

Higher-order orbifold Euler characteristics for compact Lie group actions

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We generalize the notions of the orbifold Euler characteristic and of the higher-order orbifold Euler characteristics to spaces with actions of a compact Lie group using integration with respect to the Euler characteristic instead of the summation over finite sets. We show that the equation for the generating series of the k th-order orbifold Euler characteristics of the Cartesian products of the space with the wreath products actions proved by Tamanoi for finite group actions and by Farsi and Seaton for compact Lie group actions with finite isotropy subgroups holds in this case as well.

Keywords: compact Lie group actions; orbifold Euler characteristic;
 wreath products; generating series; compact Lie group actions

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1. Introduction

Let X be a topological space (good enough; say, a quasi-projective variety) with the action of a finite group G . For a subgroup H of G , let $X^H = \{x \in X : Hx = x\}$ be the fixed-point set of H . The orbifold Euler characteristic $\chi^{\text{orb}}(X, G)$ of the G -space X is defined, for example, in [1, 8]:

$$\chi^{\text{orb}}(X, G) = \frac{1}{|G|} \sum_{\substack{(g_0, g_1) \in G \times G : \\ g_0 g_1 = g_1 g_0}} \chi(X^{\langle g_0, g_1 \rangle}) = \sum_{[g] \in G_*} \chi(X^{\langle g \rangle} / C_G(g)), \quad (1.1)$$

where G_* is the set of the conjugacy classes of the elements of G , $C_G(g) = \{h \in G : h^{-1}gh = g\}$ is the centralizer of g and $\langle g \rangle$ and $\langle g_0, g_1 \rangle$ are the subgroups generated by the corresponding elements. Here and below we use the additive Euler characteristic, i.e. the one defined through the cohomologies with compact support.

The higher-order Euler characteristics of (X, G) (alongside with some other generalizations) were defined in [1, 3, 10].

DEFINITION 1.1. The *orbifold Euler characteristic* $\chi^{(k)}(X, G)$ of order k of the G -space X is

$$\chi^{(k)}(X, G) = \frac{1}{|G|} \sum_{\substack{\mathbf{g} \in G^{k+1}: \\ g_i g_j = g_j g_i}} \chi(X^{\langle \mathbf{g} \rangle}) = \sum_{[g] \in G_*} \chi^{(k-1)}(X^{\langle g \rangle}, C_G(g)), \quad (1.2)$$

where $\mathbf{g} = (g_0, g_1, \dots, g_k)$, $\langle \mathbf{g} \rangle$ is the subgroup generated by $g_0, g_1, \dots, g_k$ and $\chi^{(0)}(X, G)$ is defined as $\chi(X/G)$.

The usual orbifold Euler characteristic $\chi^{\text{orb}}(X, G)$ is the orbifold Euler characteristic of order 1, $\chi^{(1)}(X, G)$.

A generalization of this definition for the orbifold Hodge–Deligne polynomial (for $k = 1$) was introduced by Batyrev in [2]. A ‘motivic version’ of it – the higher-order generalized Euler characteristics with values in the ring $K_0(\text{Var}_{\mathbb{C}})[\mathbb{L}^{1/m}]$, where $K_0(\text{Var}_{\mathbb{C}})$ is the Grothendieck ring of complex quasi-projective varieties, $\mathbb{L}$ is the class of the complex affine line and m runs through positive integers – was defined in [7].

Let $G^n = G \times \dots \times G$ be the Cartesian product of a group G . The symmetric group S_n acts on G^n by permutation of the factors: $s(g_1, \dots, g_n) = (g_{s^{-1}(1)}, \dots, g_{s^{-1}(n)})$. The *wreath product* $G_n = G \wr S_n$ is the semidirect product of the groups G^n and S_n defined by the described action. Namely, the multiplication in the group G_n is given by the formula $(\mathbf{g}, s)(\mathbf{h}, t) = (\mathbf{g} \cdot s(\mathbf{h}), st)$, where $\mathbf{g}, \mathbf{h} \in G^n$, $s, t \in S_n$. The group G^n is a normal subgroup of the group G_n via the identification of $\mathbf{g} \in G^n$ with $(\mathbf{g}, 1) \in G_n$. For a space X with a G -action, the corresponding action of the group G_n on the Cartesian product X^n is given by the formula

