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LINKS AND ANALYTIC INVARIANTS OF SUPERISOLATED SINGULARITIES

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Abstract

Using superisolated singularities we present examples and counterexamples to some of the most important conjectures regarding invariants of normal surface singularities. More precisely, we show that the "Seiberg-Witten invariant conjecture"(of Nicolaescu and the third author), the "Universal abelian cover conjecture" (of Neumann and Wahl) and the "Geometric genus conjecture" fail (at least at that generality in which they were formulated). Moreover, we also show that for Gorenstein singularities (even with integral homology sphere links) besides the geometric genus, the embedded dimension and the multiplicity (in particular, the Hilbert-Samuel function) also fail to be topological; and in general, the Artin cycle does not coincide with the maximal (ideal) cycle.

1. Introduction

1.1. In the last years we witness an intense effort to understand the following question: what kind of analytic invariants of an analytic complex normal surface singularity can be determined from the topology (i.e. from the link) of the singularity ? Is the link indeed sufficiently powerful to contain valuable information which would help to recover analytic invariants (like multiplicity, Hilbert-Samuel function, geometric genus), or equations (modulo equisingular deformations) ? See, e.g. [3, 4, 5, 6, 9, 11, 12, 13, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 37, 40, 41, 42, 43]. In fact, in order to give a chance to these type of questions, one has to assume two types of restrictions (see e.g. [13] and [25] for more details and examples): a topological one – e.g. that the link is a rational homology sphere – and an analytic one – e.g. that the singularity is $\mathbb{Q}$–Gorenstein. Therefore, in the sequel we will assume that the link is a rational homology sphere.

As a result of the above mentioned efforts, in the last years a large number of positive results and conjectures have appeared. Some of the conjectures were verified for large nontrivial families of singularities, a fact which created an increasing optimism. Nevertheless, some signs started to give the signal that there are special families of singularities which might create some obstructions, and whose understanding would be crucial for further progress. One of these families is the class of superisolated singularities.

The goal of this note is to present examples and counterexamples to some of the most important conjectures in this area regarding invariants of normal surface singularities (using superisolated singularities). More precisely, we show that the "Seiberg-Witten invariant conjecture"(of Nicolaescu and the third author), the "Universal abelian cover conjecture" (of Neumann and Wahl) and the "Geometric genus conjecture" fail (at least at that generality in which they were formulated); see section 3 for a short review of these conjectures. Moreover, these examples also show that for Gorenstein singularities (even with integral homology sphere links) besides the geometric genus, the embedded dimension and the multiplicity (in particular, the Hilbert-Samuel function) also fail to be topological. One of the examples also shows that, in general, the Artin cycle does not coincide with the maximal (ideal) cycle (even for complete intersections with integral homology sphere links). The main message is that all the present conjectures and knowledge must be reconsidered, rethought, reorganized in order to find the right and correct connections, directions and questions which would guide the next steps.

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Surprisingly, some of our examples are not very complicated (compared with the list of – rather different and sometimes rather complex – positive examples which verify the corresponding conjectures). E.g., they are hypersurface singularities (or their universal abelian covers). Nevertheless, they have some other rather specific properties which allow some room for anomalies.

We wish to emphasize that the failure of the conjectures (at the generality how they were formulated) puts in a different new light all those families for which the conjectures were verified: their role and importance become much stronger and dominant. Moreover, this is a clear invitation for clarification of some other new families of singularities, out of which the superisolated singularities have the first priority.

In section 2 we will set our notations and we will present some results about the invariants of superisolated hypersurface singularities. In section 3 we give the list of conjectures and problems for which we will provide examples-counterexamples in the following sections. In section 4 our strategy is the following. We start with the classification of the hypersurface superisolated singularities. By computing invariants one gets directly counterexamples for the Seiberg-Witten invariant conjecture (cf. 4.1) and the Universal abelian cover conjecture (cf. 4.3-4.4). More complicated, but more striking examples are found by considering the universal abelian cover of singularities (cf. 4.5-4.6).

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2. Hypersurface superisolated singularities

2.1. Hypersurface superisolated singularities achieved historically the reputation of being an interesting class of singularities. This class "contains" in a canonical way the theory of complex projective plane curves, which gives a series of nice examples and counterexamples. They were introduced in [16] by the first author in order to show that the μ-constant stratum in the semiuniversal deformation space of an isolated hypersurface singularity, in general, is not smooth. Later Artal-Bartolo in [1] used them to provide a counterexample for S. S.-T. Yau's conjecture (showing that, in general, the link of an isolated hypersurface surface singularity and its characteristic polynomial not determine the embedded topological type of the singular germ). On the other hand, A. Durfee's conjecture and the monodromy conjecture of J. Denef and F. Loeser has been proved for them, see [17] and [2].

