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Power structure over the Grothendieck ring of maps

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Abstract A power structure over a ring is a method to give sense to expressions of the form $(1 + a_1 t + a_2 t^2 + \cdots)^m$, where a_i, $i = 1, 2, \ldots$, and m are elements of the ring. The (natural) power structure over the Grothendieck ring of complex quasi-projective varieties appeared to be useful for a number of applications. We discuss new examples of λ-and power structures over some Grothendieck rings. The main example is for the Grothendieck ring of maps of complex quasi-projective varieties. We describe two natural λ-structures on it which lead to the same power structure. We show that this power structure is effective. In the terms of this power structure we write some equations containing classes of Hilbert–Chow morphisms. We describe some generalizations of this construction for maps of varieties with some additional structures.

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1 Introduction

A λ-*structure* (called sometimes a pre-λ-structure) on a ring R is an additive-to-multiplicative homomorphism $R \to 1 + tR[[t]]$ ([15]). A *power structure* over a ring R ([10]) is a method to give sense to expressions of the form $(1 + a_1t + a_2t^2 + \cdots)^m$ as a series in $1 + tR[[t]]$, where a_i, $i = 1, 2, \ldots$, and m are elements of the ring R. The notions of λ-structures and power structures are closely related to each other, but are not equivalent. In particular, each λ-structure on a ring defines a power structure over it, but there are, in general, many λ-structures corresponding to one and the same power structure. A "natural" power structure over the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ of complex quasi-projective varieties (see, e.g., [7]) was described in [10]. Its version for the relative case (that is, over the Grothendieck ring of complex quasi-projective varieties over a fixed variety) was defined in [12]. A power structure over the Grothendieck ring of complex quasi-projective varieties over an Abelian monoid was defined in [16]. (A particular case of the Grothendieck ring of varieties over an Abelian monoid when the monoid is the Abelian group $\mathbb{C}$ was considered in [4,6,18] under the name Grothendieck rings of varieties with exponentials.) Power structures over Grothendieck rings of varieties appear to be useful, in particular, for formulation and proof of formulae for the generating series of classes of some configuration spaces or of their invariants, see, e.g., [1,3,11,16].

An important property of the power structure over the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ which makes it useful for the mentioned applications is its *effectiveness*. This means that if all the coefficients a_i of the series $A(t) = 1 + a_1t + a_2t^2 + \cdots$ and the exponent m are classes of complex quasi-projective varieties (not of virtual varieties: differences of such classes), then all the coefficients of the series $(1 + a_1t + a_2t^2 + \cdots)^m$ are also represented by classes of complex quasi-projective varieties. This is a somewhat special property of this power structure. Another natural power structure over the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ (in fact up to now only two power structures over this ring are known) and also its natural "extension" to the Grothendieck ring of stacks ([13]) are not effective.

Here we discuss new examples of λ-and power structures over some Grothendieck rings. The main example is for the Grothendieck ring of regular maps of complex quasi-projective varieties. We describe two natural λ-structures on it which lead to the same power structure. We show that this power structure is effective. In the terms of this power structure we write some equations containing classes of Hilbert–Chow morphisms. We describe some generalizations of this construction for maps of varieties with some additional structures.

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2 Power structures and λ-structures

A *power structure* over a ring R is a method to give sense to expressions of the form $(1 + a_1 t + a_2 t^2 + \cdots)^m$, where the coefficients a_i and the exponent m are elements of R.

Definition ([10]) A *power structure* over a ring R with unity 1 is a map

$$(1 + tR[[t]]) \times R \to 1 + tR[[t]]$$

$\left((A(t), m) \mapsto (A(t))^m\right)$, $A(t) = 1 + a_1 t + a_2 t^2 + \cdots$ which possesses the properties of the exponential function, namely:

1. $(A(t))^0 = 1$,
2. $(A(t))^1 = A(t)$,
3. $(A(t) \cdot B(t))^m = (A(t))^m \cdot (B(t))^m$,
4. $(A(t))^{m+n} = (A(t))^m \cdot (A(t))^n$,
5. $(A(t))^{mn} = \left((A(t))^n\right)^m$,
6. $(1 + a_1 t + \cdots)^m = 1 + m a_1 t + \cdots$;
7. $\left(A(t^k)\right)^m = (A(t))^m \,|_{t \mapsto t^k}$.

Definition A power structure over a ring R is *finitely determined* if the fact that two series $A_1(t)$ and $A_2(t)$ from $1 + tR[[t]]$ differ by terms of order $\geq k$ (that is $A_1(t) - A_2(t) \in \mathfrak{m}^k$, where $\mathfrak{m} = \langle t \rangle \subset R[[t]]$) implies that $(A_1(t))^m - (A_2(t))^m \in \mathfrak{m}^k$.

A natural power structure over the ring $\mathbb{Z}$ of integers is defined by the usual equation for an exponent of a series (see, e.g., [17], page 40)

$$\begin{aligned}
&(1 + a_1 t + a_2 t^2 + \cdots)^m \\
&\quad = 1 + \sum_{k=1}^{\infty} \left(\sum_{\{k_i\}: \sum_i i k_i = k} \frac{m(m-1)\dots(m - \sum_i k_i + 1) \times \prod_i a_i^{k_i}}{\prod_i k_i!} \right) \cdot t^k. \quad (1)
\end{aligned}$$

The sum in the parenthesis is over all partitions of k. (Obviously this power structure is finitely determined. It is easy to show that this power structure is the only finitely determined power structure over the ring $\mathbb{Z}$.)

The Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ of complex quasi-projective varieties is the Abelian group generated by the classes $[X]$ of all quasi-projective varieties X (with the reduced scheme structures) modulo the relations

1. if varierties X and Y are isomorphic, then $[X] = [Y]$;
2. if Y is a Zariski closed subset of a variety X, them $[X] = [Y] + [X \setminus Y]$.

The multiplication in $K_0(\mathrm{Var}_{\mathbb{C}})$ is defined by the Cartesian product of varieties.

The power structure over the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ of complex quasi-projective varieties defined in [10] is given by the equation

$$(1+[A_1]t+[A_2]t^2+\cdots)^{[M]}$$
$$=1+\sum_{k=1}^{\infty}\left(\sum_{\{k_i\}:\sum_i ik_i=k}\left[\left(\left(M^{\sum_i k_i}\setminus\Delta\right)\times\prod_i A_i^{k_i}\right)\Big/\prod_i S_{k_i}\right]\right)\cdot t^k, \quad (2)$$

where $[A_i]$, $i=1,2,\ldots$, and $[M]$ are classes in $K_0(\mathrm{Var}_{\mathbb{C}})$ of complex quasi-projective varieties, Δ is the "large diagonal" in $M^{\sum_i k_i}$, that is the set of (ordered) collections of $\sum_i k_i$ points from M with at least two coinciding ones, the group S_{k_i} of permutations on k_i elements acts by simultaneous permutations on the components of the corresponding factor M^{k_i} in $M^{\sum_i k_i}=\prod_i M^{k_i}$ and on the components of the factor $A_i^{k_i}$.

Apart from the Grothendieck ring of complex quasi-projective varieties one can consider the Grothendieck semiring $S_0(\mathrm{Var}_{\mathbb{C}})$. It is defined in the same way as $K_0(\mathrm{Var}_{\mathbb{C}})$ with the word group substituted by the word semigroup. Elements of the semiring $S_0(\mathrm{Var}_{\mathbb{C}})$ are represented by "genuine" complex quasi-projective varieties, not by virtual ones (that is formal differences of varieties). One can show that two complex quasi-projective varieties X and Y represent the same element of the semiring $S_0(\mathrm{Var}_{\mathbb{C}})$ if and only if they are piece-wise isomorphic, that is if there exist decompositions $X=\bigsqcup_{i=1}^s X_i$ and $Y=\bigsqcup_{i=1}^s Y_i$ into Zariski locally closed subsets such that X_i and Y_i are isomorphic for $i=1,\ldots,s$. There is a natural map (a semiring homomorphism) from $S_0(\mathrm{Var}_{\mathbb{C}})$ to $K_0(\mathrm{Var}_{\mathbb{C}})$. (According to [2] this map is not injective.)

A power structure over the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ is called *effective* if the fact that all the coefficients a_i of the series $A(t)$ and the exponent m are represented by classes of complex quasi-projective varieties (i.e., belong to the image of the map $S_0(\mathrm{Var}_{\mathbb{C}})\to K_0(\mathrm{Var}_{\mathbb{C}})$) implies that all the coefficients of the series $(A(t))^m$ are also represented by such classes. Roughly speaking this means that the power structure can be defined over the Grothendieck semiring $S_0(\mathrm{Var}_{\mathbb{C}})$. The same concept is used for Grothendieck rings of complex quasi-projective varieties with additional structures. The effectiveness of the described power structure over the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ is clear from Eq. (2).

An equation similar to (2) was given in [16] for a power structure over the Grothendieck ring of complex quasi-projective varieties over an Abelian monoid used there.

Power structures over a ring are related to λ-structures on it. Let R be a ring with a λ-structure, that is, for each $a\in R$ there is defined a series $\lambda_a(t)=1+at+\cdots\in 1+tR[[t]]$ so that $\lambda_{a+b}(t)=\lambda_a(t)\lambda_b(t)$ (in other words one has an additive-to-multiplicative homomorphism $R\to 1+tR[[t]]$; see, e.g., [15]). A λ-structure $\lambda_a(t)$ defines a (finitely determined) power structure over R in the following way. Any power series $A(t)=1+a_1t+a_2t^2+\cdots\in 1+tR[[t]]$ can be in a unique way represented as the product $A(t)=\prod_{i=1}^{\infty}\lambda_{b_i}(t^i)$ with $b_i\in R$. (Indeed, one can see that $b_1=a_1$, b_2 is the coefficient of t^2 in $A(t)\left(\lambda_{b_1}(t)\right)^{-1}$, b_3 is the coefficient of t^3 in $A(t)\left(\lambda_{b_1}(t)\right)^{-1}\left(\lambda_{b_2}(t^2)\right)^{-1}$, ...). Then one defines the series $(A(t))^m$ by

$$(A(t))^m:=\prod_{i=1}^{\infty}\lambda_{mb_i}(t^i), \quad (3)$$

which induces a power structure over R.

Remark A λ-structure on a ring R defines maps $\mathbf{Exp} : tR[[t]] \to 1 + tR[[t]]$ and $\mathbf{Log} : 1 + tR[[t]] \to tR[[t]]$ (inverse to each other) in the following way:

$$\mathbf{Exp}(b_1 t + b_2 t^2 + \cdots) := \prod_{k \geq 1} \lambda_{b_k}(t^k)\,;$$

if $1 + a_1 t + a_2 t^2 + \cdots = \prod_{k=1}^{\infty} \lambda_{b_k}(t^k)$, then

$$\mathbf{Log}(1 + a_1 t + a_2 t^2 + \cdots) := \sum_{k=1}^{\infty} b_k t^k.$$

The map $\mathbf{Exp}$ is an additive-to-multiplicative homomorphism. The map $\mathbf{Log}$ is a multiplicative-to-additive homomorphism. Each of these maps determines the λ-structure on the ring, see, e.g., [10].

