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ON THE ZETA-FUNCTION OF A POLYNOMIAL AT INFINITY (*)

BY

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ABSTRACT. – A polynomial function defines a locally trivial fibre bundle over the complement to a finite set in the target ℂ. Objects connected with this fibration (say, monodromy operators and, in particular, the monodromy operator of the polynomial at infinity) are in some sense global. The idea of the paper is to localize computations of the zeta-functions of monodromy transformations for a polynomial, i.e., to express them in local terms. It is done with the use of the notion of Milnor fibres of the germ of a meromorphic function and the methods of calculation of the corresponding zeta-functions elaborated by the authors. It gives effective methods of computation of the zeta-function for a number of cases and a criterium for a value to be atypical at infinity.
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1. Introduction

Let P be a complex polynomial in $(n+1)$ variables. It defines a map from $\mathbb{C}^{n+1}$ to $\mathbb{C}$ which also will be denoted by P. It is known [13] that there exists a finite set $B(P) \subset \mathbb{C}$ such that the map P is a C^∞ locally trivial fibration over its complement. The monodromy transformation h of this fibration corresponding to the loop $z_0 \cdot \exp(2\pi i\tau)$ $(0 \leqslant \tau \leqslant 1)$ with $\|z_0\|$ big enough is called the *geometric monodromy at infinity* of the polynomial P. Let h_* be its action in the homology groups of the fibre (the level set) $\{P = z_0\}$.

DEFINITION. – *The* zeta-function of the monodromy at infinity *of the polynomial P is the rational function*

$$\zeta_P(t) = \prod_{q \geqslant 0} \left\{\det\left[id - th_*|_{H_q(\{P=z_0\};\mathbb{C})}\right]\right\}^{(-1)^q}.$$

Remark 1. – We use the definition from [2], which means that the zeta-function defined this way is the inverse of that used in [1].

The degree of the zeta-function (the degree of the numerator minus the degree of the denominator) is equal to the Euler characteristic χ_P of the (generic) fibre $\{P = z_0\}$. Formulae for the zeta-functions at infinity for certain polynomials were given in particular in [6,9].

The main aim of the paper is to express the zeta-function of the monodromy at infinity in local terms. At points of the infinity hyperplane, a polynomial defines germs of meromorphic functions. We use invariants of meromorphic germs [8] to describe the zeta-function of the monodromy at infinity. We also apply this techniques to monodromy transformations corresponding to finite atypical values of the polynomial.

2. Zeta-function of a polynomial via zeta-functions of meromorphic germs

A polynomial function $P : \mathbb{C}^{n+1} \to \mathbb{C}$ defines a meromorphic function P on the projective space $\mathbb{CP}^{n+1}$. At each point x of the infinity hyperplane $\mathbb{CP}^n_\infty$ the germ of the meromorphic function P has the form $F(u, x_1, \ldots, x_n)/u^d$ where $u, x_1, \ldots, x_n$ are local coordinates such that $\mathbb{CP}^n_\infty = \{u = 0\}$, F is the germ of a holomorphic function, and d is the degree of the polynomial P.

In [8], for a meromorphic germ $f = F/G$, there were defined two Milnor fibres (the zero and the infinity ones), two monodromy transformations, and thus two zeta-functions $\zeta_f^0(t)$ and $\zeta_f^\infty(t)$. Let $\zeta_{P,x}^\bullet(t)$ ($\bullet = 0$ or ∞) be the corresponding zeta-function of the germ of the meromorphic function P at the point $x \in \mathbb{CP}^n_\infty$.

For the aim of convenience, in [8] we considered only meromorphic germs $f = F/G$ with $F(0) = G(0) = 0$. At a generic point of the infinity hyperplane $\mathbb{CP}^n_\infty$ the meromorphic function P has the form $1/u^d$. For a germ of the form $f = 1/G$ with $G(0) = 0$, it is reasonable to give the following definition: its infinity Milnor fibre coincides with the (usual) Milnor fibre of the holomorphic germ G and its zero Milnor fibre is empty. Thus $\zeta_f^0(t) = 1$ and $\zeta_f^\infty(t) = \zeta_G(t)$. According to this definition, for the germ $1/u^d$, its infinity zeta-function is equal to $(1 - t^d)$.

