The universal Euler characteristic of V-manifolds *

S.M. Gusein-Zade † I. Luengo ‡ A. Melle-Hernández §

Abstract

The Euler characteristic is the only additive topological invariant for spaces of certain sort, in particular, for manifolds with some finiteness properties. A generalization of the notion of a manifold is the notion of a V-manifold. We discuss a universal additive topological invariant of V-manifolds: the universal Euler characteristic. It takes values in the ring freely generated (as a $\mathbb{Z}$-module) by isomorphism classes of finite groups. We also consider the universal Euler characteristic on the class of locally closed equivariant unions of cells in equivariant CW-complexes. We show that it is a universal additive invariant satisfying a certain "induction relation". We give Macdonald type equations for the universal Euler characteristic for V-manifolds and for cell complexes of the described type.

1 Introduction

The Euler characteristic $\chi(\cdot)$ (defined as the alternating sum of the ranks of the cohomology groups with compact support) is the only *additive* topological invariant for spaces of certain sort: see. e. g., [20], see also [8, Proposition 2]. In particular, the Euler characteristic is the only additive invariant of manifolds with some finiteness

*Math. Subject Class.: 57R18, 55M35. Keywords: finite group actions, V-manifold, orbifold, additive topological invariant, lambda-ring, Macdonald equation.

†The work of the first author (Sections 2, 5, 6) was supported by the grant 16-11-10018 of the Russian Science Foundation. Address: Moscow State University, Faculty of Mechanics and Mathematics, GSP-1, Moscow, 119991, Russia. E-mail: sabir@mccme.ru

‡The last two authors were partially supported by a competitive Spanish national grant MTM2016-76868-C2-1-P. Address: ICMAT (CSIC-UAM-UC3M-UCM), Dept. of Algebra, Geometry and Topology, Complutense University of Madrid, Plaza de Ciencias 3, Madrid, 28040, Spain. E-mail: iluengo@mat.ucm.es

§Address: Instituto de Matemática Interdisciplinar (IMI), Dept. of Algebra, Geometry and Topology, Complutense University of Madrid, Plaza de Ciencias 3, Madrid, 28040, Spain. E-mail: amelle@mat.ucm.es

properties: see below. This property has some generalizations. For example, the equivariant Euler characteristic with values in the Burnside ring $A(G)$ of a finite group G is the only additive topological invariant of spaces with G-actions (see, e. g., [8]).

A generalization of the notion of a manifold is the notion of a V-manifold (that is of a (real) orbifold: locally defined as the quotient of a manifold by a finite group action) introduced initially in [16]. There are a number of additive invariants defined for V-manifolds, e. g., the Euler–Satake characteristic: [17], the orbifold Euler characteristic: [4], [5], [1], [13], the higher order (orbifold) Euler characteristics: [2], [18], the Γ–Euler–Satake characteristic: [7].

Here we discuss the universal additive topological invariant $\chi^{\rm un}$ of V-manifolds: a sort of a universal (topological) Euler characteristic for them. It takes values in the ring $\mathcal{R}$ freely generated (as a $\mathbb{Z}$-module) by the isomorphism classes of finite groups.

We also consider the universal Euler characteristic $\chi^{\rm un}$ on the Grothendieck ring $K_0^{\rm fGr}({\rm Var}_{\mathbb{C}})$ of complex quasi-projective varieties with finite groups actions ([12]) and on the class of *equivariant cell complexes*: locally closed unions of cells in equivariant CW-complexes in the sense of [19]. In the latter case we show that $\chi^{\rm un}$ is a universal additive invariant satisfying a certain "induction relation".

The classical Euler characteristic satisfies the Macdonald equation for the generating series of the Euler characteristics of the symmetric products of a topological space X:

$$1+\sum_{n=1}^{\infty}\chi(S^nX)\cdot t^n=(1-t)^{-\chi(X)}.$$

Also one has a Macdonald type equation for the Euler characteristics of the configuration spaces of points on X. Let $M_n(X)=(X^n\setminus\Delta)/S_n$ be the configuration space of (unordered) n-tuples of points in X (Δ is the big diagonal in the Cartesian power in X^n consisting of points $\bar{x}=(x_1,\ldots x_n)\in X^n$ with at least two coinciding components). One has the equation

$$1+\sum_{n=1}^{\infty}\chi(M_nX)\cdot t^n=(1+t)^{\chi(X)}.$$

Analogues of these equations for other (additive) invariants with values in rings different from the ring of integers (e. g., for the equivariant Euler characteristic or for the generalized (motivic) Euler characteristic of complex quasi-projective varieties) are formulated in terms of power structures over the rings of values: [11, Lemma 1], [10]. A power structure over a ring is closely related with (and is defined by) a

λ-structure on it. Analogues of these equations for the universal Euler characteristic $\chi^{\rm un}$ are formulated in terms of different λ-structures on $\mathcal{R}$. We discuss these λ-structures on $\mathcal{R}$ and the corresponding power structures. We give Macdonald type equations for the universal Euler characteristic $\chi^{\rm un}$ for V-manifolds and for equivariant cell complexes.

2 Euler characteristic of manifolds

The Euler characteristic is defined for manifolds with some finiteness properties. To fix a class of such manifolds, let us consider (C^∞-) manifolds which are interiors of compact manifolds with boundaries. A submanifold of such a manifold is the interior of a (closed) submanifold in a manifold with boundary, that is of a submanifold transversal to the boundary. (We permit a submanifold to be of the same dimension as the manifold itself. In this case the submanifold is a connected component of the manifold.) In what follows we consider only manifolds from this class. Let M be a manifold and let N be a (closed) submanifold of M. One has the following additivity property of the Euler characteristic:

$$\chi(M) = \chi(N) + \chi(M \setminus N)\,.$$

(Pay attention that $M \setminus N$ is also a manifold from the described class.) One has the inverse statement.

