Grothendieck ring of varieties with finite groups actions *

S.M. Gusein-Zade † I. Luengo ‡ A. Melle-Hernández §

Abstract

We define a Grothendieck ring of varieties with finite groups actions and show that the orbifold Euler characteristic and the Euler characteristics of higher orders can be defined as homomorphisms from this ring to the ring of integers. We describe two natural λ-structures on the ring and the corresponding power structures over it and show that one of these power structures is effective. We define a Grothendieck ring of varieties with equivariant vector bundles and show that the generalized ("motivic") Euler characteristics of higher orders can be defined as homomorphisms from this ring to the Grothendieck ring of varieties extended by powers of the class of the complex affine line. We give an analogue of the Macdonald type formula for the generating series of the generalized higher order Euler characteristics of wreath products.

*Math. Subject Class.: 14F30, 18F30, 55M35. Keywords: finite group actions, complex quasi-projective varieties, Grothendieck rings, lambda-structure, power structure.

†The work of the first author (Sections 1, 2, 6, and 7) was supported by the grant 16-11-10018 of the Russian Science Foundation. Address: Moscow State University, Faculty of Mechanics and Mathematics, GSP-1, Moscow, 119991, Russia. E-mail: sabir@mccme.ru

‡The last two authors were partially supported by a competitive Spanish national grant MTM2016-76868-C2-1-P. Address: ICMAT (CSIC-UAM-UC3M-UCM), Dept. of Algebra, Geometry and Topology, Complutense University of Madrid, Plaza de Ciencias 3, Madrid, 28040, Spain. E-mail: iluengo@mat.ucm.es

§Address: Instituto de Matemática Interdisciplinar (IMI), Dept. of Algebra, Geometry and Topology, Complutense University of Madrid, Plaza de Ciencias 3, Madrid, 28040, Spain. E-mail: amelle@mat.ucm.es

1 Introduction

The Euler characteristic $\chi(\cdot)$ (defined as the alternating sum of cohomology groups with compact support) is an *additive* invariant of topological spaces (sufficiently nice, e. g., quasi-projective varieties). It can be considered as a homomorphism from the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ of quasi-projective varieties to the ring $\mathbb{Z}$ of integers. There exists a simple formula for the generating series of the Euler characteristics of the symmetric powers $S^nX = X^n/S_n$ of a variety X:

$$1 + \chi(X)t + \chi(S^2X)t^2 + \chi(S^3X)t^3 + \ldots = (1-t)^{-\chi(X)} \qquad (1)$$

(the Macdonald formula). (One can interpret this formula as the fact that the Euler characteristic is a λ-ring homomorphism between the rings $K_0(\mathrm{Var}_{\mathbb{C}})$ and $\mathbb{Z}$ endowed with natural λ-structures: see Section 3).

For a topological space X with an action of a finite group G, one has the notions of the orbifold Euler characteristic $\chi^{\mathrm{orb}}(X, G)$ (coming from physics) and of Euler characteristics $\chi^{(k)}(X, G)$ of higher orders ($\chi^{\mathrm{orb}}(X, G) = \chi^{(1)}(X, G)$). They can be considered as ($\mathbb{Z}$-valued) functions on the Grothendieck ring $K_0^G(\mathrm{Var}_{\mathbb{C}})$ of G-varieties. However, they are not ring homomorphisms from $K_0^G(\mathrm{Var}_{\mathbb{C}})$ to $\mathbb{Z}$.

In [7], there were defined generalized ("motivic") versions of the orbifold Euler characteristic and of Euler characteristics of higher orders with values in the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ of varieties extended by the rational powers of the class $\mathbb{L}$ of the complex affine line. They are defined for a non-singular (!) variety with an action of a finite group G. They are not defined as functions on a certain ring (say, on the Grothendieck ring of G-varieties). There was obtained a Macdonald type formula for the generating series of generalized higher order Euler characteristics of the Cartesian powers of a G-manifold with the actions of the wreath products G_n on them. It is formulated in terms of the so-called power structure over the ring $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^s]$. (A power structure over a ring R is a method to give sense to an expression of the form $(1 + a_1t + a_2t^2 + \ldots)^m$, where a_i and m are elements of the ring R: [5], see also Section 3.)

Here we define a Grothendieck ring $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ of varieties with finite groups actions. Elements of $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ are classes of varieties with actions of finite groups (different, in general) on subvarieties constituting partitions of them. The most important ingredient of the definition is the identification of the class of a variety with a group action with the class of the variety obtained by the induction operation (with an action of a bigger group).

We show that the orbifold Euler characteristic and the Euler characteristics of higher orders can be defined as homomorphisms from $K_0^{\rm fGr}({\rm Var}_{\mathbb{C}})$ to the ring $\mathbb{Z}$ of integers. We describe two natural λ-structures on the ring $K_0^{\rm fGr}({\rm Var}_{\mathbb{C}})$. These λ-structures define different power structures over this ring. We show that one of this power structures is not effective (see the definition in Section 3) and the other one is. We give a geometric description of the effective power structure. We define a Grothendieck ring $K_0^{\rm fGr}({\rm Vect}_{\mathbb{C}})$ of varieties with equivariant vector bundles and show that the generalized Euler characteristics of higher orders can be defined as homomorphisms from $K_0^{\rm fGr}({\rm Vect}_{\mathbb{C}})$ to $K_0({\rm Var}_{\mathbb{C}})[\mathbb{L}^s]$. We give an analogue of the formula from [7] for the generating series of the generalized higher order Euler characteristics of wreath products.

The authors are very thankful to the referee for a number of useful remarks/suggestions.

2 Orbifold Euler characteristics and their generalized versions

For a finite group G, let ${\rm Conj}\, G$ be the set of the conjugacy classes of elements of G, for an element $g \in G$ let $C_G(g) = \{h \in G : h^{-1}gh = g\}$ be the centralizer of g. For a G-space X and for a subgroup $H \subset G$, let $X^H = \{x \in X : gx = x \text{ for all } g \in H\}$ be the fixed point set of the subgroup H. The *orbifold Euler characteristic* $\chi^{orb}(X, G)$ of the G-space X is defined, e. g., in [1], [9]:

$$\chi^{orb}(X,G) = \frac{1}{|G|} \sum_{\substack{(g_0,g_1)\in G\times G:\\ g_0g_1=g_1g_0}} \chi(X^{\langle g_0,g_1\rangle}) = \sum_{[g]\in {\rm Conj}\, G} \chi(X^{\langle g\rangle}/C_G(g)) \in \mathbb{Z}\,, \quad (2)$$

where g is a representative of a class $[g]$, $\langle g\rangle$ and $\langle g_0, g_1\rangle$ are the subgroups of G generated by the corresponding elements.

The *higher order Euler characteristics* of (X, G) were defined in [1] and [3] by:

$$\chi^{(k)}(X,G) = \frac{1}{|G|} \sum_{\substack{\mathbf{g}\in G^{k+1}:\\ g_ig_j=g_jg_i}} \chi(X^{\langle \mathbf{g}\rangle}) = \sum_{[g]\in {\rm Conj}\, G} \chi^{(k-1)}(X^{\langle g\rangle}, C_G(g))\,, \quad (3)$$

where k is a positive integer (the order of the Euler characteristic), $\mathbf{g} = (g_0, g_1, \ldots, g_k)$, $\langle \mathbf{g} \rangle$ is the subgroup generated by $g_0, g_1, \ldots, g_k$, and (for the second, recurrent, definition) $\chi^{(0)}(X, G)$ is defined as the usual Euler characteristic $\chi(X/G)$ of the quotient. The orbifold Euler characteristic $\chi^{orb}(X, G)$ is the Euler characteristic $\chi^{(1)}(X, G)$ of order 1.

For a G-variety X, the cartesian power X^n carries an action of the wreath product $G_n = G^n \rtimes S_n$ generated by the natural action of the symmetric group S_n (permuting the factors) and by the natural (component-wise) action of the Cartesian power G^n. The pair (X_n, G_n) should be (or can be) considered as an analogue of the symmetric power for the pair (X, G).

For $k \geq 0$ one has the following Macdonald type formula (see [14, Theorem A])

$$1 + \sum_{n=1}^{\infty} \chi^{(k)}(X^n, G_n) \cdot t^n = \left(\prod_{r_1, \ldots, r_k \geq 1} (1 - t^{r_1 r_2 \cdots r_k})^{r_2 r_3^2 \cdots r_k^{k-1}} \right)^{-\chi^{(k)}(X,G)} . \quad (4)$$

When $k = 0$, one gets the standard Macdonald formula (Equation (1)) for the quotient X/G.

There is a (more or less) natural notion of the Grothendieck ring $K_0^G(\mathrm{Var}_{\mathbb{C}})$ of (complex quasi-projective) G-varieties such that the orbifold Euler characteristic $\chi^{\mathrm{orb}}(\cdot)$ and the Euler characteristics of higher orders $\chi^{(k)}(\cdot)$ are functions on it. *The Grothendieck ring $K_0^G(\mathrm{Var}_{\mathbb{C}})$ of complex quasi-projective G-varieties* is the Abelian group generated by the G-isomorphism classes $[X, G]$ of complex quasi-projective varieties X with G-actions modulo the relation: $[X, G] = [Y, G] + [X \setminus Y, G]$ for a Zariski closed G-invariant subvariety Y of X. The multiplication in $K_0^G(\mathrm{Var}_{\mathbb{C}})$ is defined by the Cartesian product with the diagonal G-action.

Remark. Usually, in the definition of the Grothendieck ring of complex quasi-projective G-varieties, one adds the following relation: if $E \to X$ is a G-equivariant vector bundle of rank n, then $[E] = \mathbb{L}^n \cdot [X, G]$, where $\mathbb{L}$ is the class of the complex affine line with the trivial G-action. We use the definition given, e. g., in [12]. In [2] this Grothendieck ring was also defined (alongside with the "traditional one") and was denoted by $K_0^{\prime, G}(\mathrm{Var}_{\mathbb{C}})$.

One can easily understand that $\chi^{(k)}$ are additive functions on $K_0^G(\mathrm{Var}_{\mathbb{C}})$, however, they are not multiplicative. This can be seen, e. g., from the fact that, for an Abelian G, $\chi^{(k)}(1) = |G|^k$ ($1 \in K_0^G(\mathrm{Var}_{\mathbb{C}})$ is the class of the one-point variety with the only G-action on it). Thus they are not ring homomorphisms from $K_0^G(\mathrm{Var}_{\mathbb{C}})$ to $\mathbb{Z}$. In what follows we, in particular, define a Grothendieck ring (so-called Grothendieck ring of varieties with finite groups actions) such that χ^{orb} and $\chi^{(k)}$ are ring homomorphisms from it to $\mathbb{Z}$.