Let $\mathcal{S} = \{\Xi\}$ be a prestratification of the infinity hyperplane $\mathbb{CP}^n_\infty$ (that is a partitioning of $\mathbb{CP}^n_\infty$ into semi-analytic subspaces without any regularity conditions) such that, for each stratum Ξ of $\mathcal{S}$, the infinity zeta-function $\zeta_{P,x}^\infty(t)$ does not depend on x, for $x \in \Xi$. Let us denote this zeta-function by $\zeta_\Xi^\infty(t)$ and by χ_Ξ^∞ its degree $\deg \zeta_\Xi^\infty(t)$.

THEOREM 1. –

$$\zeta_P(t) = \prod_{\Xi \in \mathcal{S}} \left[\zeta_\Xi^\infty(t)\right]^{\chi(\Xi)}, \qquad \chi_P = \sum_{\Xi \in \mathcal{S}} \chi_\Xi^\infty \cdot \chi(\Xi).$$

The *proof* is similar to that of Theorem 1 in [7]. (The formulae in this two theorems looks very similar to each other. However the Theorem from [7] described the zeta-function of a *holomorphic germ* in terms of the germs (also *holomorphic*) of its lifting to the space of a modification, while the Theorem here describes the zeta-function of a polynomial at infinity in terms of new invariants: Milnor fibres and zeta-functions of meromorphic germs. These notions were elaborated in [8] mainly in order to treat this situation.)

Remark 2. – One can write the formula for χ_P in the form of an integral with respect to the Euler characteristic

$$\chi_P = \int_{\mathbb{CP}^n_\infty} \chi_{P,x}^\infty \, d\chi$$

in the sense of Viro [14].

Remark 3. – Let P_d be the (highest) homogeneous part of degree d of the polynomial P. Then at each point $x \in \mathbb{CP}^n_\infty \setminus \{P_d = 0\}$ the germ of the meromorphic function P is of the form $1/u^d$. The set $\Xi^n = \mathbb{CP}^n_\infty \setminus \{P_d = 0\}$ can be considered as the n-dimensional stratum of a partition. It brings the factor $(1-t^d)^{\chi(\Xi^n)}$ into the zeta-function $\zeta_P(t)$.

3. Examples

3.1. Yomdin-at-infinity polynomials

This name was introduced in [4]. For a polynomial $P \in \mathbb{C}[z_0, z_1, \ldots, z_n]$, let P_i be its homogeneous part of degree i. Let a polynomial P be of the form $P = P_d + P_{d-k} +$ terms of lower degree, $k \geqslant 1$. Let us consider hypersurfaces in $\mathbb{CP}^n$ defined by $\{P_d = 0\}$ and $\{P_{d-k} = 0\}$. Let $\mathrm{Sing}(P_d)$ be the singular locus of the hypersurface $\{P_d = 0\}$ (including all points where $\{P_d = 0\}$ is not reduced). One says that P is a *Yomdine-at-infinity polynomial* if $\mathrm{Sing}(P_d) \cap \{P_{d-k} = 0\} = \emptyset$ (in particular it implies that $\mathrm{Sing}(P_d)$ is finite).

Y. Yomdin [15] has considered critical points of holomorphic functions which are local versions of such polynomials. He gave a formula for their Milnor numbers. The generic fibre (level set) of a Yomdin-at-infinity polynomial is homotopy equivalent to the bouquet of n-dimensional spheres [5]. Its Euler characteristic χ_P (or rather the (global) Milnor number) has been determined in [4]. For $k = 1$, the zeta-function of such a polynomial has been obtained in [6].