One can show that the power structure (2) over the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ corresponds to the λ-structure on it defined by the *Kapranov motivic zeta function* ([14])

$$\zeta_{[X]}(t) = 1 + [X]t + [S^2X]t^2 + [S^3X]t^3 + \cdots\,,$$

where $S^kX = X^k/S_k$ is the kth symmetric power of the variety X. In terms of the power structure one has $\zeta_{[X]}(t) = (1 + t + t^2 + \cdots)^{[X]} = (1-t)^{-[X]}$.

There are many λ-structures corresponding to the same power structure over a ring R. For any series $\lambda_1(t) = 1 + t + a_2 t^2 + \cdots$ the equation

$$\lambda_a(t) := (\lambda_1(t))^a$$

gives a λ-structure on the ring R. For example, the power structure (2) over $K_0(\mathrm{Var}_{\mathbb{C}})$ can be defined both by the Kapranov motivic zeta function and by the λ-structure

$$\lambda_{[X]}(t) := (1+t)^{[X]} = 1 + [X]t + [B_2X]t^2 + [B_3X]t^3 + \cdots\,,$$

where $B_kX := (X^k \setminus \Delta)/S_k$ is the configuration space of k distinct unordered points of X (see [10]).

Another "natural" λ-structure on the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ (*opposite* to the one defined by the Kapranov motivic zeta function $\zeta_{[X]}(t)$) is defined by the series $\zeta_{-[X]}(-t)$. One can show that the corresponding power structure over the ring $K_0(\mathrm{Var}_{\mathbb{C}})$ is not effective (see [13]). (The authors know no power structure over the ring $K_0(\mathrm{Var}_{\mathbb{C}})$ except the described two.)

Let R_1 and R_2 be rings with power structures over them. A ring homomorphism $\varphi : R_1 \to R_2$ induces the natural ring homomorphism $R_1[[t]] \to R_2[[t]]$ (also denoted by φ) by $\varphi\left(\sum_i a_i t^i\right) = \sum_i \varphi(a_i)\, t^i$. One has the following statement.

Proposition 1 ([11]) *If a ring homomorphism $\varphi : R_1 \to R_2$ is such that $\varphi\left((1-t)^{-a}\right) = (1-t)^{-\varphi(a)}$, then $\varphi\left((A(t))^m\right) = (\varphi\left(A(t)\right))^{\varphi(m)}$.*

Equations written in terms of the power structure (2) give equations for the Euler characteristics with compact support $\chi(\bullet)$ and for the *Hodge–Deligne polynomial* $e_{\bullet}(u, v)$ via the natural homomorphisms $\chi : K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}$ and $e : K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}[u, v]$. These homomorphisms are compatible with the power structures over the rings $\mathbb{Z}$ (see Eq. (1)) and $\mathbb{Z}[u, v]$, where the power structure over the latter is defined as follows.

Let $\mathbb{Z}[u_1, \ldots, u_r]$ be the ring of polynomials in r variables. Let $P(u_1, \ldots, u_r) = \sum_{\underline{k}\in\mathbb{Z}^r_{\geq 0}} p_{\underline{k}} \underline{u}^{\underline{k}} \in \mathbb{Z}[u_1, \ldots, u_r]$, where $\underline{k} = (k_1, \ldots, k_r)$, $\underline{u} = (u_1, \ldots, u_r)$, $\underline{u}^{\underline{k}} = u_1^{k_1} \cdot \ldots \cdot u_r^{k_r}$, $p_{\underline{k}} \in \mathbb{Z}$. Let

$$\lambda_P(t) := \prod_{\underline{k}\in\mathbb{Z}^r_{\geq 0}} (1 - \underline{u}^{\underline{k}} t)^{-p_{\underline{k}}},$$

where the power (with an integer exponent $-p_{\underline{k}}$) means the usual one. The series $\lambda_P(t)$ defines a λ-structure on the ring $\mathbb{Z}[u_1, \ldots, u_r]$ and therefore a power structure over it (with $\lambda_P(t) = (1 - t)^{-P}$).

Let $r = 2$, $u_1 = u$, $u_2 = v$. Let $e : K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}[u, v]$ be the ring homomorphism which sends the class $[X]$ of a quasi-projective variety X to its Hodge–Deligne polynomial $e_X(u, v) = \sum_{i,j} h_X^{ij}(-u)^i(-v)^j$. One can see that the homomorphism e respects the λ-and therefore the power structures over the source and over the target. This is shown in [5,11]: in terms of the power structures Proposition 1.2 in [5] can be rewritten as

$$e\left((1 - t)^{-[X]}\right) = (1 - t)^{-e_X(u,v)}.$$

3 Grothendieck ring of maps

Let us consider (regular) maps $f : X \to Y$ between complex quasi-projective varieties.

Definition Maps $f : X \to Y$ and $f' : X' \to Y'$ between complex quasi-projective varieties are *equivalent* if there exist isomorphisms $h_1 : X \to X'$ and $h_2 : Y \to Y'$ such that $h_2 \circ f = f' \circ h_1$.

The definition of the Grothendieck ring of maps is inspired by the one for the Grothendieck ring of varieties $K_0(\mathrm{Var}_{\mathbb{C}})$.