Let $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^s]_{s\in\mathbb{Q}}$ (or $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^s]$ for short) be the modification of the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ of quasi-projective varieties obtained by adding all rational powers of the class $\mathbb{L}$ of the complex affine line. The elements of $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^s]$ are in a bijective correspondence with the finite sums of the form $\sum_i c_i\mathbb{L}^{r_i}$, where c_i are elements of the localization $K_0(\mathrm{Var}_{\mathbb{C}})_{(\mathbb{L})}$ of the ring $K_0(\mathrm{Var}_{\mathbb{C}})$ by the class $\mathbb{L}$, r_i are different rational numbers inbetween 0 and 1: $0 \leq r_i < 1$. Thus the ring $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^s]$ contains the localization $K_0(\mathrm{Var}_{\mathbb{C}})_{(\mathbb{L})}$. It was shown (L. Borisov) that $\mathbb{L}$ is a zero divisor in $K_0(\mathrm{Var}_{\mathbb{C}})$ and therefore the natural map $K_0(\mathrm{Var}_{\mathbb{C}}) \to K_0(\mathrm{Var}_{\mathbb{C}})_{(\mathbb{L})}$ is not injective. Therefore the natural map $K_0(\mathrm{Var}_{\mathbb{C}}) \to K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^s]$ is not injective as well.

Let X be a non-singular complex quasi-projective variety of dimension d with an (algebraic) action of the group G. To define the *higher order generalized ("motivic") Euler characteristics* of the pair (X, G), one has to use the so called age (or fermion shift) $\mathrm{age}_x(g)$ of an element $g \in G$ at a fixed point x of g defined in [16], [10]. The element g acts on the tangent space T_xX as an automorphism of finite order. This action on T_xX can be represented by a diagonal matrix $\mathrm{diag}(\exp(2\pi i\theta_1), \ldots, \exp(2\pi i\theta_d))$ with $0 \leq \theta_j < 1$ for $j = 1, 2, \ldots, d$ (θ_j are rational numbers). The *age* of the element g at the point x is defined by $\mathrm{age}_x(g) = \sum_{j=1}^d \theta_j \in \mathbb{Q}_{\geq 0}$. For a rational number q, let $X_q^{\langle g\rangle}$ be the set of points $x \in X^{\langle g\rangle}$ such that $\mathrm{age}_x(g) = q$.

For a rational number $\varphi_1 \in \mathbb{Q}$, the generalized orbifold Euler characteristic of weight φ_1 of the pair (X, G) is defined by

$$[X, G]_{\varphi_1} := \sum_{[g]\in\mathrm{Conj}\, G} \sum_{q\in\mathbb{Q}} [X_q^{\langle g\rangle}/C_G(g)] \cdot \mathbb{L}^{\varphi_1 q} \in K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^s]\,. \qquad (5)$$

(See an explanation for introducing the weight φ_1 in [7]. Generalized orbifold Euler characteristic is meaningful for at least two values of the weight φ_1: 0 and 1.) Equation (5) is a reformulation of the definition from [15] given in terms of the orbifold Hodge–Deligne polynomial.

The generalized Euler characteristics of higher orders are defined recursively by an equation which is a sort of "a motivic version" of the second equality in (3) taking into account ages of elements: see [7] for the details or

Section 7 for a somewhat more general definition. Since all of them are defined only for smooth varieties, they are not functions on a certain ring (say, on a Grothendieck ring of G-varieties). In Section 7 we define a Grothendieck ring (the Grothendieck ring of varieties with equivariant vector bundles) such that (appropriately defined) generalized Euler characteristics of higher orders are ring homomorphisms from this Grothendieck ring to $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^s]$.

A Macdonald type formula for the generalized Euler characteristics of higher orders (a "motivic" version of (4)) is written in terms of the power structure over the ring $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^s]$).

3 λ-structures and power structures

A Macdonald type equation for an invariant taking values in a certain ring can be formulated in terms of a power structure over the ring of values of the invariant. Let R be a commutative ring with unity. A power structure over the ring R is a method to give sense to expressions of the form $(A(t))^m$, where $A(t) = 1 + a_1 t + a_2 t^2 + \ldots$ is a power series with the coefficients a_i from R and m is an element of R.

Definition 1 *[5] A* power structure *over the ring R is a map*

$$(1 + tR[[t]]) \times R \to 1 + tR[[t]] \qquad ((A(t), m) \mapsto (A(t))^m)$$

possessing the properties of the exponential function, namely:

(1) $(A(t))^0 = 1$;

(2) $(A(t))^1 = A(t)$;

(3) $(A(t) \cdot B(t))^m = (A(t))^m \cdot (B(t))^m$;

(4) $(A(t))^{m+n} = (A(t))^m \cdot (A(t))^n$;

(5) $(A(t))^{mn} = ((A(t))^n)^m$;

(6) $(1 + a_1 t + \ldots)^m = 1 + m a_1 t + \ldots$;

(7) $\left(A(t^k)\right)^m = (A(t))^m\big|_{t \mapsto t^k}$.

Remark. In another (more short and more formal) way this definition can be formulated as follows: see [13] for details. A power structure over R is an R-module structure on the additive group of the ring of (big) Witt vectors $W(R) := 1 + tR[[t]]$ such that the natural map $W(R) \to R$, $\sum a_i t^i \mapsto a_1$, is a morphism of R-modules and the R-module structure commutes with the Verschiebung maps $V_n : W(R) \to W(R)$, $A(t) \mapsto A(t^n)$.

Let $\mathfrak{m}$ be the ideal $tR[[t]]$ in the ring $R[[t]]$. A power structure over the ring R is *finitely determined* if, for any $k \geq 1$, the fact that $A(t) \in 1 + \mathfrak{m}^k$ implies that $(A(t))^m \in 1 + \mathfrak{m}^k$, i. e., if it respects the natural filtration $F^n W(R) = 1 + t^n R[[t]]$ on the ring of Witt vectors. (Because of the Verschiebung property, this is equivalent to being continuous with respect to the filtration $F^n W(R)$.)

The natural power structure over the ring $\mathbb{Z}$ of integers is defined by the standard formula for a power of a series:

$$\begin{aligned}(1 + a_1 t + a_2 t^2 + \ldots)^m &= \\ = \ 1 + \sum_{k=1}^{\infty} &\left(\sum_{\{k_i\}:\sum ik_i = k} \frac{m(m-1)\cdots(m - \sum_i k_i + 1) \cdot \prod_i a_i^{k_i}}{\prod_i k_i!} \right) \cdot t^k.\end{aligned}$$

A power structure over the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ of complex quasi-projective varieties was defined in [5] by the formula

$$\begin{aligned}(1 + [A_1]t + [A_2]t^2 + \ldots)^{[M]} &= \\ = \ 1 + \sum_{k=1}^{\infty} &\left(\sum_{\{k_i\}:\sum ik_i = k} \left[\left((M^{\sum_i k_i} \setminus \Delta) \times \prod_i A_i^{k_i} \right) \Big/ \prod_i S_{k_i} \right] \right) \cdot t^k, \quad (6)\end{aligned}$$

where A_i, $i = 1, 2, \ldots$, and M are quasi-projective varieties ($[A_i]$ and $[M]$ are their classes in the ring $K_0(\mathrm{Var}_{\mathbb{C}})$), Δ is the large diagonal in $M^{\sum_i k_i}$, that is, the set of (ordered) $(\sum_i k_i)$-tuples of points of M with at least two coinciding ones, the group S_{k_i} of permutations on k_i elements acts by the simultaneous permutations on the components of the corresponding factor M^{k_i} in $M^{\sum_i k_i} = \prod_i M^{k_i}$ and on the components of $A_i^{k_i}$.

Except the Grothendieck ring of complex quasi-projective varieties one can consider *the Grothendieck semiring* $S_0(\mathrm{Var}_{\mathbb{C}})$. It is defined in the same way as $K_0(\mathrm{Var}_{\mathbb{C}})$ with the word *group* substituted by the word *semigroup*. Two complex quasi-projective varieties X and Y represent the same element of the semiring $S_0(\mathrm{Var}_{\mathbb{C}})$ if and only if they are piece-wise isomorphic, that is if there exist decompositions $X = \bigsqcup_{i=1}^{s} X_i$ and $Y = \bigsqcup_{i=1}^{s} Y_i$ into Zariski locally closed subsets such that X_i and Y_i are isomorphic for $i = 1, \ldots, s$. There is a natural map (a semiring homomorphism) from $S_0(\mathrm{Var}_{\mathbb{C}})$ to $K_0(\mathrm{Var}_{\mathbb{C}})$. (It is not known whether or not this map is injective.)

A power structure over the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ is called *effective* if the fact that all the coefficients a_i of the series $A(t)$ and the exponent m are represented by classes of complex quasi-projective varieties (i. e., belong to the image of the map $S_0(\mathrm{Var}_{\mathbb{C}}) \to K_0(\mathrm{Var}_{\mathbb{C}})$) implies that all the coefficients of the series $(A(t))^m$ are also represented by such classes. (Roughly speaking this means that the power structure can be defined over the Grothendieck semiring $S_0(\mathrm{Var}_{\mathbb{C}})$.) The same concept can be considered for Grothendieck rings of complex quasi-projective varieties with additional structures. The effectiveness of the described power structure over the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ is clear from Equation (6).

The notion of a power structure over a ring is closely related with the notion of a λ-ring structure. A λ-ring structure (or a pre-λ-ring structure in a certain terminology, see, e. g., [11]) is an additive-to-multiplicative homomorphism $R \to 1 + tR[[t]], a \mapsto \lambda_a(t)$ $(\lambda_{a+b}(t) = \lambda_a(t) \cdot \lambda_b(t))$ such that $\lambda_a(t) = 1 + at + \ldots$ A λ-ring structure on a ring defines a finitely determined power structure over it in the following way. Any series $A(t) \in 1 + tR[[t]]$ can be in a unique way represented as $\prod_{i=1}^{\infty} \lambda_{b_i}(t^i)$, for some $b_i \in R$. Then one defines $(A(t))^m := \prod_{i=1}^{\infty} \lambda_{mb_i}(t^i)$. This gives a surjective map from the set of λ-structures to the set of finitely determined power structures. The preimage of a power structure consists of all λ-structures given by the formula $\lambda_a(t) = (\lambda_1(t))^a$ with an arbitrary $\lambda_1(t) = 1 + t + \sum_{i=2}^{\infty} a_i t^i$.