Let $P(z_0, z_1, \ldots, z_n) = P_d + P_{d-k} + \cdots$ be a Yomdin-at-infinity polynomial. Let $\mathrm{Sing}(P_d)$ consist of s points $Q_1, \ldots, Q_s$. One has the following natural stratification of the infinity hyperplane $\mathbb{CP}^n_\infty$:

(1) the n-dimensional stratum $\Xi^n = \mathbb{CP}^n_\infty \setminus \{P_d = 0\}$;

(2) the $(n-1)$-dimensional stratum $\Xi^{n-1} = \{P_d = 0\} \setminus \{Q_1, \ldots, Q_s\}$;

(3) the 0-dimensional strata Ξ^0_i $(i = 1, \ldots, s)$, each consisting of one point Q_i.

The Euler characteristic of the stratum Ξ^n is equal to

$$\chi\left(\mathbb{CP}^n_\infty\right) - \chi\left(\{P_d = 0\}\right) = (n+1) - \chi(n, d) + (-1)^{n-1} \sum_{i=1}^{s} \mu_i,$$

where

$$\chi(n,d) = (n+1) + \frac{(1-d)^{n+1}-1}{d}$$

is the Euler characteristic of a non-singular hypersurface of degree d in the complex projective space $\mathbb{CP}^n_\infty$, μ_i is the Milnor number of the germ of the hypersurface $\{P_d = 0\} \subset \mathbb{CP}^n_\infty$ at the point Q_i. At each point of the stratum Ξ^n, the germ of the meromorphic function P has (in some local coordinates $u, y_1, \ldots, y_n$) the form $1/u^d$ ($\mathbb{CP}^n_\infty = \{u = 0\}$) and its infinity zeta-function $\zeta^\infty_{\Xi^n}(t)$ is equal to $(1 - t^d)$.

At each point of the stratum Ξ^{n-1}, the germ of the polynomial P has (in some local coordinates $u, y_1, \ldots, y_n$) the form y_1/u^d. Its infinity zeta-function $\zeta^\infty_{\Xi^{n-1}}(t)$ is equal to 1 and thus it does not contribute a factor to the zeta-function of the polynomial P.

At a point Q_i $(i = 1, \ldots, s)$, the germ of the meromorphic function P has the form

$$\varphi(u, y_1, \ldots, y_n) = \frac{g_i(y_1, \ldots, y_n) + u^k}{u^d},$$

where g_i is a local equation of the hypersurface $\{P_d = 0\} \subset \mathbb{CP}^n_\infty$ at the point Q_i. Thus μ_i is its Milnor number.

To compute the infinity zeta-function $\zeta^\infty_\varphi(t)$ of the meromorphic germ φ, let us consider a resolution $\pi : (\mathcal{X}, \mathcal{D}) \to (\mathbb{C}^n, 0)$ of the singularity g_i, i.e., a proper modification of $(\mathbb{C}^n, 0)$ which is an isomorphism outside the origin in $\mathbb{C}^n$ and such that, at each point of the exceptional divisor $\mathcal{D}$, the lifting $g_i \circ \pi$ of the function g_i to the space $\mathcal{X}$ of the modification has (in some local coordinates) the form $y_1^{m_1} \cdot \cdots \cdot y_n^{m_n}$ $(m_i \geqslant 0)$.

Let us consider the modification

$$\widetilde{\pi} = id \times \pi : (\mathbb{C}_u \times \mathcal{X}, 0 \times \mathcal{D}) \to \big(\mathbb{C}^{n+1}, 0\big) = \big(\mathbb{C}_u \times \mathbb{C}^n, 0\big)$$