Proposition 1 *Let I be a topological invariant of manifolds which possesses the additivity property:*

$$I(M) = I(N) + I(M \setminus N)$$

for a submanifold $N \subset M$. Then $I(M) = \chi(M)a$, where $a = I(pt)$.

Proof. First let us reduce the statement to the corresponding statement for cells, i. e., for manifolds diffeomorphic to open balls. For that we shall cut a given n-dimensional manifold M^n by submanifolds into pieces diffeomorphic to cells. A one-dimensional manifold is a (finite) union of open segments and circles and there is no problem to decompose it into cells. Assume that this is possible for manifolds of dimension less than n. Let M be the interior of a manifold $\widehat{M}$ with boundary and let $f : \widehat{M} \to \mathbb{R}$ be a Morse function on $\widehat{M}$ equal to zero on the boundary $\partial\widehat{M}$ and positive on M. Let $0 < c_1 < c_2 < \ldots < c_r$ be the critical values of f (and let $c_0 = 0$). Let U_i, $i = 1, 2, \ldots, r$, be small open balls around the corresponding critical points P_i. The manifold M is the union of the manifolds $M_i = f^{-1}((c_{i-1} + \varepsilon, c_i - \varepsilon))$, $M_i' = f^{-1}((c_i - \varepsilon, c_i + \varepsilon))$ and $N_{i\pm} = f^{-1}(c_i \pm \varepsilon)$ (we take ε small enough). We have to cut these manifolds into cells (using submanifolds). For the manifolds $N_{i\pm}$ this is possible because of the assumption. The manifold M_i is diffeomorphic to the cylinder over the manifold N_{i-} and a method to cut N_{i-} into cells (by submanifolds) can be extended to M_i in an obvious way. The intersection $M_i' \cap U_i$ is diffeomorphic to a cell. The complement $M_i' \setminus U_i$ is diffeomorphic to the cylinder over $N_{i-} \setminus U_i$. A method to cut $N_{i-} \setminus U_i$ gives a method to cut $M_i' \cap U_i$. (Here we apply an obvious version of the procedure to $N_{i-} \setminus U_i$ which is a manifold with boundary.)

The additivity property permits to prove the statement for cells (open balls):

$$I(\sigma^k) = (-1)^k I(\mathrm{pt})\,. \tag{1}$$

Assume that (1) is proved for cells of dimension less than k, in particular, $I(\sigma^{k-1}) = (-1)^{k-1}I(\mathrm{pt})$. The ball σ^k can be cut by a submanifold diffeomorphic to σ^{k-1} into two manifolds diffeomorphic to σ^k. Therefore

$$I(\sigma^k) = 2I(\sigma^k) + I(\sigma^{k-1})\,,$$

what gives (1). $\square$

3 V-manifolds (real orbifolds)

Let us give some definitions in a form appropriate for a discussion below.

For a G-space X, that is a topological space X with a (left) G-action, and for an embedding $G \subset H$ (G and H are finite groups), let *the induction* $\mathrm{ind}_G^H X$ be the H-space defined as the quotient

$$\mathrm{ind}_G^H X = H \times X / \sim\,,$$

where $(h_1, x_1) \sim (h_2, x_2)$ if (and only if) there exists $g \in G$ such that $x_1 = gx_2$, $h_1 = h_2 g^{-1}$ (with an obvious H-action). As a topological space $\mathrm{ind}_G^H X$ is the union of several ($\frac{|H|}{|G|}$) copies of X. If X is, say, a (C^∞-) manifold or a complex quasi-projective variety, the space $\mathrm{ind}_G^H X$ is of the same type.

Definition 1 *An n-dimensional uniformizing system on a topological space X is a quadruple $(U, \widetilde{U}, G, \varphi)$, where U is an open subset of X, G is a finite group, $\widetilde{U}$ is a smooth (C^∞-) n-dimensional manifold with a G-action, and φ is a map $\widetilde{U} \to U$ such that $\varphi(gx) = \varphi(x)$ (that is φ factorizes through a map $p_\varphi : \widetilde{U}/G \to U$) and the corresponding map p_φ is a homeomorphism.*

Remark. In some cases one adds the condition that the fixed point set of each element of G has codimension at least two in $\widetilde{U}$. This restriction is not necessary in this paper and it is more convenient not to require it.

Definition 2 *Two uniformizing systems* $(U', \widetilde{U}', G', \varphi')$ *and* $(U'', \widetilde{U}'', G'', \varphi'')$ *on* X *are equivalent if for any point* $x \in U' \cap U''$ *there exists a neighbourhood* U *of* x *in* $U' \cap U''$*, a group* G *contained both in* G' *and in* G'' *(that is with embeddings into* G' *and into* G''*) and a uniformizing system* $(U, \widetilde{U}, G, \varphi)$ *such that the* G'*-manifolds* $\mathrm{ind}_G^{G'} \widetilde{U}$ *and* $(\varphi')^{-1}(U)$ *are isomorphic over* U *(that is by an isomorphism commuting with the projections to* U*) and the* G''*-manifolds* $\mathrm{ind}_G^{G''} \widetilde{U}$ *and* $(\varphi'')^{-1}(U)$ *are isomorphic over* U *as well.*