One can show that the power structure (6) is defined by the λ-ring structure on the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ given by the Kapranov zeta function

$$\zeta_{[M]}(t) := 1 + \sum_{k=1}^{\infty} [S^k M] \cdot t^k,$$

where $S^k M = M^k / S_k$ is the kth symmetric power of the variety M. This follows from the following equation

$$\zeta_{[M]}(t) = (1-t)^{-[M]} = (1 + t + t^2 + \ldots)^{[M]}.$$

The power structure (6) over the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ is also defined by the following λ-ring structure on it. Let $B^k M = (M^k \setminus \Delta)/S_k$ be the configuration space of k-point subsets of M (Δ is the big diagonal in M^k consisting on k-tuples of points of M with at least two coinciding ones). The series

$$\lambda_{[M]}(t) := 1 + \sum_{k=1}^{\infty} [B_k M] \cdot t^k$$

gives a λ-ring structure on the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ which defines the same power structure (6) over it. In terms of the power structure one has $\lambda_{[M]}(t) = (1+t)^{[M]}$.

Alongside with a λ-structure on a ring (defined by a series $\lambda_a(t)$) one has the so-called opposite λ-structure defined by the series $\lambda'_a(t) := (\lambda_a(-t))^{-1}$. For example, on a ring with a power structure, the λ-structure opposite to $(1+t)^a$ is the result of the substitution $t \mapsto -t$ in the series $(1+t)^{-a}$, which differs from $(1-t)^{-a}$, because, in general, a power structure does not commute with the substitution $t \mapsto -t$. In particular this is the case for the power structure over the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ described above. Two λ-structures define the same finitely determined power structure if and only if this holds for the opposite λ-structures. One can show that the power structure over the ring $K_0(\mathrm{Var}_{\mathbb{C}})$ defined by the λ-structures opposite to $\zeta_{[M]}(t)$ and $\lambda_{[M]}(t)$ is not effective: [6].

4 Grothendieck ring of varieties with finite groups actions

Definition 2 *A* quasi-projective variety $\mathcal{X}$ with a finite groups action *is a variety represented as the disjoint union of (locally closed) subvarieties X_i, $i = 1, \ldots, s$, with (left) actions of finite groups G_i on them.*

This means that $\mathcal{X}$ can be decomposed into parts with actions of (different, in general) finite groups on them. We shall write $\mathcal{X} = \bigsqcup_{i=1}^{s} (X_i, G_i)$. A partition of $\mathcal{X}$ means partitions of its components X_i as G_i-varieties. In particular a G-variety (G is a finite group) is a variety with a finite groups action. For short we will call varieties of this sort *varieties with pure actions.*

Definition 3 *Two varieties with pure actions (Z, G) and (Z', G') are* isomorphic *if there exist isomorphisms $\varphi : G \to G'$ (of finite groups) and $\psi : Z \to Z'$*

(of quasi-projective varieties) such that ψ is equivariant relative to φ, that is, $\psi(gx) = \varphi(g)\psi(x)$ for $x \in Z$, $g \in G$.

Definition 4 *Two varieties with finite groups actions $\mathcal{X}$ and $\mathcal{Y}$ are called* equivalent *if there exist partitions $\mathcal{X} = \coprod_{i=1}^{N} (X_{(i)}, G_{(i)})$ and $\mathcal{Y} = \coprod_{i=1}^{N} (Y_{(i)}, G'_{(i)})$ such that $(X_{(i)}, G_{(i)})$ is isomorphic to $(Y_{(i)}, G'_{(i)})$ for $i = 1, \ldots, N$.*

There exist a somewhat natural notion of the Grothendieck ring of varieties with a finite groups actions: see below. However, it does not really correspond to our aim. Because of that here we will use the name pre-Grothendieck ring.

Definition 5 The pre-Grothendieck ring of quasi-projective varieties with finite groups actions *is the Abelian group $\widetilde{K}_0^{\rm fGr}({\rm Var}_{\mathbb{C}})$ generated by the classes $[\mathcal{X}]$ of (quasi-projective) varieties with finite groups actions modulo the following relations:*

(1) if quasi-projective varieties with finite groups actions $\mathcal{X}$ and $\mathcal{Y}$ are equivalent, then $[\mathcal{X}] = [\mathcal{Y}]$;

(2) if $\mathcal{Y}$ is a Zariski closed subvariety invariant with respect to the groups actions of $\mathcal{X}$, then $[\mathcal{X}] = [\mathcal{Y}] + [\mathcal{X} \setminus \mathcal{Y}]$.

The multiplication in $\widetilde{K}_0^{\rm fGr}({\rm Var}_{\mathbb{C}})$ is defined by the Cartesian product of varieties with the natural finite groups action on it.

Remark. One can see that, in this definition, one can consider the group generated by the classes of varieties with pure actions, obtaining the same ring.

In particular, for varieties with pure actions, one has

$$[(Z_1, G_1)] \cdot [(Z_2, G_2)] = [(Z_1 \times Z_2, G_1 \times G_2)]$$

with the natural (diagonal) action of $G_1 \times G_2$. The unit element in $\widetilde{K}_0^{\rm fGr}({\rm Var}_{\mathbb{C}})$ is $1 = [({\rm Spec}(\,\mathbb{C}\,), (e))]$, the class of the one-point variety with the action of the group with one element. The ring $\widetilde{K}_0^{\rm fGr}({\rm Var}_{\mathbb{C}})$ (as an Abelian group) is generated by the classes $[(Z, G)]$ of varieties with pure actions.

It is easy to see that the orbifold Euler characteristic $\chi^{\rm orb}$ and the Euler characteristics $\chi^{(k)}$ of higher order can be defined as functions on the Grothendieck ring $\widetilde{K}_0^{\rm fGr}({\rm Var}_{\mathbb{C}})$. Moreover, the following statement implies that they are (ring) homomorphisms from $\widetilde{K}_0^{\rm fGr}({\rm Var}_{\mathbb{C}})$ to $\mathbb{Z}$.

Proposition 1 *(see, e. g., [14, Proposition 2–1]) For two varieties with pure actions* (Z, G) *and* (Z', G') *one has*

$$\chi^{(k)}(Z \times Z', G \times G') = \chi^{(k)}(Z, G) \cdot \chi^{(k)}(Z', G') \, .$$

Nevertheless it is not clear whether the ring $\widetilde{K}_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ can be endowed with a (natural) λ-structure and therefore it is not possible to try to consider χ^{orb} and $\chi^{(k)}$ as λ-ring homomorphisms. To make this possible we have to introduce a sort of a reduction of the ring $\widetilde{K}_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$.

Let Z be a G-variety and let H be a finite group such that $G \subset H$. There is a natural *induction operation* which produces an H-variety. Consider the following equivalence relation on $H \times Z$: $(h_1, x_1) \sim (h_2, x_2)$ $(x_i \in Z,\, h_i \in H)$ if and only if there exists $g \in G$ such that $h_2 = h_1 g^{-1}$, $x_2 = g x_1$. The quotient $\mathrm{ind}_G^H Z := (H \times Z)/\sim$ carries a natural H-action. The map $x \mapsto (1, x)$ is an embedding of Z into $\mathrm{ind}_G^H Z$ (as a G-variety).

Definition 6 The Grothendieck ring of quasi-projective varieties with finite groups actions *is the Abelian group* $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ *generated by the classes* $[\mathcal{X}]$ *of (quasi-projective) varieties with finite groups actions modulo the relations:*

(1) and (2) from Definition 5.

(3) if (Z, G) *is a* G*-variety and* G *is a subgroup of a finite group* H*, then*

$$\left[(\mathrm{ind}_G^H Z, H)\right] = \left[(Z, G)\right] \, .$$

The multiplication in $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ *is defined by the Cartesian product of varieties with the natural finite groups action on it.*

Remarks. **1.** There exist two natural ring homomorphisms $i : K_0(\mathrm{Var}_{\mathbb{C}}) \to K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ sending $[Z]$ to $[(Z, \{e\})]$ and $p : K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}}) \to K_0(\mathrm{Var}_{\mathbb{C}})$ sending $[(Z, G)]$ to $[Z/G]$. One has $p \circ i = \mathrm{id}$.
2. In [4], E. Getzler and R. Pandharipande considered the Grothendieck group of varieties with so-called $\mathcal{S}$-actions, where $\mathcal{S} = \bigsqcup_{n=0}^{\infty} S_n$ (S_n is the group of permutations of n elements):

$$K_0(\mathrm{Var}_{\mathbb{C}}, \mathcal{S}) = \prod_{n=0}^{\infty} K_0^{S_n}(\mathrm{Var}_{\mathbb{C}})$$

with the multiplication $\boxtimes$ induced by

$$[(X, S_m)] \boxtimes [(Y, S_n)] = [(\mathrm{ind}_{S_m \times S_n}^{S_{m+n}}(X \times Y), S_{m+n})].$$

One can see that modulo relation (3) this multiplication coincides with the Cartesian one. Therefore there exists a natural homomorphism

$$K_0(\mathrm{Var}_{\mathbb{C}}, \mathcal{S}) \to K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}}).$$

For a finite group H, let $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}}{}^{([H])})$ be the Grothendieck group (not ring!) generated by the classes of quasi-projective varieties with finite groups actions such that the isotropy group of each point is isomorphic to H modulo the same relations (1)–(3) as in Definition 6 of $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$.

Proposition 2 *As an Abelian group, $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ is the direct sum over the isomorphism classes $[H]$ of finite groups of the groups $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}}{}^{([H])})$.*

Proof. One has a natural (group) homomorphism $j_{[H]} : K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}}{}^{([H])}) \to K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$. Let $\mathcal{X} = \bigsqcup_{i=1}^{s}(X_i, G_i)$ be a variety with a finite groups action. Let $X_i^{([H])}$ be the set of points $x \in X_i$ such that the isotropy group G_{ix} of the G_i-action is isomorphic to H. The subset $X_i^{([H])}$ is locally Zariski closed in X_i. Indeed,

$$X_i^{([H])} = \bigsqcup_{K \subset G_i : K \in [H]} X_i^K \setminus \bigsqcup_{K \subset G_i | \exists K' \in [H] : K' \subsetneq K} X_i^K ,$$

where $X_i^K = \{x \in X_i : \forall g \in K, gx = x\}$ (the set of fixed points of a subgroup K) is Zariski closed in X_i. Let $p_{[H]}$ be the homomorphism $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}}) \to K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}}{}^{([H])})$ defined by

$$p_{[H]}([\mathcal{X}]) = \sum_{i=1}^{s} [(X_i^{([H])}, G_i)].$$

One can see that the homomorphism $p_{[H]}$ is well-defined, $p_{[H]} \circ j_{[H]} = \mathrm{id}$, $p_{[H]} \circ j_{[H']} = 0$ for $H' \ncong H$, and $\sum_{[H]} j_{[H]} \circ p_{[H]} = \mathrm{id}$. This proves the statement. $\square$

In the same way $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ is the direct sum over $n = 1, 2, \ldots$ of the Grothendieck groups of quasi-projective varieties with finite groups actions with the isotropy subgroups of points of order n.