of the space $(\mathbb{C}^{n+1}, 0)$ — the trivial extension: $(u, x) \mapsto (u, \pi(x))$. Let $\widetilde{\varphi} = \varphi \circ \widetilde{\pi}$ be the lifting of the meromorphic function φ to the space $\mathbb{C}_u \times \mathcal{X}$ of the modification $\widetilde{\pi}$. Let $\mathcal{M}^\infty_{\widetilde{\varphi}} = \widetilde{\pi}^{-1}(\mathcal{M}^\infty_\varphi)$ ($\mathcal{M}^\infty_{\widetilde{\varphi}}$ is the infinity Milnor fibre of the germ φ) be the local level set of the meromorphic function $\widetilde{\varphi}$ (close to the infinity one). In the natural way one has the (infinity) monodromy $h^\infty_{\widetilde{\varphi}}$ acting on $\mathcal{M}^\infty_{\widetilde{\varphi}}$ and its zeta-function $\zeta^\infty_{\widetilde{\varphi}}(t)$.

THEOREM 2. –

$$\zeta^{\infty}_{\widetilde{\varphi}}(t) = \left(1 - t^{d-k}\right)^{\chi(\mathcal{D})-1} \zeta^{\infty}_{\varphi}(t).$$

Proof. – The infinity monodromy transformation of the function $\widetilde{\varphi}$ can be described in the following way. Let $h^{\infty}_{\varphi} : \mathcal{M}^{\infty}_{\varphi} \to \mathcal{M}^{\infty}_{\varphi}$ be the infinity monodromy transformation of the germ φ. One can suppose that it preserves the intersection of the Milnor fibre $\mathcal{M}^{\infty}_{\varphi}$ with the line $\mathbb{C}_u \times \{0\}$. There it coincides with the infinity monodromy transformation of the restriction $\varphi|_{\mathbb{C}_u \times \{0\}} = u^k/u^d$ of the germ φ to this line, i.e., with a cyclic permutation of $(d-k)$ points. The zeta-function of a cyclic permutation of $(d-k)$ points is equal to $(1 - t^{d-k})$. The projection $\widetilde{\pi} : \mathcal{M}^{\infty}_{\widetilde{\varphi}} \to \mathcal{M}^{\infty}_{\varphi}$ is an isomorphism outside $\mathcal{M}^{\infty}_{\varphi} \cap (\mathbb{C}_u \times \{0\})$, the preimage of each point from $\mathcal{M}^{\infty}_{\varphi} \cap (\mathbb{C}_u \times \{0\})$ is isomorphic to the exceptional divisor $\mathcal{D}$. This means that the transformation (the diffeomorphism) $h^{\infty}_{\widetilde{\varphi}} : \mathcal{M}^{\infty}_{\widetilde{\varphi}} \to \mathcal{M}^{\infty}_{\widetilde{\varphi}}$ can be constructed in such a way that it preserves $\widetilde{\pi}^{-1}(\mathcal{M}^{\infty}_{\varphi} \cap (\mathbb{C}_u \times \{0\}))$ and acts on it by a cyclic permutation of $(d-k)$ copies of $\mathcal{D}$. The zeta-function of this transformation of $\{(d-k) \text{ points}\} \times \mathcal{D}$ is equal to $(1 - t^{d-k})^{\chi(\mathcal{D})}$. The result follows from the *multiplication property* of the zeta-function of a transformation (see [2] p. 94). □

For $\bar{m} = (m_1, m_2, \ldots, m_n)$ with integer $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n \geqslant 0$, let $S_{\bar{m}}$ be the set of points of the exceptional divisor $\mathcal{D}$ of the resolution π at which the lifting of the germ g_i has the form $y_1^{m_1} \cdot \cdots \cdot y_n^{m_n}$; for $m \geqslant 1$, let S_m be $S_{(m,0,\ldots,0)}$. By the formula of A'Campo [1]

$$\zeta_{g_i}(t) = \prod_{m \geqslant 1} \left(1 - t^m\right)^{\chi(S_m)}. \tag{1}$$