2.2. Definitions-Notations. A hypersurface singularity $f:(\mathbb{C}^3,0)\to(\mathbb{C},0)$, $f=f_d+f_{d+1}+\cdots$ (where f_j is homogeneous of degree j) is superisolated if the projective plane curve $C:=\{f_d=0\}\subset\mathbb{P}^2$ is reduced with isolated singularities $\{p_i\}_i$, and these points are not situated on the projective curve $\{f_{d+1}=0\}$. In this case the embedded topological type (and the equisingular type) of f does not depend on the choice of f_j's (for $j>d$, as long as f_{d+1} satisfies the above requirement), e.g. one can take $f_j=0$ for any $j>d+1$ and $f_{d+1}=l^{d+1}$ where l is a linear form not vanishing at the points $\{p_i\}_i$. We will denote by μ_i (respectively by Δ_i, with the sign choice $\Delta_i(1)=1$) the Milnor number (respectively, the characteristic polynomial) of the local plane curve singularities $(C,p_i)\subset(\mathbb{P}^2,p_i)$. For simplicity, in this note we will assume that C is irreducible. (The interested reader can adopt the next discussion easily to the general situation.)

Let M be the link of $\{f=0\}$ (with its natural orientation), $H:=H_1(M,\mathbb{Z})$; μ and p_g be the Milnor number and the geometric genus of f.

2.3. Invariants.

• [16] The *minimal resolution* of $\{f=0\}$ has only one irreducible exceptional divisor which is isomorphic to C and has self intersection $-d$. In particular, the link M of f is a rational homology sphere if and only if C is rational and all the plane curve singularities $(C,p_i)\subset(\mathbb{P}^2,p_i)$ are (locally) irreducible (i.e., C is a rational cuspidal plane curve). In particular, $\sum_i\mu_i=(d-1)(d-2)$.

If Γ_i is the minimal embedded resolution graph of $(C,p_i)\subset(\mathbb{P}^2,p_i)$ (with a unique -1 vertex v_i which supports the strict transform of (C,p_i)), then the minimal good resolution graph of $\{f=0\}$ can be constructed in the following way: consider a "central vertex" v (which corresponds to the curve C), for each i connect v with v_i by an edge, keep all the decorations of Γ_i, and add a new decoration e_v (self intersection) to v as follows. In the graphs Γ_i insert the set of multiplicities of the (reduced) plane curve singularity (i.e. the strict transform of (C,p_i) goes with multiplicity one). Let a_i be the multiplicity of the unique -1 curve of Γ_i. Then $e_v=-d-\sum_i a_i$.

• Fix any resolution graph Γ of $\{f=0\}$. Let K be the *canonical cycle* associated with Γ, and s the number of vertices. Then K^2+s does not depend on the choice of Γ, it is a topological invariant of M. In our case, it is easy to compute it at the level of the minimal resolution:

$$K^2+s=-d(d-2)^2+1.$$

• By (3.6.4) of [1], the *Milnor number* μ of f is the sum of the Milnor number of the singularity $x^d+y^d+z^d$ and $\sum_i \mu_i$. Using the above mentioned identity $\sum_i \mu_i=(d-1)(d-2)$, we get:

$$\mu=(d-1)^3+(d-1)(d-2).$$

Similarly, the *characteristic polynomial* Δ_f of f is

$$\Delta_f(t)=\frac{t^d-1}{t-1}\cdot\prod_i \Delta_i(t^{d+1}).$$

Since $\Delta_i(1)=1$, this implies that $|H|=\Delta_f(1)=d$. In fact, one can verify easily that $H=\mathbb{Z}_d$, and a possible generator of H is an elementary loop in a transversal slice to C.

• Since $12p_g=\mu-(K^2+s)$ by [14], one obtains:

$$p_g=d(d-1)(d-2)/6.$$

• One of the conjectures relates the *Seiberg-Witten invariant* $\mathbf{sw}(M)$ of M (associated with the canonical *spin*^{c} structure) with the analytic (or smoothing) invariants of the singularity. Here, by definition, $\mathbf{sw}(M)$ is the sign-refined Reidemeister-Turaev torsion $\mathcal{T}(M)$ (associated with the canonical *spin*^{c} structure) [38] normalized by the Casson-Walker invariant $\lambda(M)$ (using the convention of [15]) (cf. also with [26, 27, 28, 23, 25]). Namely, we consider:

$$\mathbf{sw}(M):=-\frac{\lambda(M)}{|H|}+\mathcal{T}(M).$$

Both invariants $\mathcal{T}(M)$ and $\lambda(M)$ can be determined from the graph (for details, see [26] or [25]). In fact, in our present case, the formula of [26] can be rewritten in the form:

$$\mathcal{T}(M)=\frac{1}{d}\sum_{\xi^d=1\neq\xi}\frac{1}{(\xi-1)^2}\cdot\prod_i \Delta_i(\xi).$$