$$((g_1, \dots, g_n), s)(x_1, \dots, x_n) = (g_1 x_{s^{-1}(1)}, \dots, g_n x_{s^{-1}(n)}),$$

where $x_1, \dots, x_n \in X$, $g_1, \dots, g_n \in G$, $s \in S_n$. One can see that (at least for compact G) the quotient X^n/G_n is naturally isomorphic to the symmetric power $S^n(X/G) = (X/G)^n/S_n$ of the quotient X/G . A formula for the generating series of the k th-order orbifold Euler characteristics of the pairs (X^n, G_n) in terms of the k th-order orbifold Euler characteristic of the G -space X was given in [10] (see also [3]); see theorem 3.3.

A generalization of this formula (for $k = 1$) for the orbifold Hodge–Deligne polynomial (for finite G) was given in [12]. The corresponding ‘motivic’ version can be found in [6]. A version of it for the generalized Euler characteristic of order k was formulated in [7].

A ‘non-finite version’ of the sum over a finite set is the integral with respect to the Euler characteristic [11] (see also [5]). Here we show that this notion permits us to define an analogue of the orbifold Euler characteristics of order k for a space X with an action of a compact Lie group G and prove that the equation from [10] for the generating series of the orbifold Euler characteristics of order k of the wreath products holds in this case as well. The case when all the isotropy subgroups of the G -action are finite was studied in [4]. (Another definition was used therein, which is not appropriate for actions with non-finite isotropy subgroups. It was also assumed that the G -space was a manifold. However, this was connected with the fact that the Farsi and Seaton worked in the framework of the orbifold theory.)

It appeared that the first equations in the definitions (1.1) and (1.2) were less convenient for the proofs of the formulae for the generating series in [10, 12] and were not appropriate for the definition of their motivic version in [7]. In what follows we shall use the second equations as the base for the generalization.

2. Orbifold Euler characteristic of order k for actions of compact Lie groups

Let X be a topological space (good enough; say, a quasi-projective variety) endowed with the action of a compact Lie group G . We assume that the action of G on X has finitely many orbit types and, moreover, the space of orbits of a fixed orbit type is good enough so that its Euler characteristic makes sense. For example, this holds if the action (i.e. the map $G \times X \rightarrow X$) is algebraic. (Each compact Lie group is a real algebraic manifold.) Let G_* be the space of the conjugacy classes of the elements of G . The space G_* is a finite CW-complex.

DEFINITION 2.1. The *orbifold Euler characteristic* of a G -space X (i.e. of the pair (X, G)) is

$$\chi^{\text{orb}}(X, G) := \int_{G_*} \chi(X^{\langle g \rangle} / C_G(g)) \, d\chi. \quad (2.1)$$

DEFINITION 2.2. The *orbifold Euler characteristic of order k* of a G -space X (i.e. of the pair (X, G)) is

$$\chi^{(k)}(X, G) := \int_{G_*} \chi^{(k-1)}(X^{\langle g \rangle}, C_G(g)) \, d\chi, \quad (2.2)$$

where $\chi^{(0)}(X, G)$ is $\chi(X/G)$.

The orbifold Euler characteristic (2.1) is the orbifold Euler characteristic of order 1: $\chi^{(1)}(X, G)$.

For a closed subgroup $H \subset G$, let $X^{(H)}$ be the set of points x in X with the isotropy subgroup $G_x = \{g \in G: gx = x\}$ coinciding with H and let $X^{([H])}$ be the set of points x with the isotropy subgroup conjugate to H . The additivity of the orbifold Euler characteristic of order k with respect to a partitioning of the space into G -invariant parts implies the following statement.

PROPOSITION 2.3. *We have*

$$\chi^{(k)}(X, G) = \sum_{[H] \in \text{Conjsub } G} \chi(X^{([H])} / G) \chi^{(k)}(G/H, G).$$

EXAMPLE 2.4. $\chi^{(k)}(S^1 / \mathbb{Z}_m, S^1) = m \chi^{(k-1)}(S^1 / \mathbb{Z}_m, S^1) = m^k$; and, for each $k > 0$, $\chi^{(k)}(S^1 / S^1, S^1) = 0$.