At a point $x \in \{0\} \times S_{\bar{m}} \subset \{0\} \times \mathcal{D}$, the lifting $\widetilde{\varphi} = \varphi \circ \widetilde{\pi}$ of the function φ has the local form $(y_1^{m_1} \cdot \cdots \cdot y_n^{m_n} + u^k)/u^d$. Thus, for fixed $\bar{m}$, the infinity zeta-function $\zeta^{\infty}_{\widetilde{\varphi},x}(t)$ of the germ of the meromorphic function $\widetilde{\varphi}$ at a point x from $\{0\} \times S_{\bar{m}}$ is one and the same. It can be determined by the Varchenko type formula from [8]. If there are more than one integers m_i different from zero, $\zeta^{\infty}_{\widetilde{\varphi},x}(t) = (1 - t^{d-k})$. For $x \in \{0\} \times S_m$,

$$\zeta^{\infty}_{\widetilde{\varphi},x}(t) = \left(1 - t^{d-k}\right)\left(1 - t^{\frac{m(d-k)}{g.c.d.(m,k)}}\right)^{-g.c.d.(m,k)}.$$

According to Theorem 1

$$\zeta_{\widetilde{\varphi}}^{\infty}(t) = \left(1 - t^{d-k}\right)^{\chi(\mathcal{D})} \prod_{m \geqslant 1} \left(1 - t^{\frac{m(d-k)}{g.c.d.(m,k)}}\right)^{-g.c.d.(m,k) \cdot \chi(S_m)}$$

and by Theorem 2

$$\zeta_{\varphi}^{\infty}(t) = \left(1 - t^{d-k}\right) \prod_{m \geqslant 1} \left(1 - t^{\frac{m(d-k)}{g.c.d.(m,k)}}\right)^{-g.c.d.(m,k) \cdot \chi(S_m)}. \tag{2}$$

The zeta-function $\zeta_h(t)$ of a transformation $h : X \to X$ of a space X into itself determines the zeta-function $\zeta_h^k(t)$ of the kth power h^k of the transformation h. In particular, if $\zeta_h(t) = \prod_{m \geqslant 1}(1 - t^m)^{a_m}$, then

$$\zeta_h^k(t) = \prod_{m \geqslant 1} \left(1 - t^{\frac{m}{g.c.d.(k,m)}}\right)^{g.c.d.(k,m) \cdot a_m}.$$

The formulae (1) and (2) mean that

$$\zeta_{\varphi}^{\infty}(t) = \left(1 - t^{d-k}\right)\left(\zeta_{g_i}^k\left(t^{d-k}\right)\right)^{-1}. \tag{3}$$

Combining the computations for the stratification $\{\Xi^n, \Xi^{n-1}, \Xi_i^0\}$ of the infinity hyperplane $\mathbb{CP}_\infty^n$, one has

THEOREM 3. – *For a Yomdin-at-infinity polynomial $P = P_d + P_{d-k} + \cdots$, its zeta-function at infinity is equal to*

$$\zeta_P(t) = \left(1 - t^d\right)^{\chi(\Xi^n)} \left(1 - t^{d-k}\right)^s \left(\prod_{i=1}^{s} \zeta_{g_i}^k\left(t^{d-k}\right)\right)^{-1},$$

where

$$\chi\left(\Xi^n\right) = \frac{1 - (1-d)^{n+1}}{d} + (-1)^{n-1} \sum_{i=1}^{s} \mu(g_i)$$

and g_i is a local equation of the hypersurface $\{P_d = 0\} \subset \mathbb{CP}_\infty^n$ at its singular point Q_i.

3.2

Let $(n+1)$ be equal to 3, $P = P_d + P_{d-k} + \cdots$, $\{P_d = 0\}$ is a curve in $\mathbb{CP}^2_\infty$. Let $C_1^{q_1} + \cdots + C_r^{q_r}$ be its decomposition into irreducible components. Let $\{P_d = 0\}_{red}$ be the reduced curve $C_1 + \cdots + C_r$ and let $\mathrm{Sing}(\{P_d = 0\}_{red})$ consist of s points $\{Q_1, \ldots, Q_s\}$. Suppose that:

(1) the curve $\{P_{d-k} = 0\}$ is reduced;

(2) $Q_i \notin \{P_{d-k} = 0\}$, $(i = 1, \ldots, s)$;

(3) for each j with $q_j > 1$, the curves C_j and $\{P_{d-k} = 0\}$ intersect transversally, i.e., the set $C_j \cap \{P_{d-k} = 0\}$ consists of $d_j(d-k)$ different points ($d_j = \deg C_j$).

