. \quad (4.1)$$

To compute  $\Delta_{C_i}(t)$  we use the same ideas as in Theorem 3.11.

**Lemma 4.4.** *For the point  $P$  at the curve  $C_1$  the ideals  $\mathcal{I}_P^k = \mathcal{O}_{\mathbb{P}^2, P}$  if  $k \leq 3$ ,  $\mathcal{I}_P^4 = \langle y^3, z \rangle \subset \mathcal{O}_{\mathbb{P}^2, P}$  and  $\mathcal{I}_P^5 = \langle y^6, z - y^3 - ay^4 - by^5 \rangle \subset \mathcal{O}_{\mathbb{P}^2, P}$ .*

**Lemma 4.5.** *For the point  $P$  at the curve  $C_2$  the ideals  $\mathcal{I}_P^k = \mathcal{O}_{\mathbb{P}^2, P}$  if  $k \leq 3$ ,  $\mathcal{I}_P^4 = \langle y^3, z - y^2 \rangle \subset \mathcal{O}_{\mathbb{P}^2, P}$  and  $\mathcal{I}_P^5 = \langle y^6, z - y^2 - y^5 \rangle \subset \mathcal{O}_{\mathbb{P}^2, P}$ .*

Thus  $\Delta_{C_i}(t) = \phi_6^{\dim \operatorname{coker} \sigma_5} \phi_3^{\dim \operatorname{coker} \sigma_4}$ .

Therefore the map  $\sigma_4$  is

$$\sigma_4 : H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \simeq \mathbb{C}^3 \rightarrow \mathcal{O}_{\mathbb{P}^2, P} / \mathcal{I}_P^4 \simeq \mathbb{C}^3,$$

and if we choose as basis of the first space  $\{1, y, z\}$  and of the second  $\{1, y, y^2\}$  then

- (1) by Lemma 4.4, for  $C_1$  the dimension  $\dim \operatorname{coker} \sigma_4 = \dim \ker \sigma_4 = 1$ .
- (2) by Lemma 4.5, for  $C_2$  the dimension  $\dim \operatorname{coker} \sigma_4 = \dim \ker \sigma_4 = 0$ .

On the other hand for the map  $\sigma_5$

$$\sigma_5 : H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) \simeq \mathbb{C}^6 \rightarrow \mathcal{O}_{\mathbb{P}^2, P} / \mathcal{I}_P^5 \simeq \mathbb{C}^6,$$

if we choose as basis of the first space  $\{1, y, z, y^2, yz, z^2\}$  and of the second  $\{1, y, y^2, y^3, y^4, y^5\}$  then we can compute

- (3) by Lemma 4.4, for  $C_1$  the dimension  $\dim \operatorname{coker} \sigma_5 = \dim \ker \sigma_5 = 1$ .
- (4) by Lemma 4.5, for  $C_2$  the dimension  $\dim \operatorname{coker} \sigma_5 = \dim \ker \sigma_5 = 0$ .

Therefore,  $\Delta_{C_1}(t) = \phi_6 \phi_3$  and  $\Delta_{C_2}(t) = 1$  and by (4.1) we have proved that the pair  $(C_1, C_2)$  is a Alexander-Zariski pair.

**Example 4.6.** Consider the superisolated surface singularities  $(V_1, 0) = (\{f_1(x, y, z) + l^7 = 0\}, 0) \subset (\mathbb{C}^3, 0)$  and  $(V_2, 0) = (\{f_2(x, y, z) + l^7 = 0\}, 0) \subset (\mathbb{C}^3, 0)$  ( $l$  is a generic hyperplane). Then the complex algebraic monodromy of  $(V_1, 0) \subset (\mathbb{C}^3, 0)$  has finite order and the complex algebraic monodromy of  $(V_2, 0) \subset (\mathbb{C}^3, 0)$  has not finite order

This answers a question proposed to us by J. Stevens: find a Zariski pair  $C_1, C_2$  such that for the corresponding SIS surface singularities  $(V_1, 0) \subset (\mathbb{C}^3, 0)$  and  $(V_2, 0) \subset (\mathbb{C}^3, 0)$  one has a finite order monodromy and the other it does not.

There are also examples of Zariski pairs which are not Alexander-Zariski pairs (see [23], [3], [4]). Some of them are distinguished by the so-called characteristic varieties introduced by Libgober [15]. These are subtori of  $(\mathbb{C}^*)^r$ ,  $r := \# \operatorname{Irr}(C)$ , which measure the excess of Betti numbers of finite Abelian coverings of the plane ramified on the curve (as Alexander polynomial does it for cyclic coverings).

**Problem 1.** How can one translate characteristic varieties of a projective curve in terms of invariants of the SIS associated to it?

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