Substituting the words *"group generated by the classes"* by the words *"semigroup generated by the classes"* in Definition 6, one gets the notion of *the Grothendieck semiring $S_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ of quasi-projective varieties with finite groups actions*. There is a natural (semiring) homomorphism $S_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}}) \to K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$. This notion permits to introduce the notion of effectiveness of a power structure defined over the ring $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$.

As in Section 2, let $\mathrm{Conj}\, G$ be the set of conjugacy classes of elements of a group G. The conjugacy class of an element $g \in G$ is denoted by $[g]$. If there are several groups containing g, we will indicate the group using the notation $[g]_G$. The centralizer of an element $g \in G$ is denoted by $C_G(g)$.

Let Z be a G-variety, let G be a subgroup of a finite group H, and let $\mathrm{ind}_G^H Z$ be the induced H-variety. If an element $h \in H$ has a non-empty fixed point set $\left(\mathrm{ind}_G^H Z\right)^{\langle h\rangle}$ (say, $(h_0, x_0) \in \left(\mathrm{ind}_G^H Z\right)^{\langle h\rangle}$), then there exists $g \in G$ such that $(hh_0g^{-1}, gx_0) = (h_0, x_0)$, i.e., $h_0^{-1}hh_0 = g$, $gx_0 = x_0$. This means that $g \in [h]_H$. Since in the definitions of the orbifold Euler characteristic and of the higher order Euler characteristics the summation runs over representatives of the conjugacy classes of elements of the group, we can assume that applying them to the H-variety $\mathrm{ind}_G^H Z$ we always take a representative of a conjugacy class of elements of H belonging to the subgroup G.

Lemma 1 *Let G, H, and Z be as above. Then, for $g \in G$, the spaces with finite groups actions $\left(\left(\mathrm{ind}_G^H Z\right)^{\langle g\rangle}, C_H(g)\right)$ and $\bigsqcup\limits_{\substack{[g']\in \mathrm{Conj}\, G:\\ [g']_H=[g]_H}} \left(\mathrm{ind}_{C_G(g')}^{C_H(g')} Z^{\langle g'\rangle}, C_H(g')\right)$ are equivalent.*

Proof. Let $(h_0, x_0) \in \mathrm{ind}_G^H Z$ be a fixed point of the action of g : this means that $(h_0, x_0) \in \left(\mathrm{ind}_G^H Z\right)^{\langle g\rangle}$. As above, there exist $g' \in G$ such that $(gh_0(g')^{-1}, g'x_0) = (h_0, x_0)$, i.e. $h_0^{-1}gh_0 = g'$, $g'x_0 = x_0$. In particular $g' \in [g]_H$. In each conjugacy class $[g'] \in \mathrm{Conj}\, G$: such that $[g']_H = [g]_H$, let us choose a representative g'.

Let $(\mathrm{ind}_G^H Z)_{[g']_G}$ be the set of points of $\left(\mathrm{ind}_G^H Z\right)^{\langle g\rangle}$ represented by pairs (h, x) with $h^{-1}gh \in [g']_G$. For $h \in H$, let $\{h\}$ be the class of h in H/G, let $Z_{\{h\}}$ be the subvariety of $\mathrm{ind}_G^H Z$ consisting of points of the form (h, x), $x \in Z$. (This subvariety depends only on the class $\{h\}$ of h.) Then $(\mathrm{ind}_G^H Z)_{[g']_G}$ is the union of the subvarieties $Z_{\{h\}}$ with $h^{-1}gh \in [g']_G$. One has $\left(\mathrm{ind}_G^H Z\right)^{\langle g\rangle} \subset \bigsqcup\limits_{[g']\in \mathrm{Conj}\, G:} (\mathrm{ind}_G^H Z)_{[g']_G}$. For each chosen g' ($g' \in [g']_G$, $[g']H = [g]_H$), let $h(g')$

be an element of H such that $(h(g')^{-1}gh(g) = g'$. (We can choose g itself as a representative of the conjugacy class $[g]$ and $h(g) = e$.) The intersection of $(\mathrm{ind}_G^H Z)_{[g']_G}$ with $Z_{\{h(g')\}}$ is, in a natural way, isomorphic to $Z^{\langle g'\rangle}$. The centralizer $C_H(g')$ acts on $\left(\mathrm{ind}_G^H Z\right)^{\langle g\rangle} \cap (\mathrm{ind}_G^H Z)_{[g']_G}$, the later is the union of the orbits of points from $\left(\mathrm{ind}_G^H Z\right)^{\langle g\rangle} \cap Z_{\{h(g')\}}$. Moreover the subgroup of $C_H(g')$ preserving $\left(\mathrm{ind}_G^H Z\right)^{\langle g\rangle} \cap Z_{\{h(g')\}}$ coincides with $C_G(g')$. Therefore $\left(\mathrm{ind}_G^H Z\right)^{\langle g\rangle} \cap (\mathrm{ind}_G^H Z)_{[g']_G}$ and $\mathrm{ind}_{C_G(g')}^{C_H(g'} Z^{\langle g'\rangle}$ are isomorphic as $C_H(g')$-varieties. Since

$$\left(\mathrm{ind}_G^H Z\right)^{\langle g\rangle} = \bigsqcup_{\substack{[g']\in \mathrm{Conj}\, G\\ [g']H=[g]_H}} \left(\mathrm{ind}_G^H Z\right)^{\langle g\rangle} \cap (\mathrm{ind}_G^H Z)_{[g']_G},$$

one has the statement. □

It is easy to see that $\mathrm{ind}_G^H Z/H = Z/G$ and therefore $\chi(\mathrm{ind}_G^H Z/H) = \chi(Z/G)$. This means that

$$\chi^{(0)}(\mathrm{ind}_G^H Z, H) = \chi^{(0)}(Z, G)\,. \tag{7}$$

Theorem 1 *Let Z be a G-variety, $G \subset H$. Then for $k \geq 0$ one has*

$$\chi^{(k)}(\mathrm{ind}_G^H Z, H) = \chi^{(k)}(Z, G)\,. \tag{8}$$

Proof. Equation (7) gives the statement for $k = 0$. Assume that Equation (8) is proven for the values of k smaller than the one under consideration. One has

$$\chi^{(k)}(\mathrm{ind}_G^H Z, H) = \sum_{[h]\in \mathrm{Conj}\, H} \chi^{(k-1)}\left((\mathrm{ind}_G^H Z)^{\langle h\rangle}, C_H(h)\right)\,.$$

It was shown that the fixed point set $(\mathrm{ind}_G^H Z)^{\langle h\rangle}$ is not empty only if there exists $g \in G$ such that $[h]_H = [g]_H$. Lemma 1 implies

$$\chi^{(k)}(\mathrm{ind}_G^H Z, H) = \sum_{[g]\in \mathrm{Conj}\, G} \chi^{(k-1)}\left((\mathrm{ind}_{C_G(g)}^{C_H(g)} Z)^{\langle g\rangle}, C_H(g)\right)\,.$$

The induction gives

$$\chi^{(k)}(\mathrm{ind}_G^H Z, H) = \sum_{[g]\in \mathrm{Conj}\, G} \chi^{(k-1)}\left(Z^{\langle g\rangle}, C_G(g)\right) = \chi^{(k)}(Z, G).$$

$\square$

Together with Proposition 1 this gives the following statement.

Corollary. The maps $\chi^{(k)} : K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}}) \to \mathbb{Z}$ are ring homomorphisms.

Remark. One can see that Lemma 1 implies that there is a well-defined ring homomorphism

$$\alpha : K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}}) \to K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$$

sending $[(X,G)]$ to $\sum\limits_{[g]\in \mathrm{Conj}\, G} [(X^{\langle g\rangle}, C_G(g))]$ and the homomorphism $\chi^{(k)}$ is equal to $\chi \circ p \circ \alpha^k$.

5 λ-structures on $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$

Let (Z,G) be a complex quasi-projective variety with a pure action (of a finite group G). The Cartesian power Z^n of the variety Z is endowed with the natural actions of the group G^n (acting component-wise) and of the group S_n (acting by permutations of the components) and therefore with the action of their semidirect product $G^n \rtimes S_n = G_n$: the wreath product.

Definition 7 *The Kapranov zeta function of* (Z,G) *is*

$$\zeta_{(Z,G)}(t) := 1 + \sum_{n=1}^{\infty} [(Z^n, G_n)]\, t^n \in 1 + tK_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})[[t]].$$

The fact that the Kapranov zeta function is well-defined on $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ follows form the following statement.

Proposition 3 *Let* (Z,G) *be a* G*-variety and let* $G \subset H$*. Then*

$$\zeta_{(Z,G)}(t) = \zeta_{(\mathrm{ind}_G^H Z, H)}(t).$$

Proof. The coefficient of t^n in $\zeta_{(\mathrm{ind}_G^H Z,H)}(t)$ is represented by $(\mathrm{ind}_G^H Z)^n$ with the corresponding H_n-action. One obviously has $(\mathrm{ind}_G^H Z)^n = \mathrm{ind}_{G^n}^{H^n} Z^n$ with the corresponding action of H_n and therefore $(\mathrm{ind}_G^H Z)^n = \mathrm{ind}_{G_n}^{H_n} Z^n$ with the corresponding action of H_n. The relation (3) in the Definition of $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ gives $[(\mathrm{ind}_{G_n}^{H_n} Z^n, H_n)] = [(Z^n, G_n)]$ what is the coefficient of t^n in $\zeta_{(Z,G)}(t)$. $\square$

The Kapranov zeta function possesses the following multiplicativity property.

Proposition 4 *Let (Z_1, G) and (Z_2, G) be quasi-projective varieties with actions of a finite group G. Then one has*

$$\zeta_{(Z_1 \sqcup Z_2, G)}(t) = \zeta_{(Z_1,G)}(t) \cdot \zeta_{(Z_2,G)}(t).$$

Proof. The coefficient of t^n in $\zeta_{(Z_1 \cup Z_2, G)}(t)$ is represented by variety $(Z_1 \cup Z_2)^n$ with the corresponding G_n action. One can see that

$$((Z_1 \cup Z_2)^n, G_n) = \bigsqcup_{k=0}^{n} \left(\mathrm{ind}_{G_k \times G_{n-k}}^{G_n} (Z_1^k \times Z_2^{n-k}), G_n \right).$$

The relation (3) in Definition 6 means that

$$\begin{aligned} [((Z_1 \cup Z_2)^n, G_n)] &= \sum_{k=0}^{n} [\big(Z_1^k \times Z_2^{n-k}, G_k \times G_{n-k}\big)] \\ &= \sum_{k=0}^{n} [(Z_1^k, G_k)][(Z_2^{n-k}, G_{n-k})]\,. \end{aligned} \tag{9}$$

The right hand side of Equation (9) is just the coefficient of t^n in $\zeta_{(Z_1,G)}(t) \cdot \zeta_{(Z_2,G)}(t)$. $\square$

Propositions 3 and 4 imply the following statement.