The generic fibre of the polynomial P is homotopy equivalent to the bouquet of 2-dimensional spheres. In this case the number of these spheres is equal to $\mu(P) = \dim_{\mathbb{C}} \mathbb{C}[x, y, z]/Jac(P)$ and is equal to

$$(d-1)^3 - k \cdot \left(\chi\left(\{P_d = 0\}\right) + d(2d - \tilde{d} - 3)\right) + k^2 \cdot (d - \tilde{d}),$$

where $\tilde{d} = d_1 + \cdots + d_r$ is the degree of the (reduced) curve $\{P_d = 0\}_{red}$, [4]. Let us consider the following partitioning of the infinity hyperplane $\mathbb{CP}^2_\infty$:

(1) the 0-dimensional stratum Ξ_i^0 consisting of one point Q_i each $(i = 1, \ldots, s)$;

(2) the 0-dimensional stratum $\Lambda_j^0 = C_j \cap \{P_{d-k} = 0\}$, for each $j = 1, \ldots, r$;

(3) the 1-dimensional stratum $\Xi_j^1 = C_j \setminus (\{Q_i\} \cup \Lambda_j^0)$, for each $j = 1, \ldots, r$;

(4) the 2-dimensional stratum $\Xi^2 = \mathbb{CP}^2_\infty \setminus \{P_d = 0\}$.

At each point of the stratum Ξ^2, the germ of the meromorphic function P has the form $(1/u^d)$ ($\mathbb{CP}^2_\infty = \{u = 0\}$). Its infinity zeta-function is equal to $(1 - t^d)$. The Euler characteristic $\chi(\Xi^2)$ of the stratum Ξ^2 is equal to

$$\chi\left(\mathbb{CP}^2_\infty\right) - \chi\left(\{P_d = 0\}\right) = 3 - 3\tilde{d} + \tilde{d}^2 - \sum_{i=1}^{s} \mu_i,$$

where μ_i is the Milnor number of the (reduced) curve $\{P_d = 0\}_{red}$ at the point Q_i.

At each point of the stratum Ξ_j^1, the germ of the meromorphic function P has the form $(y_1^{q_j} + u^k)/u^d$. Its infinity zeta-function can be determined by the Varchenko type formula from [8] and is equal to

$$\big(1-t^{d-k}\big)\big(1-t^{\frac{q_j(d-k)}{g.c.d.(q_j,k)}}\big)^{-g.c.d.(q_j,k)}.$$

The Euler characteristic of the stratum Ξ_j^1 is equal to

$$\chi(C_j)-d_j(d-k)-\sharp\big\{C_j\cap\{Q_i\colon i=1,\dots,s\}\big\}.$$

At each point of the stratum Λ_j^0, the germ of the meromorphic function P has the form $(y_1^{q_j}+u^k y_2)/u^d$. Its infinity zeta-function is equal to 1.

At a point Q_i, the germ of the meromorphic function P has the form $(g_i(y_1,y_2)+u^k)/u^d$, where $\{g_i=0\}$ is the local equation of the (non-reduced) curve $\{P_d=0\}$ at the point Q_i. Its infinity zeta-function is equal to

$$\big(1-t^{d-k}\big)\big(\zeta_{g_i}^k\big(t^{d-k}\big)\big)^{-1}.$$

Remark 4. – We can not apply the formula (3) directly since the singularity of the germ g_i is, in general, not isolated. However, it is not difficult to see that, actually, the proof of this formula uses only the fact that the singularity of the germ g_i can be resolved by a modification which is an isomorphism outside the origin. This is so for a curve singularity.