Definition The Grothendieck ring $K_0(\mathrm{Map}_{\mathbb{C}})$ of maps between complex quasi-projective varieties is the free Abelian group generated by the classes $[X \xrightarrow{f} Y]$ of maps between varieties with the reduced scheme structures modulo the relations:

1. if two maps $f : X \to Y$ and $f' : X' \to Y'$ are equivalent, then

$$[X \xrightarrow{f} Y] = [X' \xrightarrow{f'} Y'];$$

2. if $f : X \to Y$ is a map and Z is a Zariski closed subset of Y, then

$$[X \xrightarrow{f} Y] = [f^{-1}(Z) \xrightarrow{f|_{f^{-1}(Z)}} Z] + [f^{-1}(Y \setminus Z) \xrightarrow{f|_{f^{-1}(Y\setminus Z)}} Y \setminus Z];$$

3. if $f : X \to Y$ is a map and Z is a Zariski closed subset of X, then

$$[X \xrightarrow{f} Y] = [Z \xrightarrow{f|_Z} Y] + [X \setminus Z \xrightarrow{f|_{X\setminus Z}} Y].$$

The definition means that the summation in $K_0(\mathrm{Map}_{\mathbb{C}})$ can be defined by the disjoint union, that is

$$[X_1 \xrightarrow{f_1} Y_2] + [X_2 \xrightarrow{f_2} Y_2] := [X_1 \sqcup X_2 \xrightarrow{f_1 \sqcup f_2} Y_1 \sqcup Y_2].$$

The multiplication in $K_0(\mathrm{Map}_{\mathbb{C}})$ is defined by the Cartesian product:

$$[X_1 \xrightarrow{f_1} Y_2] \cdot [X_2 \xrightarrow{f_2} Y_2] := [X_1 \times X_2 \xrightarrow{f_1 \times f_2} Y_1 \times Y_2].$$

The unit **1** in $K_0(\mathrm{Map}_{\mathbb{C}})$ is represented by the identity map form a point to itself.

Remark The relation 3) applied to $Z = \emptyset$ gives that $[\emptyset \to Y] = 0$. This implies that if $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ is a map between varieties and X is a subvariety of $\mathcal{X}$, then the class of a map $f : X \to Y$, where $Y \subset \mathcal{Y}$, $f = \mathcal{F}_{|X}$ does not depend on Y if $Y \supset \mathcal{F}(X)$. Therefore, in a situation of this sort, to define the class $[X \xrightarrow{f} Y]$, one has to describe only the source X. This will be used, in particular, in the proof of Theorem 1.

There is a natural homomorphism π from $K_0(\mathrm{Map}_{\mathbb{C}})$ to the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ of complex quasi-projective varieties which sends $[X \xrightarrow{f} Y]$ to $[X]$. (The correspondence $[X \xrightarrow{f} Y] \mapsto [Y]$ is not well defined: see the Remark above.) This map has two natural "sections": injective ring homomorphisms σ_1 and σ_2 from $K_0(\mathrm{Var}_{\mathbb{C}})$ to $K_0(\mathrm{Map}_{\mathbb{C}})$ defined by$[X] \mapsto [X \to pt]$ and $[X] \mapsto [X \xrightarrow{id} X]$ respectively, where pt is a one point set. The relations 2) and 3) in the Definition of $K_0(\mathrm{Map}_{\mathbb{C}})$ show that $\sigma_i \circ \pi = \mathrm{id}$ for $i = 1, 2$.

Remarks 1. In the same way (substituting the word group by the word semigroup) one can define the Grothendieck semiring $S_0(\mathrm{Map}_{\mathbb{C}})$ of maps between complex quasi-projective varieties. The elements in $S_0(\mathrm{Map}_{\mathbb{C}})$ are represented by classes of genuine maps, not by virtual ones (that is formal differences of maps). One can show that the classes in $S_0(\mathrm{Map}_{\mathbb{C}})$ of regular maps $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are equal if and only if they are piece-wise isomorphic, that is if there exist partitions $X_1 = \bigsqcup_{i=1}^n X_{1,i}$ and $X_2 = \bigsqcup_{i=1}^n X_{2,i}$ such that the maps $f_1 : X_{1,i} \to f_1(X_{1,i})$ and $f_2 : X_{2,i} \to f_2(X_{2,i})$ are equivalent for all $i = 1, \ldots, n$.

2. One can see that the subring of the Grothendieck ring of $K_0(\mathrm{Map}_{\mathbb{C}})$ generated by the classes of maps between zero-dimensional varieties (that is between finite sets) is isomorphic to $\mathbb{Z}$. This follows from the fact that, due to the relations 2) an 3) in the Definition of $K_0(\mathrm{Map}_{\mathbb{C}})$, each class $[X \xrightarrow{f} Y]$, where X and Y are finite sets, is a multiple of the class $[pt \to pt]$.

4 λ-structures over the Grothendieck ring of maps

Let us describe two natural λ-structures over the ring $K_0(\mathrm{Map}_{\mathbb{C}})$.

For a map $f : X \to Y$, one has the natural map $S^k f : S^k X \to S^k Y$ between the kth symmetric powers of X and Y. (Pay attention that the map f does not define a map between the configuration spaces $B_k X = \left(X^k \backslash \Delta\right)/S_k$ and $B_k Y = \left(Y^k \backslash \Delta\right)/S_k$ of k distinct points on X and Y respectively.)