Using similar method as in the proof of Theorem 4.5 of [28] (i.e. Fujita's splicing formula for the Casson-Walker invariant [10], and Walker-Lescop surgery formula [15], page 13) one can proof the following identity. Let $\bar{\Delta}(t)$ be the product $\prod_i \Delta_i(t)$ symmetrized (i.e. its degree is 2δ and $\bar{\Delta}(t)=t^{-\delta}\cdot\prod_i \Delta_i(t)$). Then the Casson-Walker invariant of the link is

$$\lambda(M)=(-1/2)\bar{\Delta}(t)''(1)+(d-1)(d-2)/24.$$

In fact, in this formula one can replace $\bar{\Delta}(t)''(1)$ by $\sum_i \bar{\Delta}_i(t)''(1)$.

3. The conjectures and questions

Here we list the main conjectures and problems which have been guiding our investigation.

3.1. SWC. The Seiberg-Witten invariant conjecture. In [26] L. Nicolaescu and the third author formulated the following conjecture.

(a) If the link of a normal surface singularity is a rational homology sphere then

$$p_g\leq \mathbf{sw}(M)-(K^2+s)/8.$$

(b) Additionally, if the singularity is $\mathbb{Q}$-Gorenstein, then in (a) the equality holds.

In the case of hypersurface singularities (more generally, for smoothings of Gorenstein singularities) the identity (b) can be rewritten as $-8\mathbf{sw}(M)=\sigma$, the signature of the Milnor fiber. If the singularity is an isolated complete intersection with an integral homology sphere link, then the conjecture transforms into the identity $8\lambda(M)=\sigma$, which was conjectured by Neumann and Wahl [30] for smoothings of complete intersections (this is called the "Casson invariant conjecture", *CIC*).

The *SWC*-conjecture was verified e.g. for quotient singularities [26], for singularities with good $\mathbb{C}^*$-actions [27], hypersurface suspension singularities $g(u,v)+w^n$ with g irreducible [28]. Even more, in [23], the third author replaced $\mathbf{sw}(M)$ by the corresponding Ozsváth-Szabó invariant (which is defined via the Ozsváth-Szabó Floer homology, and which conjecturally equals $\mathbf{sw}(M)$), and verified the inequality (a) for any singularity with *almost rational* (AR in short) resolution graph. (A graph is AR, if by replacing the decoration of one of the vertices one gets a rational graph. E.g., all the rational, weakly elliptic, minimal good star-shaped graphs are AR.)

In fact, for rational singularities, even the equivariant version of the SWC was verified: [24] shows the identity of the set of Seiberg-Witten invariants of the link (parametrized by all the possible $spin^c$-structures) with the equivariant geometric genera of the universal abelian cover.

On the other hand, in [23] the author exemplifies some types of graphs which are not AR, and whose understanding would be necessary for further progress regarding the result of [loc.cit.]. These are exactly the type of graphs which are provided typically by superisolated singularities.

3.2. UACC. The universal abelian cover conjecture. The starting point of the next conjecture of Neumann and Wahl is Neumann's paper [29] which proves that the universal abelian cover of a singularity with a good $\mathbb{C}^*$-action and with $b_1(M) = 0$ is a Brieskorn complete intersection whose weights can be determined from the Seifert invariants of the link. This, and other examples worked out by Neumann and Wahl (see e.g. [32] about quotient-cusps) lead them to a rather complex program and package of conjectures [31]:

Assume that $(X, 0)$ is $\mathbb{Q}$-Gorenstein singularity with $b_1(M) = 0$. Then there exists an equisingular and equivariant deformation of the universal abelian cover of $(X, 0)$ to an isolated complete intersection singularity. Moreover, the equations of this complete intersection, together with the action of $H_1(M, \mathbb{Z})$, can be recovered from M via the "splice equations".

The main point of the above conjecture, in its detailed version, provides a clear recipe for the equations of the complete intersection singularity (the "splice equations") and the action of H on these equations. This is done in terms of the combinatorics of the resolution graph of $(X, 0)$. In order to be able to write down the equations, the graph should satisfy some arithmetical properties: the so-called *semigroup conditions* and *congruence conditions*. Their validity is part of the conjecture. The reader is invited to see all the details in [31].

In order to eliminate any confusion, we mention that in this note *equisingular deformation* means the existence of a *simultaneous equitopological resolution* as discussed in [39].

3.3. GGC. The geometric genus conjecture. Both 3.1 and 3.2 are closely related with the following more general conjecture, which was formulated as a very general guiding principle (cf. with Question (3.2) in [30], see also Problem 9.2 in [25]).

In the case of a $\mathbb{Q}$-Gorenstein singularity with $b_1(M) = 0$, the geometric genus p_g is topological (i.e. can be recovered from the oriented link).