Corollary. The Kapranov zeta function $\zeta_\bullet(t)$ defines a λ-structure on the ring $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$.

Remark. In the terms of Definition 2 the Kapranov zeta function of a variety $\mathcal{X} = \bigsqcup_{i=1}^{s} (X_i, G_i)$ with a finite groups action is

$$\zeta_{\mathcal{X}}(t) = \prod_{i=1}^{s} \zeta_{(X_i,G_i)}(t) \in 1 + tK_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})[[t]].$$

For a G-variety Z, let $\Delta_G \subset Z^n$ (the *big G-diagonal*) be the set of n-tuples $(x_1, \ldots, x_n) \in Z^n$ with at least of two of x_i from the same G-orbit. One has a natural action of the wreath product G_n on $Z^n \setminus \Delta_G$ (inherited from the action on Z^n).

Definition 8 *Let the series $\lambda_{(Z,G)}(t) \in 1 + tK_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})[[t]]$ be defined by*

$$\lambda_{(Z,G)}(t) := 1 + \sum_{n=1}^{\infty} [(Z^n \setminus \Delta_G, G_n)]\, t^n \in 1 + tK_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})[[t]].$$

Just as above the facts that the series $\lambda_\bullet(t)$ is well-defined and defines a λ-structure on the ring $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ follows from the following statements.

Proposition 5 *Let (Z_1, G) and (Z_2, G) be quasi-projective varieties with actions of a finite group G. Then one has*

$$\lambda_{(Z_1 \cup Z_2, G)}(t) = \lambda_{(Z_1,G)}(t) \cdot \lambda_{(Z_2,G)}(t).$$

Let (Z, G) be a G-variety and let $G \subset H$. Then

$$\lambda_{(Z,G)}(t) = \lambda_{(\mathrm{ind}_G^H Z, H)}(t).$$

Proof. The coefficient of t^n in $\lambda_{(Z_1 \cup Z_2, G)}(t)$ is represented by variety $(Z_1 \cup Z_2)^n \setminus \Delta_G$ with the corresponding G_n action. One has

$$((Z_1 \cup Z_2)^n \setminus \Delta_G, G_n) = \coprod_{k=0}^{n} \left(\mathrm{ind}_{G_k \times G_{n-k}}^{G_n} (Z_1^k \setminus \Delta_G) \times (Z_2^{n-k} \setminus \Delta_G), G_n \right).$$

The relation (3) in the Definition 6 of $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ gives

$$\begin{aligned} [((Z_1 \cup Z_2)^n \setminus \Delta_G, G_n)] &= \sum_{k=0}^{n} [(Z_1^k \setminus \Delta_G) \times (Z_2^{n-k} \setminus \Delta_G), G_k \times G_{n-k})] \\ &= \sum_{k=0}^{n} [(Z_1^k \setminus \Delta_G, G_k)][(Z_2^{n-k} \setminus \Delta_G, G_{n-k})]. \end{aligned}$$

The right hand side is the coefficient of t^n in $\lambda_{(Z_1,G)}(t) \cdot \lambda_{(Z_2,G)}(t)$.

The arguments for the second part of the Proposition are literally the same as for the Kapranov zeta function in Proposition 3. □

6 Power structures over $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$

The λ-structures on $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ introduced above define power structures over the ring. In all cases up to now (say in $K_0(\mathrm{Var}_{\mathbb{C}})$) the power structures defined by the analogues of the series $\zeta_\bullet(t)$ and $\lambda_\bullet(t)$ are the same. This is not the case here.

Proposition 6 *The power structures over the ring $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ defined by the series $\zeta_\bullet(t)$ and $\lambda_\bullet(t)$ differ from each other.*

Proof. In terms of the corresponding power structures (which we denote by $(A(t))_\zeta^m$ and $(A(t))_\lambda^m$ respectively) one has

$$\zeta_{(Z,G)}(t) = (\zeta_{(\mathrm{Spec}(\mathbb{C}),(e))}(t))_\zeta^{[(Z,G)]} = (1 + \sum_{n=1}^{\infty} [(\mathrm{Spec}(\mathbb{C}), S_n)]\, t^n)_\zeta^{[(Z,G)]}\,,$$

$$\lambda_{(Z,G)}(t) = (\lambda_{(\mathrm{Spec}(\mathbb{C}),(e))}(t))_\lambda^{[(Z,G)]} = (1 + [(\mathrm{Spec}(\mathbb{C}), S_1)]\, t)_\lambda^{[(Z,G)]}\,.$$

We shall show that these two series are different.

Let us compute the first terms of the series $(1 + [(\mathrm{Spec}(\mathbb{C}), S_1)]\, t)_\zeta^{[(Z,G)]}$. We have

$$\begin{aligned} 1 + [(\mathrm{Spec}(\mathbb{C}), S_1)]\, t &= (1 + [(\mathrm{Spec}(\mathbb{C}), S_1)]\, t + [(\mathrm{Spec}(\mathbb{C}), S_2)]\, t^2 + \ldots) \times \\ \times (1 - [(\mathrm{Spec}(\mathbb{C}), S_2)]\, t^2 + \ldots) &= \zeta_{(\mathrm{Spec}(\mathbb{C}),S_1)}(t) \cdot \zeta_{-[(\mathrm{Spec}(\mathbb{C}),S_2)]}(t^2) \cdot \ldots \end{aligned}$$

where the dots mean terms do not influencing the part of degree ≤ 2. Therefore

$$\begin{aligned} (1 &+ [(\mathrm{Spec}(\mathbb{C}), S_1)]\, t)_\zeta^{[(Z,G)]} = (\zeta_{[(Z,G)]}(t)) \cdot (\zeta_{[(Z,G\times S_2)]}(t^2))^{-1} \cdot \ldots \\ &= (1 + [(Z,G)]t + [(Z^2, G_2)]t^2 + \ldots)(1 - [(Z, G \times S_2)]t^2 + \ldots) \cdot \ldots \\ &= 1 + [(Z,G)]t + ([(Z^2, G_2)] - [(Z, G \times S_2)])t^2 + \ldots \end{aligned} \tag{10}$$

where S_2 acts on Z trivially. Thus one has

$$\begin{aligned} (1 &+ [(\mathrm{Spec}(\mathbb{C}), S_1)]\, t)_\zeta^{[(Z,G)]} - (1 + [(\mathrm{Spec}(\mathbb{C}), S_1)]\, t)_\lambda^{[(Z,G)]} \\ &= ([(\Delta_G, G_2)] - [(Z, G \times S_2)])t^2 \mod t^3 \end{aligned}$$

where Δ_G is the big G-diagonal in Z^2. The coefficient of t^2 is not equal to zero in $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ even for the trivial action of the group G on Z. This follows from Proposition 2 and the fact that the isotropy groups of points of the two terms have different orders. $\square$

Proposition 7 *The power structure over the ring $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ defined by the Kapranov zeta function $\zeta_\bullet(t)$ is not effective.*

Proof. Equation (10) gives that the coefficient of t^2 in $(1+[(\mathrm{Spec}(\mathbb{C}), S_1)]\, t)_\zeta^{[(\mathrm{Spec}(\mathbb{C}),G)]}$ is equal to $[(\mathrm{Spec}(\mathbb{C}), G_2)] - [(\mathrm{Spec}(\mathbb{C}), G \times S_2)]$. Proposition 2 implies that it does not belong to the image of the Grothendieck semi-ring $S_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ (since $-[((\mathrm{Spec}(\mathbb{C}), G \times S_2)]$ does not belong to it). $\square$

Theorem 2 *The power structure over* $K_0^{\rm fGr}({\rm Var}_{\mathbb{C}})$ *defined by the series* $\lambda_\bullet(t)$ *is effective.*

Proof. To prove the statement, we will give a formula for $(A(t))^m$, where $A(t) = 1 + [(A_1, G_1)]t + [(A_2, G_2)]t^2 + \ldots$, $m = [(M, G)]$, that is, for the case when the coefficients of the series $A(t)$ and the exponent are classes of varieties with pure actions. It is given by the equation

$$\begin{aligned}(A(t))^m &= \\ &= 1 + \sum_{k=1}^{\infty} \left(\sum_{\{k_i\}:\sum_i ik_i = k} \left[\left(\left(M^{\sum_i k_i} \setminus \Delta_G\right) \times \prod_i A_i^{k_i}, G_{\{k_i\}} \right) \right] \right) \cdot t^k , \quad (11)\end{aligned}$$

where the variety $\left(M^{\sum_i k_i} \setminus \Delta_G\right) \times \prod_i A_i^{k_i}$ is endowed with a finite group action in the following way. It carries the natural action of the product

$$G^{\sum_i k_i} \times \prod_i G_i^{k_i}$$

of the finite groups acting on the components of M and A_i. Besides that there is a natural action of the product $\prod_i S_{k_i}$ of permutation groups, where S_{k_i} acts simultaneously on the components of M^{k_i} and of $A_i^{k_i}$ (that is it acts by permutation on the components of $(M \times A_i)^{k_i}$). The variety $\left(M^{\sum_i k_i} \setminus \Delta_G\right) \times \prod_i A_i^{k_i}$ is endowed with the action of the group $G_{\{k_i\}}$ generated by these two actions: the semidirect product

$$\left(G^{\sum_i k_i} \times \prod_i G_i^{k_i} \right) \rtimes \left(\prod_i S_{k_i} \right) = \prod_i \left((G \times G_i)^{k^i} \rtimes S_{k_i} \right)$$

of the groups indicated above.

It is enough to prove that Equation (11) defines a power structure over the ring $K_0^{\rm fGr}({\rm Var}_{\mathbb{C}})$ and that $\lambda_{(M,G)}(t) = (1+t)^{[(M,G)]}$. After that its effectiveness is obvious. We have to verify the properties (3), (4) and (5) of Definition 6: all other properties obviously hold. For that let us give a geometric interpretation of the coefficient of t^k in Equation (11).