Definition 3 *A (*V*-manifold) atlas on a topological space* X *is a collection of* n*-dimensional uniformizing systems* $\{(U_\alpha, \widetilde{U}_\alpha, G_\alpha, \varphi_\alpha)\}$ *on* X *such that* $\bigcup_\alpha U_\alpha = X$ *and any two uniformizing systems from the collection are equivalent.*

Definition 4 *Two atlases on* X *are equivalent if their union is an atlas on* X *as well.*

Definition 5 *(see [16], [3]) An* n*-dimensional* V*-manifold* Q *is a separable Hausdorff space* $X = X_Q$ *with an equivalence class of* n*-dimensional atlases on it.*

One can define in a natural way the notion of a V-manifold with boundary: see [17], [3, Appendix]. In order to ensure that the topological characteristics discussed below are defined, one has to impose certain finiteness conditions on V-manifolds under consideration. In what follows we shall assume that all V-manifolds are interiors of compact V-manifolds with boundaries. (For short we shall call them *tame.*)

The universal Euler characteristic as well as other invariants discussed below can also be regarded as homomorphisms from the *Grothendieck ring* $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ *of finite group actions* defined in [12]. The Grothendieck ring $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ is the Abelian group generated by the classes $[(X, G)]$ of quasi-projective G-varieties for all finite groups G modulo the relations

1) if $(X, G) \cong (X', G')$ (that is if there exist a group isomorphism $\alpha : G \to G'$ and an (algebraic) isomorphism $\psi : X \to X'$ such that $\psi(gx) = \alpha(g)\psi(x)$), then $[(X, G)] = [(X', G')]$;

2) if Y is a Zariski closed G-subset of X, then $[(X, G)] = [(Y, G)] + [(X \setminus Y, G)]$;

3) if G is a subgroup of a finite group H and X is a G-variety, then $[(X,G)] = [(\mathrm{ind}_G^H X, H)]$.

The multiplication in $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ is defined by the Cartesian product of the varieties and of the groups acting on them.

It is convenient to discuss some properties of the universal Euler characteristic in a purely topological setting. For that we consider a class of "nice" topological spaces with finite group actions. The notion of an equivariant CW-complex was introduced in [19]. An equivariant CW-complex with a finite group G action is a CW-complex possessing, in particular, the following property: if $g\sigma = \sigma$ for a cell σ of the complex, then $g_{|\sigma}$ is the identity transformation.

Definition 6 *An* equivariant cell complex *is an invariant locally closed union of cells in a finite equivariant (with respect to a finite group) CW-complex.*

A quasi-projective G-variety or a (real) semialgebraic G-set (G is a finite group) can be represented as an equivariant cell complex. For an equivariant cell complex its Euler characteristic, equivariant Euler characteristic, orbifold Euler characteristic, ... are well defined: see below.

4 Additive invariants of V-manifolds

There are a number of additive invariants defined for V-manifolds.

For a G-space X (G is a finite group) and for a point $x \in X$, denote by $G_x = \{g \in G : gx = x\}$ the isotropy subgroup of x. For a subgroup $H \subset G$, denote by $X^H = \{x \in X : hx = x \text{ for all } h \in H\}$ the fixed point set of the subgroup H and by $X^{(H)} = \{x \in X : G_x = H\}$ the subspace of points with the isotropy subgroup H ($X^{(H)} \subset X^H$). For a conjugacy class $[H]$ of subgroups of G, let $X^{([H])} = \{x \in X : G_x \in [H]\}$. Let $\mathcal{G}$ be the set of the isomorphism classes of finite groups.

Let Q be a (tame) V-manifold. For each point $x \in Q$ one associates the isotropy group G_x. For a finite group G, let $Q^{(G)} = \{x \in Q : G_x \cong G\}$. One can see that the V-manifold $Q^{(G)}$ is a global quotient (under an action of the group G). Moreover, its reduction is the usual (C^∞-) manifold (with the action of the trivial group).

The *Euler–Satake characteristic* of Q ([17]) is defined by

$$\chi^{\mathrm{ES}}(Q) = \sum_{\{G\} \in \mathcal{G}} \frac{1}{|G|} \chi(Q^{(G)}). \qquad (2)$$

The *orbifold Euler characteristic* (defined in, e. g., [1], [13]) can be defined for a V-manifold by

$$\chi^{\rm orb}(Q) = \sum_{\{G\}\in\mathcal{G}} \chi^{\rm orb}(G/G, G)\cdot\chi(Q^{(G)})\,,$$

where $\chi^{\rm orb}(G/G, G)$ is the orbifold Euler characteristic of the one-point G-set G/G (in the sense of [1], [13]). If all the isotropy groups of points of the V-manifold Q are Abelian, one has

$$\chi^{\rm orb}(Q) = \sum_{\{G\}\in\mathcal{G}} |G|\cdot\chi(Q^{(G)})\,.$$

The *higher order* orbifold *Euler characteristics* $\chi^{(k)}(X, G)$ of a G-space (X, G) were defined in [2], [18]. For $k = 0, 1$, one has $\chi^{(0)}(X, G) = \chi(X/G)$, $\chi^{(1)}(X, G) = \chi^{\rm orb}(X, G)$. (We follow the numbering used in [18].) For a V-manifold they can be defined by

$$\chi^{(k)}(Q) = \sum_{\{G\}\in\mathcal{G}} \chi^{(k)}(G/G, G)\cdot\chi(Q^{(G)})\,.$$

If all the isotropy groups of points of the V-manifold Q are Abelian, one has

$$\chi^{(k)}(Q) = \sum_{\{G\}\in\mathcal{G}} |G|^k\cdot\chi(Q^{(G)})\,.$$

One can see that the Euler–Satake characteristic (2) can be regarded as the Euler characteristic of order (-1). This fits to the definition of the Γ–Euler–Satake characteristic $\chi^{\rm ES}_{\Gamma}(Q)$ of a V-manifold Q for a group Γ in [7]: for $\Gamma = \mathbb{Z}^{k+1}$ one gets the Euler characteristic of order k; for $\Gamma = \{1\}$ (i. e., $\Gamma = \mathbb{Z}^0$), one gets the Euler–Satake characteristic.