Thus one obtains

$$\zeta_P(t)=\big(1-t^d\big)^{\chi(\Xi^2)}\big(1-t^{d-k}\big)^{(3\tilde d-\tilde d^2-\tilde d(d-k)+\sum\mu_i)}$$
$$\times\prod_{j=1}^{r}\big(1-t^{\frac{q_j(d-k)}{g.c.d.(q_j,k)}}\big)^{-g.c.d.(q_j,k)\cdot\chi(\Xi_j^1)}\cdot\prod_{i=1}^{s}\big(\zeta_{g_i}^k\big(t^{d-k}\big)\big)^{-1}.$$

4. On the bifurcation set of a polynomial map

As we have mentioned, a polynomial map $P:\mathbb{C}^{n+1}\to\mathbb{C}$ defines a locally trivial fibration over the complement to a finite set in $\mathbb{C}$. The minimal set $B(P)$ with this property is called the bifurcation set of P. The bifurcation set consists of critical values of the polynomial P (in the affine part) and of atypical ("critical") values at infinity.

In order to consider a level set $\{P = c\}$, one can substitute the polynomial P by the polynomial $(P - c)$ and consider the zero level set. Thus let us consider the zero level set $V_0 = \{P = 0\} \subset \mathbb{C}^{n+1}$ of the polynomial P. Let us suppose that the level set V_0 of the polynomial P has only isolated singular points (in the affine part $\mathbb{C}^{n+1}$). For $\rho > 0$, let B_ρ be the open ball of radius ρ centered at the origin in $\mathbb{C}^{n+1}$ and $S_\rho = \partial B_\rho$ be the $(2n+1)$-dimensional sphere of radius ρ with the centre at the origin. There exists $R > 0$ such that, for all $\rho \geqslant R$, the sphere S_ρ is transversal to the level set $V_0 = \{P = 0\}$ of the polynomial map P. The restriction $P|_{\mathbb{C}^{n+1}\setminus B_R} : \mathbb{C}^{n+1} \setminus B_R \to \mathbb{C}$ of the function P to the complement of the ball B_R defines a C^∞ locally trivial fibration over a punctured neighbourhood of the origin in $\mathbb{C}$. The loop $\varepsilon_0 \cdot \exp(2\pi i \tau)$ $(0 \leqslant \tau \leqslant 1,\ \|\varepsilon_0\|$ small enough) defines the monodromy transformation $h : V_{\varepsilon_0} \setminus B_R \to V_{\varepsilon_0} \setminus B_R$. Let us denote its zeta-function $\zeta_h(t)$ by $\zeta_P^0(t)$. We use the following definition

DEFINITION. – *The value* 0 *is* atypical at infinity *for the polynomial* P *if the restriction* $P|_{\mathbb{C}^{n+1}\setminus B_R}$ *of the function* P *to the complement of the ball* B_R *is not a* C^∞ *locally trivial fibration over a neighbourhood of the origin in* $\mathbb{C}$.

Remark 5. – This definition does not depend on a choice of coordinates, i.e., it is invariant with respect to polynomial diffeomorphisms of the space $\mathbb{C}^{n+1}$. One can see that an atypical at infinity value is atypical, i.e. it belongs to the bifurcation set $B(P)$ of the polynomial P. Moreover the bifurcation set $B(P)$ is the union of the set of critical values of the polynomial P (in $\mathbb{C}^{n+1}$) and of the set of values atypical at infinity in the described sense. If the singular locus of the level set $V_0 = \{P = 0\}$ is not finite, the value 0 hardly can be considered as typical at infinity. Thus, one should consider this definition as a (possible) general definition of a value atypical at infinity. In fact the same definition was used in [10] for polynomial functions of two variables.