Let $\Gamma_A := \coprod_{i=1}^k A_i$ and let $I_A : \Gamma_A \to \mathbb{Z}$ be the tautological function on Γ_A which sends the component A_i to i. The coefficient of t^k in Equation (11) is the configuration space of pairs (K, Ψ), where K is an ordered finite subset of M and Ψ is a map from K to Γ_A such that $I_A(\Psi(x)) \le I_A(\Psi(y))$ for

$x < y$ (that is several first points of K (let us denote their number by k_1) are mapped to A_1, several subsequent ones (number of then being k_2) are mapped to A_2, ...) and

$$\sum_{x \in K} I_A(\Psi(x)) = \sum_i i k_i = k.$$

The group $G^{\sum_i k_i} \times \prod_i G_i^{k_i}$ acts on this configuration space: $G^{\sum_i k_i}$ acts on the source and $\prod_i G_i^{k_i}$ acts on the image. The group $S_{\sum_i k_i}$ acts by simultaneous permutations on points of K sent to A_i and on there images. This gives an action of the group $G_{\{k_i\}}$.

Property (3). Let $B(t) = 1 + [(B_1, G_1')]t + [(B_2, G_2')]t^2 + \ldots$ Let $C_j = \bigsqcup_{i=0}^{j} (A_i \times B_{j-i})$ be the variety with a finite groups action representing the coefficient of t^j in the product $A(t)B(t)$. (Here $A_0 = B_0 = 1 = [(\mathrm{Spec}(\mathbb{C}), (e))]$).

The coefficient of t^k in $(A(t)B(t))^m$ is represented by the configuration space L_k (L for "left") of pairs (K, Ψ), where K is an ordered finite subset of M and Ψ is a map from K to $\Gamma_C = \bigsqcup_{i,j} A_i \times B_j$ such that $\sum_{x \in K} I_C(\Psi(x)) = k$. Such pair is defined by two pairs (K', Ψ'), $\Psi' : K' \to \Gamma_A$, and (K'', Ψ''), $\Psi'' : K'' \to \Gamma_B$, where $K = K' \cup K''$. The coefficient of t^k in $(A(t))^m \cdot (B(t))^m$ is represented by the configuration space R_k (R for "right") of quadruples $((K', \Psi'), (K'', \Psi''))$, where K' and K'' are finite ordered subsets of M, $\Psi' : K' \to \Gamma_A$, $\Psi'' : K'' \to \Gamma_B$ and $\sum_{x \in K'} I_A(\Psi'(x)) + \sum_{x \in K''} I_B(\Psi''(x)) = k$.

Modulo orderings of the sets K, K' and K'' (i. e. after factorization by the corresponding permutations) the varieties L_k and R_k are equal. However them themselves differ from each other and even the groups acting on them are different. In order to prove the property, we shall distinguish parts of L_k and R_k which can be identified (without factorization by permutations) and such that L_k and R_k are (disjoint) unions of varieties obtained from these parts by induction operations. The relation (3) in Definition 6 will imply that the classes of L_k and R_k in $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ coincide.

We have $L_k = \bigsqcup_{\bar{k}: \sum i k_i = k} L_{\bar{k}}$. Let $k_{i_\ell} (\ell = 1, 2, \ldots)$ be integers such that $\sum_\ell k_{i_\ell} = k_i$. Let $L_{\{k_{i_\ell}\}}$ be the subvariety of $L_{\bar{k}}$ consisting of pairs (K, Ψ) such that among k_i points of K mapped into C_i there are k_{i_ℓ} points x the first components $\pi_1 \circ \Psi(x)$ of whose image under Ψ belong to A_ℓ (and thus the second component belongs to $B_{i-\ell}$).

Let $\widehat{L}_{\{k_{i_\ell}\}}$ be the subvariety of $L_{\{k_{i_\ell}\}}$ consisting of pairs $(K, \Psi) \subset L_{\{k_{i_\ell}\}}$ such that the points of K (of fixed multiplicity i) are ordered, say, in the following way: if $\ell_1 < \ell_2$, then those points x for whom $\pi_1(\Psi(x)) \in A_{\ell_1}$

precede those for whom $\pi_1(\Psi(x)) \in A_{\ell_2}$, (the order of the points whose images under $\pi_1 \circ \Psi$ lie in the same A_ℓ is arbitrary).

Also we have $R_k = \sqcup_{\bar{k}:\sum_i ik_i=k} R_{\bar{k}}$. For a collection $\{k_{i_\ell}\}$, let $R_{\{k_{i_\ell}\}}$ be the corresponding subvariety of $R_{\bar{k}}$. Let $\widehat{R}_{\{k_{i_\ell}\}}$ be the subvariety of $R_{\{k_{i_\ell}\}}$ of the quadruples $((K', \Psi'), (K'', \Psi''))$ such that the points of K' (of K'' respectively) of fixed multiplicity ℓ are ordered, say, in the following way: if $i_1 < i_2$, then those for whom $\Psi(x) \in C_{i_1}$ precede those for whom $\Psi(x) \in C_{i_2}$, the order of the points whose images under Ψ lie in the same C_i is arbitrary.

The varieties $\widehat{L}_{\{k_{i_l}\}}$ and $\widehat{R}_{\{k_{i_l}\}}$ carry natural actions of the group

$$\prod_{i,\ell} S_{k_{i_\ell}}$$

and they are isomorphic as $\prod_{i,\ell} S_{k_{i_\ell}}$-varieties.

Moreover, as a $\prod_i S_{k_i}$-variety:

$$L_{\{k_{i_\ell}\}} = \mathrm{ind}_{\prod_{i,\ell} S_{k_{i_\ell}}}^{\prod_i S_{k_i}} \widehat{L}_{\{k_{i_\ell}\}}.$$

Let $m_\ell := \sum_i k_{i_\ell}, m'_\ell := \sum_i k_{i_{i-\ell}}$. The group $S_{k_{i_\ell}}$ is embedded into $(\prod_\ell S_{m_\ell}) \times (\prod_{\ell'} S_{m_{\ell'}})$ permuting the corresponding k_{i_ℓ} elements among those permuted by S_{m_ℓ} and the corresponding k_{i_ℓ} elements among those permuted by $S_{m_{\ell'}}$ simultaneously. Then as a $(\prod_\ell S_{m_\ell}) \times (\prod_{\ell'} S_{m_{\ell'}})$-variety

$$R_{\{k_{i_\ell}\}} = \mathrm{ind}_{\prod_{i,\ell} S_{k_{i_\ell}}}^{(\prod_\ell S_{m_\ell})\times(\prod_{\ell'} S_{m_{\ell'}})} \widehat{R}_{\{k_{i_\ell}\}}.$$

The actions of the products of the groups G, G_i and G'_i are obviously coordinated.

Modulo the relation (3) in Definition 6 this means that $[L_{\{k_{i_\ell}\}}] = [R_{\{k_{i_\ell}\}}]$ and therefore $[L_k] = [R_k]$.

Property (4). Let $n = (N, G)$. The coefficient of t^k in $(A(t))^{m+n}$ is represented by the configuration space L_k of pairs (K, Ψ), where K is a finite subset of $M \sqcup N$ and $\Psi : K \to \Gamma_A$ is a map such that $\sum_{x \in K} I_A(\Psi(x)) = k$. For a collection $\{k_i\}$, $i = 1, 2, \ldots$, $\sum_i ik_i = k$, (that is for a partition of k), let k'_i, $i = 1, 2, \ldots$, be integers such that $0 \le k'_i \le k_i$. Let $L_{\{k_i\}\{k'_i\}}$ be the subvariety of L_k consisting of pairs (K, Ψ) such that among k_i points of K of multiplicity i, the number of points belonging to M is equal to k'_i for $i = 1, 2, \ldots$ (and thus $(k_i - k'_i)$ points of K of multiplicity i belong to N). Let $\widehat{L}_{\{k_i\}\{k'_i\}}$ be the subvariety of $L_{\{k_i\}\{k'_i\}}$ consisting of pairs $(K, \psi) \in L_{\{k_i\}\{k'_i\}}$

such that the points of K of fixed multiplicity i are ordered in the following way: first the points of M, then the points of N.

The coefficient of t^k in $(A(t))^m(A(t))^n$ is represented by the configuration space R_k of quadruples $((K', \Psi'), (K'', \Psi''))$, where K' is an ordered finite subset of M, K'' is an ordered finite subset of N, $\Psi' : K' \to \Gamma_A$, $\Psi'' : K'' \to \Gamma_A$,

$$\sum_{x \in K'} I_A(\Psi'(x)) + \sum_{x \in K''} I_A(\Psi''(x)) = k\,.$$

Let $R_{\{k_i\}\{k'_i\}}$ be the subvariety of R_k consisting of quadruples $((K', \Psi'), (K'', \Psi''))$ with the number of points of multiplicity i in K' equal to k'_i and the number of points of multiplicity i in K'' equal to $(k_i - k'_i)$.

The $\left(\prod_i S_{k'_i} \times \prod_i S_{k_i - k'_i}\right)$-varieties $\widehat{L}_{\{k_i\}\{k'_i\}}$ and $R_{\{k_i\}\{k'_i\}}$ are isomorphic,

$$L_{\{k_i\}\{k'_i\}} = \mathrm{ind}_{\prod_i S_{k'_i} \times \prod_i S_{k_i - k'_i}}^{\prod_i S_{k_i}} \widehat{L}_{\{k_i\}\{k'_i\}}$$

as a $\prod_i S_{k_i}$-variety. Again the actions of the products of the groups G, G' and G_i on these varieties are obviously coordinated. Therefore

$$[L_{\{k_i\}\{k'_i\}}] = [R_{\{k_i\}\{k'_i\}}]\,.$$

This implies that $[L_k] = [R_k]$.

Property (5). The coefficient of t^k in $(A(t))^{mn}$ is represented by the configuration space L_k of pairs (K, Ψ), where K is a finite subset of $M \times N$, $\Psi : K \to \Gamma_A$ is a map such that $\sum_{x \in K} I_A(\Psi(x)) = k$. For a fixed function $k(\overline{s})$ ($\overline{s} = (s_1, s_2, \ldots)$) with non-negative integer values and with $\sum_{\overline{s}} \left(k(\overline{s}) \sum_i i s_i\right) = k$, let $L_{k(\bullet)}$ be the subvariety of L_k consisting of the pairs (K, ψ) such that the projection of K to M consists of $\sum_{\overline{s}} k(\overline{s})$ points and $k(\overline{s})$ of them are such that the preimage (in K) of each of them contains s_1 points of multiplicity 1, s_2 points of multiplicity 2, ...

Let $\widehat{L}_{k(\bullet)}$ be the subvariety of $L_{k(\bullet)}$ consisting of pairs $(K, \psi) \in L_{k(\bullet)}$ such that the points of K of fixed multiplicity i are ordered in the following way. One takes an arbitrary order of their projections to M and the order of points with different projections is the one in M (the order of points with the same projection can be arbitrary).