Here we mention the following positive result of Pinkham [35]: If a singularity with $b_1(M) = 0$ has a good $\mathbb{C}^*$-action, then its geometric genus can be computed explicitly from the resolution graph. (Moreover, by [29], such a singularity is $\mathbb{Q}$-Gorenstein.)

The crucial testing case for the above GGC is the case of the star-shaped resolution graphs: *is it true that if the minimal good resolution graph of a $\mathbb{Q}$-Gorenstein singularity is star-shaped, then its geometric genus is the same as the number predicted by Pinkham's formula ?*

3.4. Other analytic invariants. Similar questions were raised for several other discrete analytic invariants as well:

For what family of $\mathbb{Q}$-Gorenstein singularities (with $b_1(M) = 0$) are the invariants like the multiplicity, embedded dimension, Hilbert-Samuel function, maximal cycle (etc.) topological ?

For different positive cases and comments, see e.g. [25]. (The fact that the embedded dimension can jump in a topological constant family – even in a positive-weight deformation of a weighted homogeneous singularity – was known by experts.)

The examples of the next sections provide negative answers to all of the above conjectures (SWC, $UACC$ and GGC) and all the analytic invariants listed in 3.4 (already in Gorenstein case).

4. Examples

4.1. Examples/counterexamples for the SWC-conjecture. Some of the next examples show that the SWC-conjecture, in general, is not true. For this, we consider superisolated singularities $f = f_d + l^{d+1}$. Below, any singular point (C, p_i) will be identified by its multiplicity sequence. Since the number of occurrences of the multiplicity 1 in the multiplicity sequence equals the last multiplicity greater than 1, we omit the multiplicity 1: we denote such a sequence by $[m_0, \ldots, m_l]$ where $m_0 \geq m_1 \geq \cdots \geq m_l > m_{l+1} = 1$ for a suitable $l \geq 0$. In fact, we will write $[\hat{m}_{0_{r_0}}, \ldots, \hat{m}_{k_{r_k}}]$ for a multiplicity sequence which means that the multiplicity $\hat{m}_i$ occurs r_i times for $i = 0, \ldots, k$. For example, $[4_2, 2_3]$ means $[4, 4, 2, 2, 2, 1, 1]$.

If C has only one singularity with sequence $[d-1]$, then f has the same invariants as the weighted homogeneous singularity $zx^{d-1} + y^d + z^{d+1}$, hence it satisfies the conjecture by [27]. (Probably it is worth to mention that not all the rational cuspidal curves of degree d with one cusp and multiplicity sequence $[d-1]$ are projectively equivalent. E.g., for $d = 4$ there are two projectively non-equivalent curves: $\{x^4 - x^3y + y^3z = 0\}$ and $\{x^4 - y^3z = 0\}$, cf. [18], page 135.)

If $d = 3$, then C has a unique singularity of type [2]. If $d = 4$, then there are four possibilities; the corresponding multiplicity sequences of the singular points $\{p_i\}_i$ of C are [3]; $[2_3]$; $[2_2], [2]$ and $[2], [2], [2]$. By a verification, in all these cases, the conjecture is again true. (For the classification of the cuspidal rational curves with small degree, see e.g. the book of Namba [18].)

If $d = 5$, then $p_g = 10$, $K^2 + s = -44$. Let N be the number of singular points of C. The next table shows for all the possible multiplicity sequences the validity of the conjecture. When the conjecture fails, we put in parenthesis the value $-\lambda/|H| + \mathcal{T} - (K^2 + s)/8$ (which can be compared with the value of p_g).

(1) $N = 1$

	type of cusp	*conj*
C_1	$[4]$	*True*
C_2	$[2_6]$	*True*

(2) $N = 2$

	type of cusps	*conj*
C_3	$[3,2]$, $[2_2]$	*True*
C_4	$[3]$, $[2_3]$	*False* (8)
C_5	$[2_2]$, $[2_4]$	*False* (8)

(3) $N = 3$

	type of cusps	*conj*
C_6	$[3]$, $[2_2]$, $[2]$	*False* (8)
C_7	$[2_2]$, $[2_2]$, $[2_2]$	*False* (4)

(4) $N = 4$

	type of cusps	*conj*
C_8	$[2_3]$, $[2]$,$[2]$, $[2]$	*False* (2)

If $d = 6$ then $p_g = 20$. The classification of multiplicity sequences of rational cuspidal plane curves of degree 6 with N singular points is given by the following list, see e.g. Fenske's paper [7].