Definition The *Kapranov motivic zeta function of a map* $f : X \to Y$ is defined by

$$\zeta_{[X \xrightarrow{f} Y]}(t) := \mathbf{1} + \sum_{k \geq 1} [S^k X \xrightarrow{S^k f} S^k Y] \cdot t^k \in \mathbf{1} + t K_0(\mathrm{Map}_{\mathbb{C}})[[t]]. \tag{4}$$

Proposition 2 *The Kapranov motivic zeta function defines a λ-structure on the ring* $K_0(\mathrm{Map}_{\mathbb{C}})$.

Proof It is necessary to show that, for two maps $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, one has

$$\zeta_{[X_1 \sqcup X_2 \xrightarrow{f_1 \sqcup f_2} Y_1 \sqcup Y_2]}(t) = \zeta_{[X_1 \xrightarrow{f_1} Y_1]}(t) \cdot \zeta_{[X_2 \xrightarrow{f_2} Y_2]}(t).$$

This follows from the relation

$$S^k(X_1 \sqcup X_2) \xrightarrow{S^k(f_1 \sqcup f_2)} S^k(Y_1 \sqcup Y_2) = \bigsqcup_{i=0}^{k} \left(S^i X_1 \xrightarrow{S^i(f_1)} S^i Y_1 \right) \times \left(S^{k-i} X_2 \xrightarrow{S^{k-i}(f_2)} S^{k-i} Y_2 \right).$$

This relation is a consequence of the fact that $S^k(X_1 \sqcup X_2) = \bigsqcup_{i=0}^{k} S^i X_1 \times S^{k-i} X_2$. □

Let $B_k X := (X^k \backslash \Delta)/S_k$ be the configuration space of collections of k different points in X (Δ is the big diagonal in X_k consisting of k-tuples of points of X with at least two coinciding ones). For a map $f : X \to Y$, one has the corresponding map $B_k f : B_k X \to S^k Y$ from the configuration space of k distinct points on X to the kth symmetric power of the variety Y. Let

$$\lambda_{[X \xrightarrow{f} Y]}(t) := \mathbf{1} + \sum_{k \geq 1} [B_k X \xrightarrow{B_k f} S^k Y] \cdot t^k \in \mathbf{1} + t K_0(\mathrm{Map}_{\mathbb{C}})[[t]]. \tag{5}$$

Proposition 3 *The series* $\lambda_{[X\xrightarrow{f}Y]}(t)$ *defines a* λ*-structure on the ring* $K_0(\mathrm{Map}_{\mathbb{C}})$.

Proof Just as in Proposition 2, for two maps $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$, one has

$$\begin{aligned} &B_k\,(X_1 \sqcup X_2) \overset{B_k(f_1\sqcup f_2)}{\longrightarrow} S^k\,(Y_1 \sqcup Y_2) \\ &= \bigsqcup_{i=0}^{k} (B_i X_1 \overset{B_i f_1}{\longrightarrow} S^i Y_1) \times (B_{k-i} X_2 \overset{B_{k-i} f_2}{\longrightarrow} S^{k-i} Y_2). \end{aligned}$$

□

5 A power structure over the ring $K_0(\mathrm{Map}_{\mathbb{C}})$

Let us define a power structure over the ring $K_0(\mathrm{Map}_{\mathbb{C}})$.

For a series

$$A(t) := \mathbf{1} + [X_1 \overset{f_1}{\to} Y_1]t + [X_2 \overset{f_2}{\to} Y_2]t^2 + \cdots \in \mathbf{1} + tK_0(\mathrm{Map}_{\mathbb{C}})[[t]],$$

and for an element $m = [M \overset{f}{\to} N] \in K_0(\mathrm{Map}_{\mathbb{C}})$, let us define $(A(t))^m$ as

$$\mathbf{1} + \sum_{k=1}^{\infty} \left(\sum_{\underline{k}:\sum_i ik_i = k} \left[\left(\left(M^{\sum_i k_i} \setminus \Delta \overset{f^{\sum_i k_i}}{\to} N^{\sum_i k_i} \right) \times \prod_i (X_i \overset{f_i}{\to} Y_i)^{k_i} \right) \Big/ \prod_i S_{k_i} \right] \right) \cdot t^k, \tag{6}$$

where $\underline{k} = \{k_i : i \in \mathbb{Z}_{>0}, k_i \in \mathbb{Z}_{\geq 0}\}$, Δ is the "large diagonal" in $M^{\sum_i k_i}$ which consists of $(\sum_i k_i)$-tuples of points of M with at least two coinciding ones, the permutation group S_{k_i} acts simultaneously on the components of the factors M^{k_i} and N^{k_i} in $M^{\sum_i k_i} \setminus \Delta$ and in $N^{\sum_i k_i}$ and on the components of $(X_i \overset{f_i}{\to} Y_i)^{k_i}$.

In Eq. (6), by

$$\left(\left(M^{\sum_i k_i} \setminus \Delta \overset{f^{\sum_i k_i}}{\to} N^{\sum_i k_i} \right) \times \prod_i (X_i \overset{f_i}{\to} Y_i)^{k_i} \right) \Big/ \prod_i S_{k_i},$$

we mean the map

$$\left(\left(M^{\sum_i k_i} \setminus \Delta \right) \times \prod_i X_i^{k_i} \right) \Big/ \prod_i S_{k_i} \to \left(N^{\sum_i k_i} \times \prod_i Y_i^{k_i} \right) \Big/ \prod_i S_{k_i}$$

induced by f and f_i. (Pay attention that the action of the group $\prod_i S_{k_i}$ on the source $\left(M^{\sum_i k_i} \setminus \Delta \right) \times \prod_i X_i$ is free.)