EXAMPLE 2.5. The conjugacy classes of the elements of the group $O(2)$ are $[T_\alpha]$, $0 \leq \alpha \leq \pi$, and $[S]$, where $T_\alpha \in SO(2)$ is the rotation by the angle α , $S \in O(2) \setminus SO(2)$ is the symmetry with respect to a line. The centralizer of T_α is $O(2)$

for $\alpha = 0, \pi$ and $\mathrm{SO}(2)$ for $0 < \alpha < \pi$. Therefore, one has, for m odd,

$$\begin{aligned}\chi^{(k)}(\mathrm{O}(2)/\mathbb{Z}_m, \mathrm{O}(2)) &= \chi^{(k-1)}(\mathrm{O}(2)/\mathbb{Z}_m, \mathrm{O}(2)) + \frac{m-1}{2} \chi^{(k-1)}(\mathrm{O}(2)/\mathbb{Z}_m, \mathrm{SO}(2)) \\ &= \chi^{(k-1)}(\mathrm{O}(2)/\mathbb{Z}_m, \mathrm{O}(2)) + (m-1)m^{k-1} \\ &= m^k;\end{aligned}$$

for m even,

$$\begin{aligned}\chi^{(k)}(\mathrm{O}(2)/\mathbb{Z}_m, \mathrm{O}(2)) &= 2\chi^{(k-1)}(\mathrm{O}(2)/\mathbb{Z}_m, \mathrm{O}(2)) \\ &\quad + \frac{m-2}{2} \chi^{(k-1)}(\mathrm{O}(2)/\mathbb{Z}_m, \mathrm{SO}(2)) \\ &= 2\chi^{(k-1)}(\mathrm{O}(2)/\mathbb{Z}_m, \mathrm{O}(2)) + (m-2)m^{k-1} \\ &= m^k; \\ \chi^{(k)}(\mathrm{O}(2)/\mathrm{SO}(2), \mathrm{O}(2)) &= 2\chi^{(k-1)}(\mathrm{O}(2)/\mathrm{SO}(2), \mathrm{O}(2)) \\ &\quad - \chi^{(k-1)}(\mathrm{O}(2)/\mathrm{SO}(2), \mathrm{SO}(2)) \\ &= 2\chi^{(k-1)}(\mathrm{O}(2)/\mathrm{SO}(2), \mathrm{O}(2)) \\ &= 2^k.\end{aligned}$$

3. Generating series of the orbifold Euler characteristics of the wreath products

To prove (3.2) for the generating series of the orbifold Euler characteristics of order k for the Cartesian products of a G -space with the wreath products actions, we shall use two technical lemmas (cf. [7, 10]).

LEMMA 3.1. *Let X and X' be two spaces with actions of compact Lie groups G' and G'' , respectively. Then $X' \times X''$ is a $G' \times G''$ -space and one has*

$$\chi^{(k)}(X' \times X'', G' \times G'') = \chi^{(k)}(X', G') \cdot \chi^{(k)}(X'', G''). \quad (3.1)$$

The proof is obvious.

LEMMA 3.2 (cf. [10, lemma 4-1]). *Let X be a G -space and let c be an element of the centre of G acting trivially on X . Let $G \cdot \langle a \rangle$ be the group generated by G and the additional element a commuting with all the elements of G and such that $\langle a \rangle \cap G = \langle c \rangle$, $c = a^r$. The space X can be regarded as a $(G \cdot \langle a \rangle)$ -space if one assumes that a acts trivially on X . In the situation described one has*

$$\chi^{(k)}(X, G \cdot \langle a \rangle) = r^k \cdot \chi^{(k)}(X, G).$$

Proof. We shall use the induction on k . For $k = 0$ this is obvious (since $\chi^{(0)}(X, G) = \chi(X/G)$). Each conjugacy class of the elements from $G \cdot \langle a \rangle$ is of the form $[g]a^s$, where $[g] \in G_*$, $0 \leq s < r$. The fixed-point set of ga^s coincides with $X^{\langle g \rangle}$, i.e. $X^{\langle ga^s \rangle} =$