(1) $N = 1$

	type of cusp	*conj*
C_1	$[5]$	*True*
C_2	$[4, 2_4]$	*True*
C_3	$[3_3, 2]$	*True*

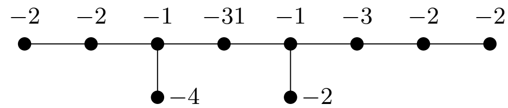
(2) $N = 2$

	type of cusps	*conj*
C_4	$[3_3]$, $[2]$	*True*
C_5	$[3_2, 2]$, $[3]$	*True*
C_6	$[3_2]$, $[3, 2]$	*True*
C_7	$[4, 2_3]$, $[2]$	*True*
C_8	$[4, 2_2]$, $[2_2]$	*True*
C_9	$[4]$, $[2_4]$	*False* (18)

(3) $N = 3$

	type of cusps	*conj*
C_{10}	$[4]$, $[2_3]$, $[2]$	*True*
C_{11}	$[4]$, $[2_2]$, $[2_2]$	*True*

For the convenience of the reader, we make the example $d = 5$, $N = 2$, case C_4, more explicit. In this case the minimal good resolution graph has the form



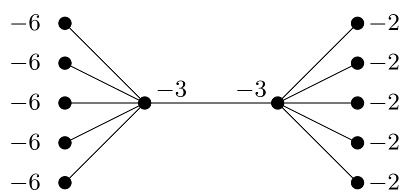
By [26], or by the above formulae, $-\lambda(M) = 21/2$ and $\mathcal{T}(M) = 2/5$, hence $\mathbf{sw}(M) - (K^2 + s)/8 = 8$.

Notice that above, in all the cases when part (b) of the SWC-conjecture fails (i.e. $p_g \neq \mathbf{sw}(M) - (K^2 + s)/8$), part (a) of 3.1 fails as well: the topological candidate becomes strict *smaller* than p_g.

4.1.1. The authors analyzed even higher degree curves C present in the literature, but were not able to find any counterexample with $N = 1$. Although this very paper shows how cautious one should be with formulating conjectures, still, *we predict that for $N = 1$ the SWC is actually true.* This conjecture is also supported by its verification for a series of non-trivial families, e.g. for irreducible curves C of *Abhyankar-Moh-Suzuki type*. They are characterized by the existence of a line $L \subset \mathbb{P}^2$ such that $C \setminus L$ is isomorphic to $\mathbb{C}$ (or, $C \cap L$ is the unique singular point of C). (Notice that not any curve with $N = 1$ satisfies this property, e.g. the Yoshihara quintic – C_2 with $[2_6]$ in our table in 4.1 – does not.) We also verified the above conjecture for all the cases when the singular point has exactly one characteristic pair. Since the techniques involved in these verifications are rather different from the spirit of the present note, they will be presented in another article [8].

In fact, since any hypersurface superisolated singularity with $N = 1$ is AR (in the sense of [23]), the inequality 3.1(a) is valid for them by [23], 9.5(a).

4.2. Remark. Analyzing the above examples 4.1, one can ask: why the class of superisolated singularities is so special? In the spirit of [23] (i.e. thinking about non-AR graphs) we can notice that if we want to transform the above superisolated graphs into rational graphs by replacing the original self intersection numbers by more negative ones, then we have to do this for many vertices (at least for N vertices of type v_i). Is the presence of these "bad" vertices the reason for the above anomalies? The answer probably is that not just this: one can produce easily suspension singularities (which verify the conjecture by [28]) with more than one "bad" vertex. For example, let $g(x, y)$ be the irreducible plane curve singularity with Newton pairs $(p_1, q_1) = (5, 6)$ and $(p_2, q_2) = (2, 5)$. Then the resolution graph of the suspension hypersurface singularity $f(x, y, z) = g(x, y) + z^5$ is the following (for the corresponding algorithm, see [19], Appendix):



In this case the graph has two "bad" vertices, $H = \mathbb{Z}_6^4 \oplus \mathbb{Z}_2^4$, $K^2 + s = -244$, $\mu = 416$, $p_g = 55$, $-\lambda/|H| = 61/18$, $\mathcal{T} = 190/9$, and $\mathbf{sw} = 49/2 = -\sigma/8$ (for different formulas and details regarding suspension singularities, see e.g. [28]).

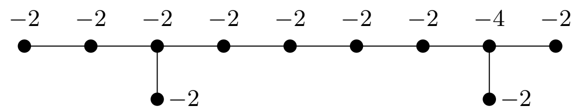
4.3. Counterexamples for $UACC$. Working with superisolated singularities one sees easily that already the construction of the "splicing equations" is obstructed: in general, the *semigroup condition is not satisfied.* More precisely, consider the splice diagram associated with the resolution graph of a hypersurface superisolated singularity. Then, if $N \geq 3$, that decoration of any edge of type $[v, v_i]$ which is closer to v is 1. This should be situated in the semigroup generated by the decorations of the leaves (which are all strict greater than 1), a fact which is not true.