All these characteristics possess the additivity and the multiplicativity properties: if Q' is a (closed) V-submanifold of a V-manifold Q, one has $\chi^\bullet(Q) = \chi^\bullet(Q') + \chi^\bullet(Q\setminus Q')$; if Q_1 and Q_2 are V-manifolds, one has $\chi^\bullet(Q_1\times Q_2) = \chi^\bullet(Q_1)\cdot\chi^\bullet(Q_2)$. (Here $\chi^\bullet$ means $\chi^{\rm ES}$, $\chi^{\rm orb}$, …)

All these invariants can be defined on the Grothendieck ring $K_0^{\rm fGr}({\rm Var}_{\mathbb{C}})$ of quasi-projective varieties with finite groups actions so that they are ring homomorphism from $K_0^{\rm fGr}({\rm Var}_{\mathbb{C}})$ to the ring $\mathbb{Z}$ of integers (see [12] where this is shown for the orbifold Euler characteristic and for the Euler characteristics of higher orders). Moreover, all of them can be defined for equivariant cell complexes. For example, if X is an equivariant cell complex with an action of a finite group G, then its orbifold Euler characteristic can be defined by the equation

$$\chi^{\rm orb}(X, G) = \frac{1}{|G|}\sum_{\substack{(g,h)\in G\times G:\\ gh=hg}} \chi(X^{\langle g,h\rangle})\,,$$

where $\langle g, h\rangle$ is the subgroup of G generated by g and h, or by the equation

$$\chi^{\mathrm{orb}}(X, G) = \sum_{[H]\in \mathrm{conjsub}\, G} \chi(X^{([H])}/G)\chi^{\mathrm{orb}}(G/H, G)\,,$$

where $\mathrm{conjsub}\, G$ is the set of conjugacy classes of subgroups of G.

5 The universal Euler characteristic

Let $\mathcal{G}$ be the set of all isomorphisms classes of finite groups and let $\mathcal{R}$ be be the free Abelian group generated by the elements T^G corresponding to the classes $\{G\} \in \mathcal{R}$. We shall write an element of $\mathcal{R}$ as a finite sum of the form $\sum_{\{G\}\in\mathcal{G}} a_G T^G$, where $a_G \in \mathbb{Z}$. One has a natural multiplication on $\mathcal{R}$ defined by $T^{G'} \cdot T^{G''} = T^{G'\times G''}$. Thus $\mathcal{R}$ is a ring. According to the Krull–Schmidt theorem each finite group has a unique representation as the Cartesian product of indecomposable groups. Let $\mathcal{G}_{ind}$ be the set of the isomorphisms classes of indecomposable finite groups. The Krull–Schmidt theorem implies that $\mathcal{R}$ is the polynomial ring $\mathbb{Z}[T_G]$ in the variables T_G corresponding to the isomorphisms classes of the indecomposable finite groups. (If a finite group G has the decomposition $G \cong \prod_{i=1}^{r} G(i)$ with indecomposable $G(i)$, one has $T^G = \prod_{i=1}^{r} T_{G(i)}$.)

Definition 7 *The universal Euler characteristic of (tame) V-manifold Q is defined by*

$$\chi^{\mathrm{un}}(Q) = \sum_{\{G\}\in\mathcal{G}} \chi(Q^{(G)}) T^G \in \mathcal{R}. \tag{3}$$

It is not difficult to see that χ^{un} is an additive and multiplicative invariant of V-manifolds.

Another interpretation of the ring $\mathcal{R}$ is the following one: it is the subring of the Grothendieck ring $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ of quasi-projective varieties with finite groups actions generated by the finite sets, i. e., by zero-dimensional varieties. In terms of the description/definition of the ring $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ in [12], an element $a = \sum_{\{G\}\in\mathcal{G}} a_G T^G \in \mathcal{R}$ can be represented by the (virtual) set consisting of $\sum_{G\in\mathcal{G}} a_G$ points so that on a_G of them (a_G may be negative) one has the trivial action of the group G. In terms of the description given above, an element a can be represented by a pair (X, G^*), where G^* is a group containing all the groups G with $a_G \neq 0$ and X is the union over $\{G\} \in \mathcal{G}$ of the (finite) G^*-sets which consist of a_G copies of G^*/G with the natural G^*-action.

The universal Euler characteristic $\chi^{\mathrm{un}}(\bullet)$ can be defined for elements of the Grothendieck ring $K_0^{\mathrm{fGr}}(\mathrm{Var}_{\mathbb{C}})$ of quasi-projective varieties with finite groups actions.