Let us give an interpretation of the coefficients at t^k in Eq. (6) similar to the one in [10]. Let $X_0 = pt$, let Γ be the disjoint union $\bigsqcup_{i=0}^{\infty} X_i$ and let $I : \Gamma \to \mathbb{Z}$ be the tautological function on Γ which sends the component X_i to i. A representative of the coefficient at t^k in the source part of (6) can be identified with the configuration space of maps $\psi : M \to \Gamma$ such that $\sum_{x\in M} I(\psi(x)) = k$. This means that there are finitely many $x \in M$ such that $\psi(x) \notin X_0$. Taking into account only the points $x \in M$ with $\psi(x) \notin X_0$, it is possible to say that we consider collections of (non-coinciding) particles on M with positive integer charges and with the space of internal states of a particle with charge i parametrized by the variety X_i, $i = 1, 2, \ldots$ The coefficient of t^k in (6) is represented by the configuration space of collections of particles with the *total charge* k (cf. [8]). These data correspond to the source of the map. The target is represented by a similar configuration space of particles on N (images of the particles on M under the map f) whose locations are permitted to coincide and whose spaces of internal states are parametrized by the varieties Y_i. The map from the source to the target is defined in the obvious way (and is determined by the maps f and f_i).

Theorem 1 *Equation* (6) *defines a power structure over the ring* $K_0(\mathrm{Map}_{\mathbb{C}})$.

Proof We have to prove the properties (3)–(5) from the definition of a power structure (cf. [10]). Due to the Remark after the definition of the ring $K_0(\mathrm{Map}_{\mathbb{C}})$ in Sect. 3, we have to controle only the source parts of the relations.

(3) Let

$$A(t) := \mathbf{1} + [X_1 \xrightarrow{f_1} Y_1]t + [X_2 \xrightarrow{f_2} Y_2]t^2 + \cdots,$$
$$B(t) := \mathbf{1} + [X'_1 \xrightarrow{f'_1} Y'_1]t + [X'_2 \xrightarrow{f'_2} Y'_2]t^2 + \cdots,$$

and $m = [M \xrightarrow{f} N]$. The coefficient of t^s in $A(t)B(t)$ is equal to

$$\sum_{i=0}^{s}[X_i \times X'_{s-i} \to Y_i \times Y'_{s-i}].$$

The coefficient of t^k in $(A(t)B(t))^m$ is represented by the configuration space of maps ψ from M to

$$\bigsqcup_{s=0}^{\infty}\bigsqcup_{i=0}^{s}(X_i \times X'_{s-i}) = \bigsqcup_{i,j=0}^{\infty}(X_i \times X'_j) = \left(\bigsqcup_{i=0}^{\infty} X_i\right) \times \left(\bigsqcup_{j=0}^{\infty} X'_j\right)$$

such that

$$\sum_{x\in M}\big(I(\pi_1 \circ \psi(x)) + I'(\pi_2 \circ \psi(x))\big) = k.$$

Here π_1 and π_2 are the projections of $\left(\bigsqcup_{i=0}^{\infty} X_i\right) \times \left(\bigsqcup_{j=0}^{\infty} X'_j\right)$ to the first and to the second factors respectively. This is the union for $\ell = 0, 1, \ldots, k$ of the products of the configuration spaces of maps $\psi_1 : M \to \bigsqcup_{i=0}^{\infty} X_i$ and of maps $\psi_2 : M \to \bigsqcup_{j=0}^{\infty} X'_j$

with $\sum_{x\in M} I(\psi_1(x)) = \ell$ and $\sum_{x\in M} I'(\psi_2(x)) = k-\ell$ respectively. This is just a description of the coefficient of t^k in $(A(t))^m \cdot (B(t))^m$.

(4) Let $n = [P \stackrel{g}{\to} Q]$. We have $m+n = [M \sqcup P \stackrel{f\sqcup g}{\to} N \sqcup Q]$. The coefficient of t^k in $(A(t))^{m+n}$ is represented by the configuration space of maps ψ from $M \sqcup P$ to $\bigsqcup_{i=0}^{\infty} X_i$ such that $\sum_{x\in M\bigsqcup P} I(\psi(x)) = k$. This is the union for $\ell = 0, 1, \dots, k$ of the products of the configuration spaces of maps $\psi_1 : M \to \bigsqcup_{i=0}^{\infty} X_i$ and of maps $\psi_2 : P \to \bigsqcup_{i=0}^{\infty} X_i$ with $\sum_{x\in M} I(\psi_1(x)) = \ell$ and $\sum_{x\in P} I(\psi_2(x)) = k-\ell$ respectively. This is just a description of the coefficient of t^k in $(A(t))^m \cdot (A(t))^n$.

(5) Let, as above, $n = [P \stackrel{g}{\to} Q]$. We have $mn = [M \times P \stackrel{f\times g}{\to} N \times Q]$. The coefficient of t^s in $(A(t))^n$ is represented by the configuration space of maps $\psi : P \to \bigsqcup_{i=0}^{\infty} X_i$ such that $\sum_{x\in P} I(\psi(x)) = s$ (s is *the total charge* of the map ψ). The coefficient of t^k in $\left((A(t))^n\right)^m$ is represented by the configuration space of maps $\check{\psi}$ from M to the union of the configuration spaces described above with the sum of weights equal to k. Such maps are in one-to-one correspondence with the maps $\widehat{\psi} : M \times P \to \bigsqcup_{i=0}^{\infty} X_i$ such that $\sum_{x\in M\times P} I(\widehat{\psi}(x)) = k$. This is just a description of the coefficient of t^k in $(A(t))^{mn}$. □

The power structure (6) is obviously effective.