This means that the algorithm [31] *which provides the equations of the complete intersection singularity predicted by the $UACC$ is not working, since to write the splice complete intersection equations one needs the semigroup condition satisfied. In other words, that conjectured complete intersection singularity whose deformation should contain the universal abelian cover, in general, does not exist.*

(But, of course, this does not imply that the universal abelian cover cannot be a complete intersection; it might be, but not of splice type.)

The simplest counterexample appears when $d = 4$ and C is the Steiner quartic (the unique quartic in the plane with three [2]-cusps).

4.4. A suspension type counterexample. In fact, the phenomenon 4.3 is not really specific to superisolated singularities. One can construct hypersurface suspension singularities with the same property. E.g. if one takes the hypersurface singularity $\{z^2 = (y + x^2)(y^3 + x^{11})\}$, then its link is a rational homology sphere (with first homology $\mathbb{Z}_4$), but its minimal plumbing graph does not satisfy the semigroup conditions (since the E_8-subgraph has determinant 1). The resolution graph is



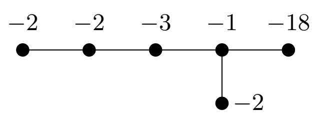
More sophisticated counterexamples are provided by considering universal abelian covers.

4.5. Counterexample: The case $d = 4$ with multiplicity sequence $[2_3]$, and its universal abelian cover. In this section, we make more explicit the invariants of the superisolated singularity with $d = 4$ when C has only one singular point of type $[2_3]$. In this case C is projectively equivalent to the projective curve $(zy - x^2)^2 = xy^3$, with parametrization $[t : s] \mapsto [t^2s^2 : t^4 : s^4 + t^3s]$ (see [18], page 146). Hence a possible choice for f is

$$f = (zy - x^2)^2 - xy^3 + z^5.$$

As we already mentioned $p_g(X, 0) = 4$. The resolution graph Γ is

$\Gamma :$



Since the graph is a star-shaped, the same resolution graph can be realized by a weighted homogeneous singularity $(X_w, 0)$ as well. In the present case this is an isolated complete intersection in $(\mathbb{C}^4, 0)$ with two equations:

$$(X_w, 0) = \begin{cases} yz = x^2 \\ z^5 + t^2 - xy^3 = 0. \end{cases}$$

The corresponding weights of the coordinates (x, y, z, t) are: (16,18,14,35). By Pinkham's formula [35] one gets that $p_g(X_w, 0) = 4$ as well.

Now, one can use the result of Neumann and Wahl (cf. [30] (3.3)) which guarantees that a Gorenstein singularity (with the same link as a weighted homogeneous singularity $(X_w, 0)$) is an equisingular deformation of $(X_w, 0)$ if and only if its p_g equals the number predicted by Pinkham's formula. (In [30] (3.3) the result is stated for integral homology spheres links, but the proof works without modification for rational homology sphere links as well.)

In particular, the superisolated singularity $(X, 0)$ is an equisingular deformation of $(X_w, 0)$. In this particular case this deformation can be written easily: the pair of equations $yz - x^2 = \lambda t$, and $z^5 + t^2 - xy^3 = 0$ – for the parameter $\lambda \neq 0$ – is equivalent to $(X, 0)$. (Here, if one wishes to emphasize the compatibility of the weights with the deformation, one should notice that λ has weight -3).

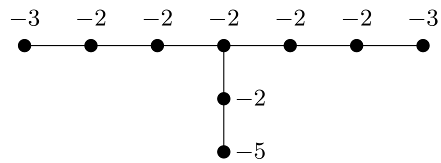
Notice also that the two singularities $(X, 0)$ and $(X_w, 0)$ *have the same multiplicity* (which equals 4), but clearly have *different embedded dimensions – hence different Hilbert-Samuel functions*.

Next, we wish to analyze the corresponding universal abelian covers.

The universal abelian cover $(X_w^{ab}, 0)$ of $(X_w, 0)$ is easy (cf. also with [29]). It is a hypersurface Brieskorn singularity $\{u^7 = v^{18} + w^2\}$. (Notice that this equation is exactly the "splice equation" predicted by the Neumann-Wahl construction, cf. 3.2). The action of (a generator of) $H = \mathbb{Z}_4 = \{\zeta \in \mathbb{C} : \zeta^4 = 1\}$ on the coordinates (u, v, w) is $u \mapsto -u$, $v \mapsto iv$ and $w \mapsto -iw$. Taking the invarints $x := uv^2$, $y := u^2$, $z := v^4$ and $t := vw$, we get that $\{u^7 = v^{18} + w^2\}/H$ has exactly those equations what we provided for $(X_w, 0)$ above.