Moreover it can be defined for equivariant cell complexes. For an equivariant cell complex X with an action of a finite group G, $\chi^{\rm un}(X,G)$ can be defined by the equation

$$\chi^{\rm un}(X,G) = \sum_{[H]\in {\rm conjsub}\, G} \chi(X^{([H])}/G)T^H.$$

In other terms it can be defined in the following way. Let Σ_k be the set of cells of dimension k in X. It is a finite G-set. Then

$$\chi^{\rm un}(X,G) = \sum_k (-1)^k[(\Sigma_k, G)] \in \mathcal{R}.$$

One can show that the universal Euler characteristic of equivariant cell complexes possesses the following properties:

1) additivity: if $(Y;G)$ is a closed G-invariant subcomplex of (X,G), then

$$\chi^{\rm un}(X,G) = \chi^{\rm un}(Y,G) + \chi^{\rm un}(X\setminus Y, G)\,;$$

2) multiplicativity: if (X',G') and (X'',G'') are two equivariant cell complexes, then

$$\chi^{\rm un}(X'\times X'', G'\times G'') = \chi^{\rm un}(X',G')\cdot\chi^{\rm un}(X'',G'')\,;$$

3) the induction relation: if G is a subgroup of H, then

$$\chi^{\rm un}({\rm ind}_G^H X, H) = \chi^{\rm un}(X,G).$$

The relations 1) and 3) permit to define the universal Euler characteristic $\chi^{\rm un}$ as a function on the Grothendieck ring $K_0^{\rm fGr}({\rm Var}_{\mathbb{C}})$ (with values in $\mathcal{R}$). The relations 1) and 2) mean that it is a ring homomorphism to $\mathcal{R}$.

Let us give a statement which explains the word "universal" in the name of $\chi^{\rm un}$.

Theorem 1 *If I is an additive invariant of (tame) V-manifolds with values in an Abelian group R, then there exists a unique homomorphism of Abelian groups $r:\mathcal{R}\to R$ such that $I(\bullet) = r(\chi^{\rm un}(\bullet))$. If R is a ring and I is multiplicative, then r is a ring homomorphism.*

Proof. Let Q be a tame V-manifold. For a finite group (or rather for an isomorphism class of finite groups) G, let $Q^{(G)}$ be the set of points $x\in X_Q$ with isotropy group isomorphic to G. The fact that Q is assumed to be tame implies that there are finitely many classes G such that $Q^{(G)}\neq\emptyset$. ($Q^{(G)}$ is a non-closed V-submanifold of Q whose reduction is a usual C^∞-manifold.)

The set $\mathcal{G}$ of isomorphism classes of finite groups is a partially order set. Let G be a minimal element from $\mathcal{G}$ with $Q^{(G)} \neq \emptyset$. By additivity we have $I(Q) = I(Q^{(G)}) + I(Q \setminus Q^{(G)})$. Iterating this equation we get $I(Q) = \sum_{G \in \mathcal{G}} I(Q^{(G)})$. Since the reduction of $Q^{(G)}$ is the usual C^∞-manifold, due to Proposition 1 we have $I(Q^{(G)}) = \chi(Q^{(G)})\tau_G$ with $\tau_G = I(T^G) \in R$. It is easy to see that the group homomorphism $r : \mathcal{R} \to R$ which sends the universal Euler characteristic $\chi^{\rm un}(\bullet)$ to $I(\bullet)$ is defined by $r(T^G) = \tau_G$.

The multiplicativity of r, for I being multiplicative, is obvious. $\square$

Also $\chi^{\rm un}(\bullet)$ possesses the following universality properties for equivariant cell complexes.

Theorem 2 *Let I be an additive invariant of equivariant cell complexes with values in an Abelian group R possessing the induction property: $I(\mathrm{ind}_G^H X, H) = I(X, G)$ for finite groups $G \subset H$. Then there exists a unique homomorphism of Abelian groups $r : \mathcal{R} \to R$ such that $I(\cdot,\cdot) = r(\chi^{\rm un}(\cdot,\cdot))$. If R is a ring and I is multiplicative, then r is a ring homomorphism.*

Proof. Let Σ be the (finite) set of cells in an equivariant cell complex (X, G) (Σ is a G-set). The additivity property of I implies that $I(X, G) = \sum_{[\sigma] \in \Sigma/G} I(G\sigma, G)$, where $\sigma = \sigma^k$ is an open cell (of certain dimension k) in X: a representative of the orbit $[\sigma]$, $G\sigma$ is the union $\cup_{g \in G} g\sigma$ of the G-shifts of σ. Let G_{σ^k} be the isotropy group of the cell σ^k. (Let us recall that G_σ acts trivially on σ.) We have $G\sigma^k = (G/G_{\sigma^k}) \times \sigma^k$, where G/G_{σ^k} is a finite G-set. Just as in the proof of Proposition 1 we have

$$I(G\sigma^k, G) = (-1)^k I((G/G_{\sigma^k}) \times \{pt\}, G).$$

The induction property implies that $I((G/G_{\sigma^k}) \times \{pt\}, G) = I(G_{\sigma^k}/G_{\sigma^k}) \times \{pt\}, G_{\sigma^k})$. Let us denote $I(G/G \times \{pt\}, G)$ by τ_G. It is easy to see that the group homomorphism $r : \mathcal{R} \to R$ which sends $\chi^{\rm un}(\cdot,\cdot)$ to $I(\cdot,\cdot)$ is defined by $r(T^G) = \tau_G$. The multiplicativity of r in the case when I is multiplicative is obvious. $\square$

6 λ-structures on the ring $\mathcal{R}$ and the corresponding power structures

An analogue of the Macdonald equation for an (additive and multiplicative) invariant with values in a ring R can be formulated in terms of the so called *power structure* over the ring R: [9]. A power structure over a ring R is a method to define an expression of the form $(1 + a_1 t + a_2 t^2 + \ldots)^m$ with $a_i, m \in R$ as a power series form $1+tR[[t]]$ so that all properties of the usual exponential function hold. The notion of a power structure over a ring R is related with the notion of a λ-structure (sometimes called a pre-λ-structure) on R: [14].