$X^{\langle g \rangle}$ (since a acts trivially). The centralizer $C_{G \cdot \langle a \rangle}(ga^s)$ is $C_G(g) \cdot \langle a \rangle$. Therefore,

$$\begin{aligned}\chi^{(k)}(X, G \cdot \langle a \rangle) &= \int_{(G \cdot \langle a \rangle)_*} \chi^{(k-1)}(X^{\langle ga^s \rangle}, C_{G \cdot \langle a \rangle}(ga^s)) \, d\chi \\ &= r \int_{G_*} \chi^{(k-1)}(X^{\langle g \rangle}, C_G(g) \cdot \langle a \rangle) \, d\chi \\ &= r \cdot r^{k-1} \int_{G_*} \chi^{(k-1)}(X^{\langle g \rangle}, C_G(g)) \, d\chi = r^k \chi^{(k)}(X, G).\end{aligned}$$

□

THEOREM 3.3. *We have*

$$\sum_{n \geq 0} \chi^{(k)}(X^n, G_n) \cdot t^n = \left(\prod_{r_1, \dots, r_k \geq 1} (1 - t^{r_1 r_2 \dots r_k})^{r_2 r_3^2 \dots r_k^{k-1}} \right)^{-\chi^{(k)}(X, G)}. \quad (3.2)$$

Proof. The proof will use the induction on the order k similar to those in [7, 10].

For $k = 0$ one has $\chi^{(0)}(X, G) = \chi(X/G)$, $X^n/G_n \cong S^n(X/G)$ and (3.2) is a particular case (for $Y = X/G$) of the well-known Macdonald formula (see [9]):

$$\sum_{n \geq 0} \chi(S^n Y) \cdot t^n = (1 - t)^{-\chi(Y)}.$$

Suppose that the statement holds for the orbifold Euler characteristic of order $(k-1)$.

Let $A_q := \{[g] \in G_* : \chi^{(k-1)}(X^{\langle g \rangle}, C_G(g)) = q\}$. One has $G_* = \bigsqcup_q A_q$. Due to our assumptions, only finitely many subspaces A_q are not empty. According to the definition (2.2), $\chi^{(k)}(X, G) = \sum_q q \chi(A_q)$.

We have

$$\sum_{n \geq 0} \chi^{(k)}(X^n, G_n) \cdot t^n = \sum_{n \geq 0} \int_{(G_n)_*} \chi^{(k-1)}((X^n)^{\langle (\mathbf{g}, s) \rangle}, C_{G_n}((\mathbf{g}, s))) \, d\chi \cdot t^n.$$

A description of the conjugacy classes $[(\mathbf{g}, s)]$ in G_n can be found, for example, in [10]. The conjugacy class of an element $a = (\mathbf{g}, s) \in G_n$ ($\mathbf{g} = (g_1, \dots, g_n)$, $s \in S_n$) is completely characterized by its type. Let $z = (i_1, \dots, i_r)$ be one of the cycles in the permutation s . The *cycle product* of the element a corresponding to the cycle z is the product $g_{i_r} g_{i_{r-1}} \dots g_{i_1} \in G$. (The conjugacy class of the cycle product is well defined by the element $\mathbf{g}$ and the cycle z of the permutation s .) For $[c] \in G_*$ and $r \geq 1$, let $m_r(c)$ be the number of r -cycles in the permutation s whose cycle products belong to $[c]$. (There are finitely many pairs (c, r) with $m_r(c) \neq 0$.) One has

$$\sum_{[c] \in G_*, r \geq 1} r m_r(c) = n.$$

The collection $\{m_r(c)\}_{r,c}$ (or equivalently, the map $G_* \rightarrow \mathbb{Z}_{\geq 0}^\infty$, $[c] \mapsto (m_1(c), m_2(c), \dots)$) is called the *type* of the element $a = (\mathbf{g}, s) \in G_n$. Two elements of the group G_n are conjugate to each other if and only if they are of the same type.