Notice that the resolution graphs of both universal abelian covers are the same (which is exactly the plumbing diagram of the universal abelian cover of the common link M). In this case it is:

Γ^{ab} :



This graph has $K^2 + s = -18$. Since the Brieskorn singularity $u^7 = v^{18} + w^2$ has Milnor number $6 \cdot 17 = 102$, we get by Laufer's formula that its geometric genus is $p_g(-u^7 + v^{18} + w^2) = 10$.

Next, we analyze the universal abelian cover of the superisolated singularity $(X, 0)$ and estimate its geometric genus.

In our original study of examples of 4.5 and 4.6, the authors had the faulty impression that the equisingular deformation existing at the level of $(X, 0)$ lifts to an equisingular and equivariant deformation at the level of the universal abelian cover. But when we showed J. Wahl the example 4.6, he recognized that this could indeed not occur, and outlined a proof: any equivariant positive weight deformation of the universal abelian cover of $(X_w, 0)$ gives a family of quotients of constant embedding dimension 6.

In the sequel we present an alternate proof of the non-existence of such deformation (using Fact B below, which is interesting by its own, and hopefully can be applied in different similar situations as well).

Fact A. *The universal abelian cover $(X^{ab}, 0)$ of $(X, 0)$ is not in the μ-constant deformation space of $(X^{ab}_w, 0) = \{w^2 + v^{18} - u^7\}$. In particular, there is no equisingular deformation from $(X^{ab}, 0)$ to $(X^{ab}_w, 0)$.*

In fact, what we will prove is the following:

Fact B. *Assume that $\mathbb{Z}_4$ acts freely in codimension 1 on a hypersurface germ which in some coordinates has the form $w^2 + (\deg \geq 5)$. Then if the quotient is a hypersurface with multiplicity greater than 2, then the tangent cone of the quotient is reducible.*

Proof. (1) Notice that any germ in the semiuniversal deformation of $(X^{ab}_w, 0)$ (modulo a coordinate change) can be written in the form $w^2 + g(u, v)$. Assume that $(X^{ab}, 0)$ is given by $(\{f^{ab} = 0\}, 0) \subset (\mathbb{C}^3, 0)$ and f is in a μ-constant deformation of $(X^{ab}_w, 0)$. Hence f^{ab} itself, in some coordinates, has this form such that the plane curve singularities $u^7 - v^{18}$ and $g(u, v)$ have the same embedded topological types. Therefore, all the monomials of g have degree at least 7.
(2) We consider the action of $\mathbb{Z}_4$ on $\{f^{ab} = 0\}$. Since $\{f^{ab} = 0\}$ is singular with tangent space $\mathbb{C}^3$, the action induces an action on this cotangent space m/m^2 and on the exact sequence $0 \to m^2 \to m \to m/m^2 \to 0$ (here m is the maximal ideal of $\mathbb{C}[u, v, w]/(f^{ab})$). Since $\mathbb{Z}_4$ is finite, this sequence equivariantly splits, hence the singularity has an equivariant embedding into its tangent space. In other words, by a change of local coordinates, we can assume that that the action extends to a linear action of $\mathbb{C}^3$. Since the group is cyclic (with distinguished generator ϵ), we can even assume that the linear action on $\mathbb{C}^3$ is diagonal.

Since the space $\{f^{ab} = 0\}$ is invariant to the action, f^{ab} is an eigenfunction of ϵ (coinvariant). Since f^{ab} (in any coordinates) has the form $l^2 + (\deg \geq 3)$, where l is a linear form, l^2 is also an eigenfunction of ϵ with the same eigenvalue λ as f^{ab}. Since l^2 is a square, we get that $\lambda = \pm 1$. Moreover, if l involves more coordinates with nonzero coefficients, then the action of ϵ on all of them should be the same, hence by another linear change of variables, and keeping the diagonal form of ϵ, we can transform l into one of the coordinates. We will denote the coordinates constructed in this way by w, u, v.

The action of ϵ has the form $diag(i^{a_1}, i^{a_2}, i^{a_3})$. Since the action on $\{f^{ab} = 0\}$ is free in codimension 1, one gets $\#\{j : a_j \text{ even}\} \leq 1$.
(3) Consider the projection $p : \mathbb{C}^3 \to \mathbb{C}^3/\mathbb{Z}_4$. If g is a function vanishing along $\{f^{ab} = 0\}/\mathbb{Z}_4$, then $g \circ p$ is an invariant function of form $f^{ab}h$. In particular, in order to obtain all the equations of $\{f^{ab} = 0\}/\mathbb{Z}_4$, we have to multiply f^{ab} with such coinvariants h which make $f^{ab}h$ invariant, and express $f^{ab}h$ in terms of principal invariants. If f^{ab} itself is invariant, it provides basically only one equation, namely its expression in terms of the principal invariants.
(4) Assume that there is an a_j (say a_1) multiple of 4. Then (modulo some symmetry) there are the following possibilities:
(4.1) $\epsilon = diag(1, i, i)$ (i.e. $\epsilon(w) = w$, $\epsilon(u) = iu$, $\epsilon(v) = iv$). The principal invariants in $\mathbb{C}[w, u, v]$ of the action are $I = \{w, v^4, v^3u, v^2u^2, vu^3, u^4\}$, hence $embdim(\mathbb{C}^3/\mathbb{Z}_4) = 6$. Recall that $f^{ab} = l^2 + (\deg \geq 3)$, where l is one of the coordinates.
(4.1.1) Assume that $f^{ab} = w^2 + (\deg \geq 3)$. Since f^{ab} in some coordinates has the from $\bar{w}^2 + (\deg \geq 5)$ (cf. part (1)), f^{ab} in variables (w, u, v) can be written as