We shall describe two λ-structures on the ring $\mathcal{R}$ adapted for the formulation of Macdonald type equations for the symmetric products and for the configuration spaces respectively. As it was explained, $\mathcal{R}$ is the ring of polynomials in the variables T_G corresponding to the isomorphism classes $\{G\}$ of indecomposable finite groups. The standard λ-structure on the polynomial ring $\mathbb{Z}[x_1, x_2, x_3, \ldots]$ (see, e. g., [14]) is defined in the following way: for

$$p(\overline{x}) = \sum_{\overline{k}} p_{\overline{k}} \overline{x}^{\overline{k}} \in \mathbb{Z}[x_1, x_2, x_3, \ldots],$$

one gives

$$\lambda_{p(\overline{x})}(t) = \prod_{\overline{k}} \left(1 - \overline{x}^{\overline{k}} t\right)^{-p_{\overline{k}}}. \tag{4}$$

Equation (4) follows directly from the equation for the λ-series corresponding to a monomial:

$$\lambda_{\overline{x}^{\overline{k}}}(t) = \left(1 - \overline{x}^{\overline{k}} t\right)^{-1}.$$

Natural λ-structures on the ring $\mathcal{R}$ are different ones. To define a λ-structure on $\mathcal{R}$, one can define the λ-series, say $\nu_{T^G}(t)$, for a monomial T^G from $\mathcal{R}$. Namely, if the series $\nu_{T^G}(t)$ is defined for all $\{G\} \in \mathcal{G}$ (so that $\nu_{T^G}(t) = 1 + T^G t + \ldots$), then the λ-series for an element $A = \sum_{\{G\} \in \mathcal{G}} a_G T^G \in \mathcal{R}$ ($a_G \in \mathbb{Z}$) is defined by

$$\nu_A(t) = \prod_{\{G\} \in \mathcal{G}} \left(\nu_{T^G}(t)\right)^{a_G}.$$

Let us first describe the λ-structure on $\mathcal{R}$ corresponding to the symmetric products of spaces. We shall call it the *symmetric product λ-structure*. This structure will be defined by a λ-series $\zeta_\bullet(t)$. For a finite group G, let $G_n = G \wr S_n = G^n \rtimes S_n$ be the corresponding wreath product. Let us define $\zeta_{T^G}(t)$ for the monomial T^G by the equation

$$\zeta_{T^G}(t) = 1 + \sum_{n=1}^{\infty} T^{G_n} t^n. \tag{5}$$

In particular,

$$\zeta_1(t) = 1 + \sum_{n=1}^{\infty} T^{S_n} t^n.$$

Remark. Pay attention that all the coefficients of the λ-series for a monomial are monomials as well.

Let us (partially) describe the series (5) in terms of the variables T_G for the polynomial ring $\mathcal{R}$ (G runs through isomorphism classes of indecomposable finite groups). For such a description the variables corresponding to the Abelian groups play a special role. Let $A_{p,k} \cong \mathbb{Z}_{p^k}$ (p is prime, $k \geq 1$) be the indecomposable finite Abelian groups (cyclic groups of orders p^k). For a group $G = \prod_{p,k}(A_{p,k})^{l_{p,k}} \prod_i G(i)^{k_i}$ with non-Abelian indecomposable finite groups $G(i)$ and for $n > 1$, one has

$$(\prod_{p,k} A_{p,k}^{l_{p,k}} \prod_i G(i)^{k_i})_n \cong \left(\prod_{p,k:p\not| n} A_{p,k}^{l_{p,k}}\right) \times \widehat{G}(n),$$

where $\widehat{G}(n)$ is an indecomposable non-Abelian group (depending on G of course): see [15] and also more precise statements in [6]. Therefore

$$\zeta_{\prod_{p,k} T_{A_{p,k}}^{l_{p,k}} \prod_i T_{G(i)}^{k_i}}(t) = 1 + \left(\prod_{p,k} T_{A_{p,k}}^{l_{p,k}} \prod_i T_{G(i)}^{k_i}\right) t + \sum_{n=2}^{\infty} \left(T_{\widehat{G}(n)} \prod_{p,k:p\not| n} T_{A_{p,k}}^{l_{p,k}}\right) t^n . \quad (6)$$

The described λ-structure on the ring $\mathcal{R}$ defines (in the usual way: see [9], [10]) a power structure over $\mathcal{R}$. We shall call it the *symmetric product power structure*. Let us recall that according to the construction of the power structure one has

$$(\zeta_1(t))^{T^G} = \zeta_{T^G}(t) \,.$$

The other λ-structure on $\mathcal{R}$ corresponds to the configuration spaces. We shall call it the *configuration space λ-structure*. This structure will be defined by a λ-series $\lambda_\bullet(t)$. As above it is sufficient to define this series for monomials. Let

$$\lambda_{T^G}(t) = 1 + T^G t \,. \quad (7)$$

In particular, $\lambda_1(t) = 1 + t$.

This λ-structure on $\mathcal{R}$ defines the corresponding *configuration space power structure* over $\mathcal{R}$. The described power structures (the symmetric product and the configuration space ones) over $\mathcal{R}$ are different: see computations in [12, page 17].