For an element $a = (\mathbf{g}, s) \in G_n$, let the number of different $c \in G_*$ that are cycle products of a be equal to ℓ . One has a map from $\bigsqcup_{n \geq 0} (G_n)_*$ to $\bigsqcup_{\ell \geq 0} ((G_*^\ell \setminus \Delta)/S_\ell)$,

where Δ is the ‘large diagonal’ in G_*^ℓ , i.e. the space of points $(c_1, \dots, c_\ell) \in G_*^\ell$ with at least two coinciding components. The restriction of this map to $(G_n)_*$ has finite preimages of points. However, the preimage of each point is the set $(\mathbb{Z}^\infty \setminus \{\mathbf{0}\})^\ell$ consisting of all possible finite sequences $(m_1(c_i), m_2(c_i), \dots)$, for $i = 1, \dots, \ell$, such that for each i the sequence $m_1(c_i), m_2(c_i), \dots$ is different from $\mathbf{0} = (0, 0, \dots)$. Let us denote the set of inappropriate sequences $\{m_r(c_i)\}$ by “ $\underline{0}$ ”. Thus,

$$\begin{aligned} & \sum_{n \geq 0} \chi^{(k)}(X^n, G_n) \cdot t^n \\ &= \sum_{\ell \geq 0} \frac{1}{\ell!} \int_{G_*^\ell \setminus \Delta} \sum_{\{m_r(c_i)\} \setminus \underline{0}} \chi^{(k-1)}((X^n)^{\langle(\mathbf{g}, s)\rangle}, C_{G_n}((\mathbf{g}, s))) \cdot t^{\sum_{i,r} r m_r(c_i)} d\chi, \end{aligned}$$

where $(\mathbf{g}, s)$ is a representative of the conjugacy class of the elements of G_n ($n = \sum_{i,r} r m_r(c_i)$) with the type defined by $(c_1, \dots, c_\ell) \in G_*^\ell \setminus \Delta$ and by the sequences $\{m_r(c_i)\}$. In [10] it was shown that the space $(X^n)^{\langle(\mathbf{g}, s)\rangle}$ is canonically isomorphic to the product

$$\prod_{i=1}^{\ell} \prod_{r \geq 1} (X^{\langle c_i \rangle})^{m_r(c_i)} \quad (3.3)$$

and the centralizer $C_{G_n}((\mathbf{g}, s))$ is isomorphic to the product

$$\prod_{i=1}^{\ell} \prod_{r \geq 1} \{(C_G(c_i) \cdot \langle a_{r,c_i} \rangle) \wr S_{m_r(c_i)}\},$$

where the factors in the group act on the product (3.3) componentwise, $C_G(c_i) \cdot \langle a_{r,c_i} \rangle$ is the group generated by $C_G(c_i)$ and an element $a_{r,c_i} \in G_n$ commuting with all the elements of $C_G(c_i)$ and such that $a_{r,c_i}^r = c_i$, $\langle a_{r,c_i} \rangle \cap C_G(c_i) = \langle c_i \rangle$, a_{r,c_i} acts on $(X^{\langle c_i \rangle})^{m_r(c_i)}$ trivially. Therefore,

$$\begin{aligned} & \sum_{n \geq 0} \chi^{(k)}(X^n, G_n) \cdot t^n \\ &= \sum_{\ell \geq 0} \frac{1}{\ell!} \int_{G_*^\ell \setminus \Delta} \sum \chi^{(k-1)} \left(\prod_{i=1}^{\ell} \prod_{r \geq 1} (X^{\langle c_i \rangle})^{m_r(c_i)}, \right. \\ & \quad \left. \prod_{i=1}^{\ell} \prod_{r \geq 1} \{(C_G(c_i) \cdot \langle a_{r,c_i} \rangle) \wr S_{m_r(c_i)}\} \right) \cdot t^{\sum_{i,r} r m_r(c_i)} d\chi \\ &= \sum_{\ell \geq 0} \frac{1}{\ell!} \int_{G_*^\ell \setminus \Delta} \sum_{i=1}^{\ell} \prod_{r \geq 1} \chi^{(k-1)}((X^{\langle c_i \rangle})^{m_r(c_i)}, \{(C_G(c_i) \cdot \langle a_{r,c_i} \rangle) \wr S_{m_r(c_i)}\}) \\ & \quad \cdot t^{\sum_{i,r} r m_r(c_i)} d\chi \\ &= \sum_{\ell \geq 0} \frac{1}{\ell!} \int_{G_*^\ell \setminus \Delta} \prod_{i=1}^{\ell} \left[\prod_{r \geq 1} \left(\sum_{m=0}^{\infty} \chi^{(k-1)}((X^{\langle c_i \rangle})^m, (C_G(c_i) \cdot \langle a_{r,c_i} \rangle) \wr S_m) t^{rm} \right) - 1 \right] d\chi, \end{aligned}$$