The coefficient of of t^k in $((A(t))^n)^m$ is represented by the configuration space R_k of the following patterns: a finite subset of points of M with different multiplicities (with an order of them) with a finite subset of N (also ordered) associated to each of them and a map of the latter to Γ_A. Such a pattern defines a finite subset of $M \times N$. For a function $k(\overline{s})$ ($\overline{s} = (s_1, s_2, \ldots)$), let $R_{k(\bullet)}$ be the subvariety of R_k consisting of the patterns with the composition of the corresponding finite subset of $M \times N$ of the type described above.

Both on $\widehat{L}_{k(\bullet)}$ and on $R_{k(\bullet)}$ one has the natural action of a semidirect product $S_{k(\bullet)}$ of the group $\prod_{\overline{s}} S_{k(\overline{s})}$ (acting on the projections to M) and of the group $\prod_{\overline{s}} \left(\prod_i S_{s_i}\right)^{k(\overline{s})}$ (acting on the preimages of points in M). The group $S_{k(\bullet)}$ is embedded into $\prod_i S_{\sum_{\overline{s}} k(s_i)s_i}$. As $S_{k(\bullet)}$-varieties $\widehat{L}_{k(\bullet)}$ and $R_{k(\bullet)}$ are isomorphic. Moreover,

$$L_{k(\bullet)} = \mathrm{ind}_{S_{k(\bullet)}}^{\prod_i S_{\sum_{\overline{s}} k(s_i)s_i}} \widehat{L}_{k(\bullet)} \,.$$

Therefore $[L_{k_{(\bullet)}}] = [R_{k_{(\bullet)}}]$ and thus $[L_k] = [R_k]$.

To show that the power structure (11) is defined by the series $\lambda_{(Z,G)}(t)$, we have to prove that (in terms of the power structure)

$$\lambda_{(Z,G)}(t) = (\lambda_1(t))^{[(Z,G)]} = (1+t)^{[(Z,G)]} \,.$$

The only non-empty summand in the coefficient of t^k in (11) corresponds to $k_1 = k$, $k_i = 0$ for $i > 1$ and is represented by the variety $Z^k \setminus \Delta_G$ with the action of the corresponding wreath product. This proves the statement. $\square$

Remark. The λ-structures on $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ defined by the series $\zeta_\bullet(t)$ and $\lambda_\bullet(t)$ commute with the corresponding structures on $K_0(\mathrm{Var}_{\mathbb{C}})$ through the ring homomorphism $p : K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}}) \to K_0(\mathrm{Var}_{\mathbb{C}})$. As any λ-ring, $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ carries the λ-structures opposite to those defined by the series $\zeta_\bullet(t)$ and $\lambda_\bullet(t)$. The fact that the power structure over $K_0(\mathrm{Var}_{\mathbb{C}})$ defined by the λ-structures opposite to those defined by the analogues of the series $\zeta_\bullet(t)$ and $\lambda_\bullet(t)$ is not effective ([6]) implies that the power structures over $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ defined by the λ-structures opposite to $\zeta_\bullet(t)$ and $\lambda_\bullet(t)$ are not effective as well.

7 Grothendieck ring of varieties with equivariant vector bundles

An equivariant vector bundle over a complex quasi-projective G-variety Z is a ($\mathbb{C}$-)vector bundle $p : E \to Z$ with an action of the group G on E commuting with the action on Z and preserving the vector bundle structure. We shall denote it by (Z, E, G). The notion of an isomorphism of two varieties with equivariant vector bundles similar to that in Definition 3 is clear.

Definition 9 A quasi-projective variety $\mathcal{X}$ with an action of finite groups and with an equivariant vector bundle E over it *is a variety represented as the disjoint union of subvarieties* X_i, $i = 1, \ldots, s$, *with actions of finite groups* G_i *on them and with equivariant vector bundles* E_i *over them (of different ranks in general).*

Definition 10 *Two varieties* $\mathcal{X}$ *and* $\mathcal{Y}$ *with finite groups actions and with equivariant vector bundles* E *and* E' *over them are* equivalent *if there exist partitions* $(\mathcal{X}, E) = \coprod_{i=1}^{N} (X_{(i)}, E_{(i)}, G_{(i)})$ *and* $(\mathcal{Y}, E') = \coprod_{i=1}^{N} (Y_{(i)}, E'_{(i)}, G'_{(i)})$ *of them such that* $(X_{(i)}, E_{(i)}, G_{(i)})$ *is isomorphic to* $(Y_{(i)}, E'_{(i)}, G'_{(i)})$ *for* $i = 1, \ldots, N$.

Definition 11 The Grothendieck ring of varieties with equivariant vector bundles *is the Abelian group* $K_0^{\rm fGr}(\mathrm{Vect}_{\mathbb{C}})$ *generated by the classes* $[(\mathcal{X}, E)]$ *of complex (quasi-projective) varieties with finite groups actions and with equivariant vector bundles over them modulo the relations:*

(1) if varieties $(\mathcal{X}, E)$ *and* $(\mathcal{Y}, E')$ *with finite groups actions and with equivariant vector bundles* E *and* E' *over them are equivalent, then* $[(\mathcal{X}, E)] = [(\mathcal{Y}, E')]$;

(2) if $\mathcal{Y}$ *is a Zariski closed subvariety of* $\mathcal{X}$ *invariant with respect to the groups action, then* $[(\mathcal{X}, E)] = [(\mathcal{Y}, E_{|\mathcal{Y}})] + [(\mathcal{X} \setminus \mathcal{Y}, E_{|\mathcal{X}\setminus\mathcal{Y}})]$;

(3) if (Z, E, G) *is a* G*-variety with an equivariant vector bundle and* G *is a subgroup of a finite group* H*, then*

$$\left[(\mathrm{ind}_G^H Z, \mathrm{ind}_G^H E, H)\right] = [(Z, E, G)] \ .$$

The multiplication in $K_0^{\rm fGr}(\mathrm{Vect}_{\mathbb{C}})$ *is defined by the Cartesian product of varieties with the natural finite groups action and with the sum of the corresponding vector bundles over it.*

In other words

$$[(Z_1, E_1, G_1)] \cdot [(Z_2, E_2, G_2)] = [(Z_1 \times Z_2, E_1 \times E_2, G_1 \times G_2)] \qquad (12)$$

(with the natural action of $G_1 \times G_2$).

Remark. There are natural ring homomorphisms $i^{\mathrm{v}}: K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}}) \to K_0^{\mathrm{fGr}}(\mathrm{Vect}_{\mathbb{C}})$ sending the class of a G-variety Z to the same variety with the (trivial) vector bundle of rank 0 and $p^{\mathrm{v}}: K_0^{\mathrm{fGr}}(\mathrm{Vect}_{\mathbb{C}}) \to K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$: forgetting the vector bundle. One obviously has $p^{\mathrm{v}} \circ i^{\mathrm{v}} = \mathrm{id}$.

Let (Z, E, G) be a G-variety with an equivariant vector bundle over it. Let $x \in Z$ be a point fixed by an element $g \in G$. The element g acts on the fibre E_x of the vector bundle as an operator of finite order. Therefore its action can be represented by a diagonal $(d_x \times d_x)$-matrix ($d_x = \dim E_x$) with the diagonal entries $\exp(2\pi i q_j)$, $j = 1, \ldots, d_x$, where $0 \le q_j < 1$.

Definition 12 *(cf. [16], [10]) The* age *(or the* fermion shift*)* $\mathrm{age}_x(g)$ *of the element* g *at the point* x *is* $\sum\limits_{j=1}^{d_x} q_j \in \mathbb{Q}$.

Let φ be a (rational) number. As above, for an element $g \in G$, let $Z^{\langle g \rangle}$ be the fixed point set of g. For a rational number q, let $Z_q^{\langle g \rangle}$ be the subset of $Z^{\langle g \rangle}$ consisting of the points x with $\mathrm{age}_x(g) = q$. (The subset $Z_q^{\langle g \rangle}$ is the union of some of the components of $Z^{\langle g \rangle}$.)

As above (Section 2), let $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^s]$ be the modification of the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ of quasi-projective varieties by adding all rational powers of the class $\mathbb{L}$ of the complex affine line. The (standard) power structure over the ring $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^s]$ can be defined through a λ-structure on it using the equation:

$$\zeta_{\mathbb{L}^s[M]}(t) = \zeta_{[M]}(\mathbb{L}^s t), \tag{13}$$

which holds for the Kapranov zeta function on the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$. This equation (used for integer values of s) defines a λ-structure on the localization $K_0(\mathrm{Var}_{\mathbb{C}})_{(\mathbb{L})}$: see [5]. For an element $c = \sum_i c_i \mathbb{L}^{r_i} \in K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^s]$ one defines $\zeta_c(t)$ by

$$\zeta_c(t) := \prod_i \zeta_{c_i}(\mathbb{L}^{r_i} t).$$

The λ-structure on the ring $K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^s]$ defines a power structure over it in the standard way ([5]): a series $A(t) \in 1 + tK_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^s][[t]]$ has a unique representation in the form $A(t) = \prod_{i=1}^{\infty} \zeta_{b_i}(t^i)$ with $b_i \in K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^s]$ and one defines $(A(t))^m$, $m \in K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^s]$, by

$$(A(t))^m := \prod_{i=1}^{\infty} \zeta_{mb_i}(t^i).$$

This definition together with the equation (13) implies that

$$(A(\mathbb{L}^s t))^m = (A(t))^m\,|_{t\mapsto \mathbb{L}^s t}\,.$$

Definition 13 The generalized orbifold Euler characteristic of (Z,E,G) of weight φ_1 *is defined by*

$$[Z,E,G]_{\varphi_1} := \sum_{[g]\in \mathrm{Conj}\, G}\ \sum_{q\in Q} [Z_q^{\langle g\rangle}/C_G(g)]\cdot \mathbb{L}^{\varphi_1 q} \in K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^s]\,. \tag{14}$$

We have to show that the generalized orbifold Euler characteristic is well defined by the class of (Z,E,G) in $K_0^{\mathrm{fGr}}(\mathrm{Vect}_{\mathbb{C}})$. Essentially we have to show that, for a G-variety Z with an equivariant vector bundle E and for a finite group H such that $G\subset H$, we have

$$[Z,E,G]_{\varphi_1} = [\mathrm{ind}_G^H Z, \mathrm{ind}_G^H E, H]_{\varphi_1}\,.$$

We will show this a little bit later for higher order generalized Euler characteristics introduced below as well. (The generalized orbifold Euler characteristic is one of them.)

Let $\overline{\varphi}=(\varphi_1,\varphi_2,\ldots)$ be a fixed sequence of rational numbers.