From [12] and the interpretation of $\mathcal{R}$ given above it follows that the configuration space power structure over $\mathcal{R}$ is effective in the following sense. Let $\mathcal{R}_+$ be the subsemiring of $\mathcal{R}$ consisting of polynomials in T_G with non-negatives coefficients. The effectiveness of the power structure means that if a_i and m are from $\mathcal{R}_+$, then all the coefficients of the series $(1 + a_1 t + a_2 t^2 + \ldots)^m$ belong to $\mathcal{R}_+$ as well.

Remark. The fact that this power structure is effective is not a direct consequence of the equation (7) for the λ-series. The effectiveness of the configuration space power structure is a consequence of an explicit equation for it: see [12, Equation 10]. The symmetric product power structure over $\mathcal{R}$ is not effective: see again [12, page 17].

7 Macdonald type equations for the universal Euler characteristics for symmetric products

The nth symmetric product S^nQ of a (tame) V-manifold Q is the V-manifold defined in the following way. The topological space endowed with the V-manifold structure S^nQ is the nth symmetric product S^nX_Q of the corresponding space X_Q for the V-manifold Q. Let

$$\underline{x} = (x_1, \ldots, x_1, \ldots, x_s, \ldots, x_s),$$

where x_i, $i = 1, \ldots, s$, is a point of X_Q with the multiplicity k_i in $\underline{x}$, $\sum_{i=1}^{s} k_i = n$, $x_i \neq x_j$ for $i \neq j$, be a point of S^nX_Q and let $(U_i, \tilde{U}_i, G(i), \varphi_i)$ be uniformizing systems for neighbourhoods U_i of the points x_i such that $U_i \cap U_j = \emptyset$ for $i \neq j$. Then the orbifold structure on S^nQ in a neighbourhood of $\underline{x}$ is defined by the uniformizing system $(S^{k_1}U_1 \times \ldots \times S^{k_s}U_s, \tilde{U}_1^{k_1} \times \ldots \times \tilde{U}_s^{k_s}, (G(1))_{k_1} \times \ldots \times (G(s))_{k_s}, \bar{\varphi})$, where $(G(i))_{k_i}$ is the wreath product $G(i) \wr S_{k_i}$ acting on the Cartesian power $\widetilde{U}_i^{k_i}$ in the usual way, $\bar{\varphi} = (\varphi_1 \times \ldots \times \varphi_1 \times \ldots \times \varphi_s \times \ldots \times \varphi_s)$.

Theorem 3 *For a (tame) V-manifold Q one has*

$$1 + \sum_{n=1}^{\infty} \chi^{\rm un}(S^nQ)t^n = \zeta_{\chi^{\rm un}(Q)}(t) = (\zeta_1(t))^{\chi^{\rm un}(Q)}, \qquad (8)$$

where the right hand side is written in terms of the symmetric product power structure over $\mathcal{R}$.

Let us recall that, if $\chi^{\rm un}(Q) = \sum_{\{G\}\in\mathcal{G}} a_G T^G$, then

$$\zeta_{\chi^{\rm un}(Q)}(t) = \prod_{\{G\}\in\mathcal{G}} \left(1 + T^G t + T^{G_2}t^2 + T^{G_3}t^3 + \ldots\right)^{a_G}.$$

Proof. Let us denote the left hand side of Equation (8) by $\xi_Q(t)$. If Q' is a closed V-submanifold of Q, one has $\xi_Q(t) = \xi_{Q'}(t)\xi_{Q\setminus Q'}(t)$. (This follows from the fact that S^nQ is the disjoint union of the V-manifolds $S^kQ' \times S^{n-k}(Q \setminus Q')$ for $0 \leq k \leq n$.) The representation of Q as the disjoint union of the V-submanifolds $Q^{(G)}$ for $G \in \mathcal{G}$

permits to prove the statement for $Q = MT^G$, where M is a C^∞-manifold (with the action of the trivial group). A representation of M as a cell complex permits to prove the statement for $Q = \sigma^k T^G$, where σ^k is an open cell of dimension k. The fact that a k-dimensional cell can be represented as the union of two k-dimensional cells and one $(k-1)$-dimensional cell implies that

$$\xi_{\sigma^k T^G}(t) = \left(\xi_{\sigma^0 T^G}(t)\right)^{(-1)^k}.$$

Therefore it is sufficient to show (8) for $Q = \sigma^0 T^G$. In this case we have

$$\xi_{\sigma^0 T^G}(t) = 1 + T^G t + T^{G_2} t^2 + T^{G_3} t^3 + \ldots = \zeta_{T^G}(t) = \left(\zeta_1(t)\right)^{T^G} = \left(\zeta_1(t)\right)^{\chi^{\rm un}(\sigma^0 T^G)}.$$

□

One has a Macdonald type equation for equivariant cell complexes (and therefore for representatives (X, G) of elements of the Grothendieck ring $K_0^{\rm fGr}({\rm Var}_{\mathbb{C}})$). For an equivariant cell complex (X, G), let (X^n, G_n) be the Cartesian power of the complex X with the standard action of the wreath product G_n.

Theorem 4 *For an equivariant cell complex (X, G), one has*

$$1 + \sum_{n=1}^{\infty} \chi^{\rm un}(X^n, G_n) t^n = \zeta_{\chi^{\rm un}(X,G)}(t) = \left(\zeta_1(t)\right)^{\chi^{\rm un}(X,G)}.$$

(the right hand side is written in terms of the symmetric product power structure).

The **proof** is essentially the same as the one of Theorem 3 with the only difference that the general case is reduced not to $\sigma^k T^G$, but to $\sigma^k \times (G/G_{\sigma^k})$.