where the sums without limits are over $\{m_r(c_i)\} \setminus \underline{0}$, and for each $i = 1, \dots, \ell$ the summand 1 is subtracted since not all $m_r(c_i)$ should be equal to zero. Using

induction one has

$$\begin{aligned}
& \sum_{n \geq 0} \chi^{(k)}(X^n, G_n) \cdot t^n \\
&= \sum_{\ell \geq 0} \frac{1}{\ell!} \int_{G_*^\ell \setminus \Delta} \prod_{i=1}^{\ell} \\
& \quad \left[\prod_{r \geq 1} \left(\left(\prod_{r_1, \dots, r_{k-1} \geq 1} (1 - t^{r r_1 \dots r_{k-1}}) \right)^{r_2 \cdot r_3^2 \dots r_{k-1}^{k-2}} \right)^{-\chi^{(k-1)}(X^{\langle c_i \rangle}, C_G(c_i) \cdot \langle a_{r, c_i} \rangle)} - 1 \right] d\chi \\
&= \sum_{\ell \geq 0} \frac{1}{\ell!} \int_{G_*^\ell \setminus \Delta} \prod_{i=1}^{\ell} \\
& \quad \left[\prod_{r \geq 1} \left(\left(\prod_{r_1, \dots, r_{k-1} \geq 1} (1 - t^{r r_1 \dots r_{k-1}}) \right)^{r_2 \cdot r_3^2 \dots r_{k-1}^{k-2}} \right)^{-r^{k-1} \chi^{(k-1)}(X^{\langle c_i \rangle}, C_G(c_i))} - 1 \right] d\chi.
\end{aligned}$$

One has

$$G_*^\ell \setminus \Delta = \prod_{\{\ell_q\}: \sum \ell_q = \ell} \frac{\ell!}{\prod \ell_q!} \prod_q (A_q^{\ell_q} \setminus \Delta),$$

where the coefficient $\ell! / \prod \ell_q!$ is the number of possible decompositions of ℓ elements into groups of sizes ℓ_q . Therefore,

$$\begin{aligned}
& \sum_{n \geq 0} \chi^{(k)}(X^n, G_n) \cdot t^n \\
&= \prod_q \sum_{\ell_q \geq 0} \frac{1}{\ell_q!} \int_{A_q^{\ell_q} \setminus \Delta} \prod_{i=1}^{\ell_q} \\
& \quad \left[\prod_{r \geq 1} \left(\left(\prod_{r_1, \dots, r_{k-1} \geq 1} (1 - t^{r r_1 \dots r_{k-1}}) \right)^{r_2 \cdot r_3^2 \dots r_{k-1}^{k-2}} \right)^{-r^{k-1} \chi^{(k-1)}(X^{\langle c_i \rangle}, C_G(c_i))} - 1 \right] d\chi \\
&= \prod_q \left(\sum_{\ell_q=0}^{\infty} \frac{\chi(A_q)(\chi(A_q) - 1) \dots (\chi(A_q) - \ell_q + 1)}{\ell_q!} \right. \\
& \quad \times \left. \left[\prod_{r_1, \dots, r_k \geq 1} ((1 - t^{r_1 \dots r_k})^{r_2 \cdot r_3^2 \dots r_k^{k-2}})^{-r_k^{k-1} q} - 1 \right]^{\ell_q} \right) \\
&= \prod_q \left(\prod_{r_1, \dots, r_k \geq 1} (1 - t^{r_1 \dots r_k})^{-r_2 \cdot r_3^2 \dots r_k^{k-1} q} \right)^{\chi(A_q)} \\
&= \left(\prod_{r_1, \dots, r_k \geq 1} (1 - t^{r_1 \dots r_k})^{r_2 \cdot r_3^2 \dots r_k^{k-1}} \right)^{-\sum_q q \chi(A_q)} \\
&= \left(\prod_{r_1, \dots, r_k \geq 1} (1 - t^{r_1 \dots r_k})^{r_2 \cdot r_3^2 \dots r_k^{k-1}} \right)^{-\chi^{(k)}(X, G)}.
\end{aligned}$$

In the intermediate step we use the standard formula

$$\sum_{\ell} \frac{M(M-1)\cdots(M-\ell+1)}{\ell!} T^{\ell} = (1+T)^M.$$

□

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