Theorem 2 *The power structure* (6) *is defined by each of the* λ*-structures given by the series* $\zeta_{[X\stackrel{f}{\to}Y]}(t)$ *and* $\lambda_{[X\stackrel{f}{\to}Y]}(t)$.

Proof We have to show that Eq. (6) gives:

$$(\mathbf{1}+t+t^2+\cdots)^{[M\stackrel{f}{\to}N]} = \zeta_{[M\stackrel{f}{\to}N]}(t), \tag{7}$$

$$(\mathbf{1}+t)^{[M\stackrel{f}{\to}N]} = \lambda_{[M\stackrel{f}{\to}N]}(t). \tag{8}$$

The coefficient of t^k in $(\mathbf{1}+t+t^2+\cdots)^{[M\stackrel{f}{\to}N]}$ is represented by the configuration space of finite subsets of points in M with (positive) multiplicities (since the map from M to $\bigsqcup_{i=0}^{\infty} pt_i$ is defined by the subset of points $x \in M$ such that $I(\psi(x)) \neq 0$ and by their multiplicities $I(\psi(x))$). This means that this coefficient is equal to $[S^k M \stackrel{S^k f}{\to} S^k N]$. This proves (7).

The only non-empty summand in the coefficient of t^k in $(\mathbf{1}+t)^{[M\stackrel{f}{\to}N]}$ corresponds to the partition $k_1 = k$, $k_i = 0$ for $i > 1$, and is represented by the map

$$(M^k \backslash \Delta)/S_k = B_k M \to N^k/S_k = S^k N.$$

This proves (8). □

Remark One can see that the both injections σ_1 and σ_2 defined in Sect. 3 respect the power structures over the source and the target. For a fixed variety S one has the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}|S})$ of varieties over S (generated by the classes of maps

$f : X \to S$) and a natural map $K_0(\mathrm{Var}_{\mathbb{C}|S}) \to \mathrm{K}_0(\mathrm{Map}_{\mathbb{C}})$. This map is not a ring homomorphism.

6 Generating series of Hilbert–Chow morphisms

Let $\mathbb{L}_v \in K_0(\mathrm{Map}_{\mathbb{C}})$ be the class of the map $\mathbb{A}^1_{\mathbb{C}} \to pt$ from the complex affine line to a one point set: the image of $\mathbb{L}$ under the inclusion $\sigma_1 : K_0(\mathrm{Var}_{\mathbb{C}}) \subset K_0(\mathrm{Map}_{\mathbb{C}})$ defined by $[X] \mapsto [X \to pt]$. (Do not mix $\mathbb{L}_v$ with the the image $\mathbb{L}_h$ of the element $\mathbb{L} \in K_0(\mathrm{Var}_{\mathbb{C}})$ under the other embedding defined by $[X] \mapsto [X \stackrel{id}{\to} X]$.)

For a non-singular d-dimensional quasi-projective variety X, let Hilb^k_X be the Hilbert scheme of zero-dimensional subschemes of length k in X. One has the Hilbert-Chow morphism $\pi_k : \mathrm{Hilb}^k_X \to S^k X$ to the kth symmetric power of X. Let $\mathrm{Hilb}^k_{\mathbb{C}^d,0}$ be the Hilbert scheme of zero-dimensional subschemes of length k in $\mathbb{C}^d$ supported at the origin.

Theorem 3 *Let X be a non-singular d-dimensional quasi-projective variety. In the Grothendieck ring $K_0(\mathrm{Map}_{\mathbb{C}})$ one has*

$$\mathbf{1} + \sum_{k=1}^{\infty} [\mathrm{Hilb}^k_X \stackrel{\pi_k}{\to} S^k X] \cdot t^k = \left(\mathbf{1} + \sum_{k=1}^{\infty} [\mathrm{Hilb}^k_{\mathbb{C}^d,0} \to pt] \cdot t^k\right)^{[X \stackrel{id}{\to} X]}. \tag{9}$$

Proof The arguments of [11] show that there exists a (finite) Zariski open covering $\{U_i\}$ of X ($X = \bigcup_{i \in I_0} U_i$) such that a zero-dimensional subscheme of length k in U_i is determined by a (finite) collection of points of U_i with a subscheme from $\bigsqcup_{q=0}^{\infty} \mathrm{Hilb}^q_{\mathbb{C}^d,0}$ associated to each of them so that the sum of their lengths q is equal to k. The same holds for zero-dimensional subschemes of any Zariski open subset of U_i, in particular, for zero-dimensional subschemes of the intersection $U_I = \bigcap_{i \in I} U_i$ with $I \subset I_0$, $I \neq \emptyset$. Alongside with the geometric description (6) of the power structure over $K_0(\mathrm{Map}_{\mathbb{C}})$ this gives

$$\mathbf{1} + \sum_{k=1}^{\infty} [\mathrm{Hilb}^k_{U_I} \stackrel{\pi_k}{\to} S^k X] \cdot t^k = \left(\mathbf{1} + \sum_{k=1}^{\infty} [\mathrm{Hilb}^k_{\mathbb{C}^d,0} \to pt] \cdot t^k\right)^{[U_I \stackrel{id}{\to} U_I]}.$$