8 Macdonald type equations for the universal Euler characteristics for configuration spaces

For a V-manifold Q, its nth configuration space $M_n Q$ is the V-manifold defined in the following way. The topological space endowed with the V-manifold structure $M_n Q$ is the nth configuration space $M_n X_Q = (X_Q^n \setminus \Delta)/S_n \subset S^n X_Q$ of the corresponding space X_Q for the V-manifold Q. The V-manifold structure on it is inherited from $S^n X_Q$. (Pay attention that one has to define uniformizing systems only for points $\underline{x} = (x_1, \ldots, x_n)$ with $x_i \neq x_j$ for $i \neq j$.)

Theorem 5 *For a (tame) V-manifold Q, one has*

$$1+\sum_{n=1}^{\infty}\chi^{\rm un}(M_nQ)t^n = \lambda_{\chi^{\rm un}(Q)}(t) = (1+t)^{\chi^{\rm un}(Q)}, \qquad (9)$$

where the right hand side is written in terms of the configuration space power structure over $\mathcal{R}$.

Let us recall that $\lambda_1(t) = 1+t$; if $\chi^{\rm un}(Q) = \sum_{\{G\}\in\mathcal{G}} a_G T^G$, then

$$\lambda_{\chi^{\rm un}(Q)}(t) = \prod_{\{G\}\in\mathcal{G}} \left(1+T^G t\right)^{a_G}.$$

An analogue of this equation for equivariant cell complex is given in the following statement. For equivariant cell complex (X, G) and for $n \geq 1$, let $\Delta_G \subset X^n$ be the big G-diagonal in X^n consisting of (ordered) n-tuples $(x_1, \ldots, x_n) \in X^n$ with at least two of the components x_i from the same G-orbit.

Theorem 6 *For an equivariant cell complex (X, G), one has*

$$1+\sum_{n=1}^{\infty}\chi^{\rm un}(X^n\setminus\Delta_G, G_n)t^n = \lambda_{\chi^{\rm un}(X,G)}(t) = (1+t)^{\chi^{\rm un}(X,G)}.$$

(the right hand side is written in the terms of the configuration space power structure).

Proofs of Theorems 5 and 6 are minor modifications of those of Theorems 3 and 4.

References

[1] M. Atiyah, G. Segal. On equivariant Euler characteristics. J. Geom. Phys. 6 (1989), no.4, 671–677.

[2] J. Bryan, J. Fulman. Orbifold Euler characteristics and the number of commuting m-tuples in the symmetric groups. Ann. Comb. 2 (1998), no.1, 1–6.

[3] W. Chen, Y. Ruan. Orbifold Gromov-Witten theory. in: Orbifolds in mathematics and physics (Madison, WI, 2001), 25–85, Contemp. Math., 310, Amer. Math. Soc., Providence, RI, 2002.

[4] L. Dixon, J. Harvey, C. Vafa, E. Witten. Strings on orbifolds. Nuclear Phys. B 261 (1985), no.4, 678–686.

[5] L. Dixon, J. Harvey, C. Vafa, E. Witten. Strings on orbifolds. II. Nuclear Phys. B 274 (1986), no.2, 285–314.

[6] M.R. Dixon, T.A. Fournelle. The indecomposability of certain wreath products indexed by partially ordered sets. Arch. Math. 43 (1984), 193–207.

[7] C. Farsi, Ch. Seaton. Generalized orbifold Euler characteristics for general orbifolds and wreath products. Algebr. Geom. Topol. 11 (2011), no.1, 523–551.

[8] S.M. Gusein-Zade. Equivariant analogues of the Euler characteristic and Macdonald type formulas. Russian Math. Surveys 72 (2017), no.1, 1–32.

[9] S.M. Gusein-Zade, I. Luengo, A. Melle-Hernández. A power structure over the Grothendieck ring of varieties. Math. Res. Lett. 11 (2004), 49–57.

[10] S.M. Gusein-Zade, I. Luengo, A. Melle-Hernández. On the power structure over the Grothendieck ring of varieties and its applications. Proc. Steklov Inst. Math. 258 (2007), no.1, 53–64.

[11] S.M. Gusein-Zade, I. Luengo, A. Melle-Hernández. Equivariant versions of higher order orbifold Euler characteristics. Mosc. Math. J. 16 (2016), no.4, 751–765.

[12] S.M. Gusein-Zade, I. Luengo, A. Melle-Hernández. Grothendieck ring of varieties with finite groups actions. arXiv: 1706.00918.

[13] F. Hirzebruch, T. Höfer. On the Euler number of an orbifold. Math. Ann. 286 (1990), no.1–3, 255–260.

[14] D. Knutson. λ-rings and the representation theory of the symmetric group. Lecture Notes in Mathematics, 308, Springer-Verlag, Berlin – New York, 1973.

[15] P.M. Neumann. On the structure of standard wreath products of groups. Math. Z. 84 (1964), 343–373.

[16] I. Satake. On a generalization of the notion of manifold. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 359–363.

[17] I. Satake. The Gauss-Bonnet theorem for V-manifolds. J. Math. Soc. Japan 9 (1957), 464–492.

[18] H. Tamanoi. Generalized orbifold Euler characteristic of symmetric products and equivariant Morava K-theory. Algebr. Geom. Topol. 1 (2001), 115–141.

[19] T. tom Dieck. Transformation groups. De Gruyter Studies in Mathematics, 8, Walter de Gruyter & Co., Berlin, 1987.

[20] C.E. Watts. On the Euler characteristic of polyhedra. Proc. Amer. Math. Soc. 13 (1962), 304–306.

Translated by S.M.Gusein-Zade.