Let $\mathcal{S}$ be a prestratification of the infinity hyperplane $\mathbb{CP}^n_\infty$ such that, for each stratum Ξ of $\mathcal{S}$, the zero zeta-function $\zeta^0_{P,x}(t)$ of the germ of the meromorphic function P at a point $x \in \mathbb{CP}^n_\infty$ does not depend on the point x, for $x \in \Xi$ (let it be $\zeta^0_\Xi(t)$ and let its degree be χ^0_Ξ).

THEOREM 4. –

$$\zeta_P^0(t) = \prod_{\Xi\in\mathcal{S}} \left[\zeta_\Xi^0(t)\right]^{\chi(\Xi)}, \qquad \chi(V_{\varepsilon_0}\setminus B_R) = \sum_{\Xi\in\mathcal{S}} \chi_\Xi^0 \cdot \chi(\Xi).$$

The *proof* is essentially the same as that of Theorem 1. Since the Euler characteristic of the set $V_0 \setminus B_R$ is equal to 0, one has

COROLLARY 1. – *If* $\zeta_P^0(t) \not\equiv 1$, *then the value* 0 *is atypical at infinity for the polynomial* P.

In several papers (see, e.g., [3,11,12]) there was considered an integer $\lambda_P(c)(c\in\mathbb{C})$ such that

$$\chi\left(\{P=c\}\right) = \chi\left(\{P=c+\varepsilon\}\right) + (-1)^{n+1}\left(\sum \mu_i + \lambda_P(c)\right),$$

where μ_i are the Milnor numbers of the (isolated) singular points of the level set $\{P=c\}\subset\mathbb{C}^{n+1}$. Theorem 4 gives the following formula for this invariant:

COROLLARY 2. –

$$\begin{aligned}\lambda_P(0) &= (-1)^n \deg\zeta_P^0(t)\\ &= (-1)^n \sum_{\Xi\in\mathcal{S}} \chi_\Xi^0\cdot\chi(\Xi)\left(= (-1)^n \int_{\mathbb{CP}^n_\infty} \chi_{P,x}^0\, d\chi\right).\end{aligned}$$

Example. – Let $P(x,y,z) = x^a y^b(x^c y^d - z^{c+d}) + z$, $(ad-bc)\neq 0$, and let $D = \deg(P) = a+b+c+d$. The curve $\{P_D = 0\}\subset\mathbb{CP}^2_\infty$ consists on three components: the line $C_1 = \{x=0\}$ with multiplicity a, the line $C_2 = \{y=0\}$ with multiplicity b, and the reduced curve $C_3 = \{x^c y^d - z^{c+d} = 0\}$. Let $Q_1 = C_2\cap C_3 = (1:0:0)$, $Q_2 = C_1\cap C_3 = (0:1:0)$, $Q_3 = C_1\cap C_2 = (0:0:1)$. At each point x of the infinity hyperplane $\mathbb{CP}^2_\infty$ except Q_1 and Q_2, one has $\zeta^0_{P,x}(t) = 1$. At the point Q_1, the germ of the meromorphic function P has the form $(y^b(y^d - z^{c+d}) + zu^{D-1})/u^D$. Its zero zeta-function can be obtained by the Varchenko type formula from [8]. If $(ad-bc)<0$, then $\zeta^0_{P,Q_1}(t) = 1$. If $(ad-bc)>0$, then

$$\zeta^0_{P,Q_1}(t) = \left(1 - t^{\frac{ad-bc}{G.C.D.}}\right)^{G.C.D.},$$

where $G.C.D. = g.c.d.(c,d)\cdot g.c.d.(\frac{ad-bc}{g.c.d.(c,d)}, D-1)$. At the point Q_2, we have just the symmetric situation. Finally

$$\zeta_P^0(t) = \left(1 - t^{\frac{|ad-bc|}{G.C.D.}}\right)^{G.C.D.}.$$

It means that the value 0 is atypical at infinity. In the same way $\zeta_{P-c}^0(t) = 1$, for $c \neq 0$.

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