Using the inclusion-exclusion formula one gets

$$\begin{aligned} &\mathbf{1} + \sum_{k=1}^{\infty} [\mathrm{Hilb}^k_X \stackrel{\pi_k}{\to} S^k X] \cdot t^k \\ &= \prod_{I \subset I_0,\ I \neq \emptyset} \left(\mathbf{1} + \sum_{k=1}^{\infty} [\mathrm{Hilb}^k_{U_I} \stackrel{\pi_k}{\to} S^k X] \cdot t^k\right)^{(-1)^{|I|-1}} \end{aligned}$$

$$= \left(\mathbf{1} + \sum_{k=1}^{\infty} [\mathrm{Hilb}^k_{\mathbb{C}^d,0} \to pt] \cdot t^k\right)^{\sum_{I\subset I_0,\ I\neq\emptyset} (-1)^{|I|-1}[U_I \overset{id}{\to} U_I]}$$

$$= \left(\mathbf{1} + \sum_{k=1}^{\infty} [\mathrm{Hilb}^k_{\mathbb{C}^d,0} \to pt] \cdot t^k\right)^{[X \overset{id}{\to} X]}.$$

This proves (9). □

For $d = 2$ one has

$$\mathbf{1} + \sum_{k=1}^{\infty} [\mathrm{Hilb}^k_{\mathbb{C}^2,0} \to pt] \cdot t^k = \prod_{i=1}^{\infty} (\mathbf{1} - \mathbb{L}_v^{i-1} t^i) \in \mathbf{1} + K_0(\mathrm{Map}_{\mathbb{C}})[[t]].$$

(This is a trivial reformulation of the equation

$$1 + \sum_{k=1}^{\infty} [\mathrm{Hilb}^k_{\mathbb{C}^2,0}] \cdot t^k = \prod_{i=1}^{\infty} (1 - \mathbb{L}^{i-1} t^i) \in 1 + K_0(\mathrm{Var}_{\mathbb{C}})[[t]]$$

proved in [9]). This implies the following statement.

Corollary *For a non-singular quasi-projective surface X, one has*

$$\mathbf{1}+\sum_{k=1}^{\infty} [\mathrm{Hilb}^k_X \overset{\pi_k}{\to} S^k X]\cdot t^k = \left(\prod_{i=1}^{\infty} (\mathbf{1} - \mathbb{L}_v^{i-1} t^i)\right)^{[X \overset{id}{\to} X]} \in \mathbf{1}+K_0(\mathrm{Map}_{\mathbb{C}})[[t]]. \quad (10)$$

7 Versions of the described power structure

One can see that analogues of the power structure on the Grothendieck ring of maps $K_0(\mathrm{Map}_{\mathbb{C}})$ defined by Eq. (6) exist in the following settings.

1. *The relative setting* The Grothendieck group $K_0(\mathrm{Map}_{\mathbb{C}}/\varphi)$ of maps over a fixed map $\varphi : S_1 \to S_2$ is defined as the Grothendieck group generated by the classes of commutative diagrams of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ S_1 \arrow[r, "\varphi"] & S_2 \end{tikzcd}$$

   with the natural analogues of the relations 1)–3). The multiplication in $K_0(\mathrm{Map}_{\mathbb{C}}/\varphi)$ is defined by the fibre products over S_1 and S_2. An analogue of Eq. (6) defines an (effective) power structure over $K_0(\mathrm{Map}_{\mathbb{C}}/\varphi)$. In this analogue all the products (including $M^{\sum k_i}$ and $N^{\sum k_i}$ considered as products of $\sum k_i$ copies of M and of N respectively) are fibre products over S_1 or S_2.

2. *The equivariant setting* For a finite group G, the Grothendieck ring $K_0^G(\mathrm{Map}_{\mathbb{C}})$ of G-equivariant maps is defined as the Grothendieck ring generated by the classes $[X \xrightarrow{f} Y]$, where X and Y are complex quasi-projective G-varieties and f is a G-equivariant map. All the maps in (6) are G-equivariant.
3. *The relative setting over an Abelian monoid* Let $\mathfrak{M}$ be an Abelian monoid with zero. As in the relative setting above, the Grothendieck group $K_0(\mathrm{Map}_{\mathbb{C}}/\mathfrak{M})$ of maps over the monoid $\mathfrak{M}$ is defined as the Grothendieck group generated by the classes of commutative diagrams of the form

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle p_X} & & \downarrow{\scriptstyle p_Y} \\ \mathfrak{M} & \xrightarrow{\mathrm{id}} & \mathfrak{M} \end{array}$$

   The difference is in the definitions of the multiplication in $K_0(\mathrm{Map}_{\mathbb{C}}/\mathfrak{M})$ and of the maps to $\mathfrak{M}$ of the summands in Eq. (6). The multiplication is defined via the map $\mathfrak{M}\times\mathfrak{M} \to \mathfrak{M}$ applied to the usual Cartesian product (with the target $\mathfrak{M}\times\mathfrak{M}$). To define the map to $\mathfrak{M}$ on the summands of (6), it is useful to use consider them as configuration spaces of particles on M with some charges and some weights. The weights of a particle $s \in M$ of charge n (and thus of the internal state ϕ from the variety X_n) is defined as $np_M(s) + p_{X_n}(\phi)$, where p_M and p_{X_n} are the maps from the corresponding varieties to $\mathfrak{M}$. The weight of a collection of particles is defined as the sum of the weights of the individual particles. Cf. [16].
4. *The relative setting over a morphism of Abelian monoids* One can see that the definition of the Grothendieck ring and of the power structure over it from **3** can be extended to maps over a fixed morphism of Abelian monoids $\varphi : \mathfrak{M}_1 \to \mathfrak{M}_2$.

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