Definition 14 The generalized Euler characteristic *of* (Z,E,G) *of order* k *of weight* $\overline{\varphi}$ *is defined by*

$$[Z,E,G]_{\overline{\varphi}}^k = \sum_{[g]\in \mathrm{Conj}\, G}\ \sum_{q\in \mathbb{Q}} [Z_q^{\langle g\rangle}, E_{|Z_q^{\langle g\rangle}}, C_G(g)]_{\overline{\varphi}}^{k-1}\cdot \mathbb{L}^{\varphi_k q} \in K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^s]\,,$$

where $[Z,E,G]_{\overline{\varphi}}^1 = [Z,E,G]_{\overline{\varphi}}$ *is the generalized orbifold Euler characteristic.*

Alternatively one can start from $k=0$, defining $[Z,E,G]_{\overline{\varphi}}^0$ as $[Z/G]\in K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^s]$. (It does not depend on the equivariant vector bundle E.) This will be used in the proof of the Theorem 3 below.

Remark. One can see, that for $k>1$ (that is for all the generalized Euler characteristics except the orbifold one), the definition is shorter (and simpler) than that in [8] for non-singular G-varieties. One can say that the described setting is to some extend more natural for the definition.

Theorem 3 *For a fixed* $\overline{\varphi}$*, the generalized orbifold Euler characteristic and the generalized Euler characteristics of higher orders are well-defined ring homomorphisms* $K_0^{\mathrm{fGr}}(\mathrm{Vect}_{\mathbb{C}})\to K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^s]$.

Proof. In fact we have to prove only that the generalized Euler characteristics of higher orders are well defined as functions on $K_0^{\rm fGr}({\rm Vect}_{\mathbb{C}})$. After that their additivity and multiplicativity are obvious. (The latter is essentially Lemma 1 in [7].) Thus we have to show that, for a G-variety Z with an equivariant vector bundle E over it and for a finite group $H\supset G$, one has

$$[{\rm ind}_G^H Z, {\rm ind}_G^H E, H]^k_{\overline{\varphi}} = [Z, E, G]^k_{\overline{\varphi}}\,. \tag{15}$$

For $k=0$ this simply means that $({\rm ind}_G^H Z)/H = Z/G$. Assume that (15) is proven for all values of k smaller than the one under consideration. Using Lemma 1 and the induction we get

$$[{\rm ind}_G^H Z, {\rm ind}_G^H E, H]^k_{\overline{\varphi}} = \sum_{[g]\in {\rm Conj}\, H}\sum_{q\in\mathbb{Q}}[({\rm ind}_G^H Z)^{\langle g\rangle}, {\rm ind}_G^H E, C_H(g)]^{k-1}_{\overline{\varphi}}\cdot \mathbb{L}^{\varphi_k q} =$$
$$\sum_{[g]\in {\rm Conj}\, G}\sum_{q\in\mathbb{Q}}[({\rm ind}_{C_G(g)}^{C_H(g)} Z)^{\langle g\rangle}_q, {\rm ind}_{C_G(g)}^{C_H(g)} E, C_H(g)]^{k-1}_{\overline{\varphi}}\cdot \mathbb{L}^{\varphi_k q} =$$
$$\sum_{[g]\in {\rm Conj}\, G}\sum_{q\in\mathbb{Q}}[Z^{\langle g\rangle}, E, C_G(g)]^{k-1}_{\overline{\varphi}}\cdot \mathbb{L}^{\varphi_k q} = [Z, E, G]^k_{\overline{\varphi}}.$$

□

Let (Z,E,G) be a G-variety with an equivariant vector bundle, and let $Z_{(d)}=\{x\in Z : \dim E_x = d\}$. (There are finitely many d with non-empty $Z_{(d)}$.) One has $[(Z,E,G)] = \sum_d [(Z_{(d)}, E_{|Z_{(d)}}, G)] \in K_0^{\rm fGr}({\rm Vect}_{\mathbb{C}})$.

Theorem 4 *One has*

$$1+\sum_{n=1}^{\infty}[Z^n, E^n, G_n]^k_{\overline{\varphi}}\cdot t^n = \prod_d\left(\prod_{r_1,\ldots,r_k\ge 1}\left(1-\mathbb{L}^{\Phi_k(\underline{r})d/2}\cdot t^{r_1r_2\cdots r_k}\right)^{r_2r_3^2\cdots r_k^{k-1}}\right)^{-[(Z_d, E_{|Z_d}, G)]^k_{\overline{\varphi}}},$$

where

$$\Phi_k(r_1,\ldots,r_k) = \varphi_1(r_1-1)+\varphi_2 r_1(r_2-1)+\ldots+\varphi_k r_1r_2\cdots r_{k-1}(r_k-1).$$

Proof. Since $[\cdot]^k_{\overline{\varphi}}$ is a homomorphism from $K_0^{\rm fGr}({\rm Vect}_{\mathbb{C}})$ to $K_0({\rm Var}_{\mathbb{C}})[\mathbb{L}^s]$, one has

$$1+\sum_{n=1}^{\infty}[(Z^n, E^n, G_n)]^k_{\overline{\varphi}}\cdot t^n = \prod_d\left(1+\sum_{n=1}^{\infty}[Z_d^n, E^n_{|Z_d}, G_n]^k_{\overline{\varphi}}\cdot t^n\right).$$

For the G-variety Z_d with an equivariant vector bundle $E_{|Z_d}$ of constant rank d the arguments of [7, Theorem 1] give

$$1+\sum_{n=1}^{\infty}[(Z_d^n, E^n_{|Z_d}, G_n)]^k_{\varphi}\cdot t^n = \left(\prod_{r_1,\ldots,r_k\geq 1}\left(1-\mathbb{L}^{\Phi_k(\underline{r})d/2}\cdot t^{r_1r_2\cdots r_k}\right)^{r_2r_3^2\cdots r_k^{k-1}}\right)^{-[(Z_d,E_{|Z_d},G)]^k_{\varphi}}.$$

(The only difference with [7, Theorem 1] consists in the fact that the age $F^{(\mathbf{g},s)}$ of an element $(\mathbf{g},s)\in G_n$ with $s=(1,2,\ldots,n)$, $\mathbf{g}=(g_1,g_2,\ldots,g_n)$, $g_1g_2\cdots g_n=c$ (taken from [15, Theorem 3.1] and equal to $F^c+\frac{(n-1)d}{2}$) is computed not in the tangent space to a d-dimensional manifold, but in the fibre of the vector bundle.) This proves the statement. $\square$

8 λ-structures on $K_0^{\mathrm{fGr}}(\mathrm{Vect}_{\mathbb{C}})$ and power structures over it

Similar to the case of the Grothendieck ring $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$, there are two natural λ-structures on the Grothendieck ring $K_0^{\mathrm{fGr}}(\mathrm{Vect}_{\mathbb{C}})$. They are defined by analogues of the series $\zeta_\bullet(t)$ and $\lambda_\bullet(t)$. Since each element of $K_0^{\mathrm{fGr}}(\mathrm{Vect}_{\mathbb{C}})$ can be represented by a G-variety with an equivariant vector bundles (with a certain finite group G), we can define these series for them. Let (Z,E,G) be a G-variety with an equivariant vector bundle E over it. Define the following two series:

$$\begin{aligned}\zeta_{(Z,E,G)}(t) &:= 1+\sum_{n=1}^{\infty}[(Z^n,E^n,G_n)]\cdot t^n,\\ \lambda_{(Z,E,G)}(t) &:= 1+\sum_{n=1}^{\infty}[(Z^n\setminus\Delta_G, E^n_{|Z^n\setminus\Delta_G}, G_n)]\cdot t^n.\end{aligned}$$

Proposition 8 *The series $\zeta_{(Z,E,G)}(t)$ and $\lambda_{(Z,E,G)}(t)$ depend only on the classes $[(Z,E,G)]$ in $K_0^{\mathrm{fGr}}(\mathrm{Vect}_{\mathbb{C}})$.*

Proof. This follows from the equations (for $H\supset G$)

$$\begin{aligned}&[(\mathrm{ind}_{G_n}^{H_n}Z^n, \mathrm{ind}_{G_n}^{H_n}E^n, H_n)] = [(Z^n,E^n,G_n)]\,,\\ &[(\mathrm{ind}_{G_n}^{H_n}Z^n\setminus\Delta_{H_n}, \mathrm{ind}_{G_n}^{H_n}E^n_{Z^n\setminus\Delta_{G_n}}, H_n)] = [(Z^n\setminus\Delta_{G_n}, E^n_{Z^n\setminus\Delta_{G_n}}, G_n)]\,,\end{aligned}$$

whose proofs are almost the same as in Propositions 3 and 5. $\square$

The fact that these series define λ-structures on the ring $K_0^{\rm fGr}({\rm Vect}_{\mathbb{C}})$ follows from the following statement.

Proposition 9 *One has*

$$\begin{aligned}
\zeta_{(Z_1\sqcup Z_2, E_1\sqcup E_2, G)}(t) &= \zeta_{(Z_1,E_1,G)}(t)\cdot\zeta_{(Z_2,E_2,G)}(t)\,,\\
\lambda_{(Z_1\sqcup Z_2, E_1\sqcup E_2, G)}(t) &= \lambda_{(Z_1,E_1,G)}(t)\cdot\lambda_{(Z_2,E_2,G)}(t)\,.
\end{aligned}$$

The proof is almost the same as in Propositions 4 and 5.

The reductions of the series $\zeta_\bullet(t)$ and $\lambda_\bullet(t)$ under the natural homomorphism $p^{\rm v}: K_0^{\rm fGr}({\rm Vect}_{\mathbb{C}}) \to K_0^{\rm fGr}({\rm Var}_{\mathbb{C}})$ coincide with the λ-structures on $K_0^{\rm fGr}({\rm Var}_{\mathbb{C}})$ discussed in Section 5. Since these two λ-structures on the ring $K_0^{\rm fGr}({\rm Var}_{\mathbb{C}})$ lead to different power structures, the same holds for the λ-structures defined by $\zeta_\bullet(t)$ and $\lambda_\bullet(t)$ on the ring $K_0^{\rm fGr}({\rm Vect}_{\mathbb{C}})$. Again as in the case of the Grothendieck ring $K_0^{\rm fGr}({\rm Var}_{\mathbb{C}})$ the power structure defined by $\zeta_\bullet(t)$ is not effective, and the one defined by $\lambda_\bullet(t)$ is. (To show the effectiveness of the power structure defined by the series $\lambda_\bullet(t)$, one can use an analogue of Equation (11). Equivariant vector bundles over the summands in the right hand side of it are defined in the obvious way.) The power structures over $K_0^{\rm fGr}({\rm Vect}_{\mathbb{C}})$ defined by the λ-structures opposite to $\zeta_\bullet(t)$ and $\lambda_\bullet(t)$ are not effective.

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