$$(*) \qquad f^{ab}=(w+h_2+h_3)^2+(\deg\geq 5)=w^2+2wh_2+h_2^2+2wh_3+(\deg\geq 5),$$

where $\deg(h_j)=j$. In this case f^{ab} and w are invariants, hence the same is valid for wh_2 and $h_2^2+2wh_3$ as well. Hence $f^{ab}=w^2+aw^3+bw^4+(\deg\geq 5)$; in particular f^{ab} expressed in terms of the invariants I has no linear term. Therefore, $embdim\{f^{ab}=0\}/\mathbb{Z}_4=6$.
(4.1.2) Assume that $f^{ab}=u^2+(\deg\geq 3)$ (the case $f^{ab}=v^2+\cdots$ is symmetric). Then the principal invariants u^4, vu^3, v^2u^2 can be eliminated using the equation of f^{ab}. The remaining relevant principal invariants are $x:=w, y:=v^4, z:=v^3u$. Hence $\{f^{ab}=0\}/\mathbb{Z}_4$ can be embedded into $(\mathbb{C}^3,0)$. Next we analyze its equation. Notice that in this case $\lambda=-1$. Therefore, if $m=w^\alpha u^\beta v^\gamma$ is a monomial of f^{ab} with nonzero coefficient, then $\beta+\gamma=4t_m+2$ for some $t_m\geq 0$. If we multiply this monomial by v^{4k+6}, we get the invariant $w^\alpha u^\beta v^{\gamma+4k+6}$. Notice that the inequality $\gamma+4k+6\geq 3\beta$ is equivalent with $k\geq \beta-t_m-2$, hence if we take $k_0:=\max_m(\beta-t_m-2)$, then $mv^{4k_0+6}=x^\alpha z^\beta y^{t_m-\beta+2+k_0}$. In particular, $u^2v^{4k_0+6}=z^2y^{k_0}$. In other words, if $f^{ab}=\sum_m a_m m$, then the wanted equation of the quotient in $(\mathbb{C}^3,0)$ is $\sum_m a_m x^\alpha z^\beta y^{t_m-\beta+2+k_0}$.

Finally notice that if the (w,u,v)-degree of m is $\alpha+\beta+\gamma>2$ (i.e. if m is any monomial different from u^2), then the (x,y,z)-degree of mv^{4k_0+6} is $\alpha+\beta+t_m-\beta+2+k_0>2+k_0$. This shows that $\{f^{ab}=0\}/\mathbb{Z}_4$ in $(\mathbb{C}^3,0)$ (with coordinates x,y,z) has the equation $z^2y^{k_0}+$ higher degree terms. This, by any coordinate change, is not equivalent with the superisolated hypersurface singularity f_4+f_5 (because its tangent cone is reducible).
(4.2) Assume that ϵ acts on (w,u,v) diagonally via $diag(1,i,-i)$. The set of principal invariants are $I=\{w,v^4,uv,u^4\}$.
(4.2.1) If $f^{ab}=w^2+\ldots$, then using the same notation as in $(*)$, h_2 is invariant, hence f^{ab}, expressed in terms of the principal invariants, has no linear term. In particular, $embdim\{f^{ab}=0\}/\mathbb{Z}_4=4$.
(4.2.2) If $f^{ab}=u^2+(\deg\geq 3)$, we proceed as in (4.1.2). The invariant u^4 can be eliminated, the other relevant invariants are $x:=w, y:=v^4$ and $z:=uv$ and the quotient can be embedded in $(\mathbb{C}^3,0)$.

Define $r_\gamma\in\{0,1\}$ such that $\gamma-r_\gamma=2c_\gamma$ is even. Notice that $\lambda=-1$. Then, if $m=w^\alpha u^\beta v^\gamma$ is a monomial of f^{ab}, then $\beta+\gamma=2(2t_m+r_\gamma+1)$ for some $t_m\geq 0$. Set $k_0:=\max_m(t_m-c_\gamma)$. Then $mv^{4k_0+2}=x^\alpha z^\beta y^{k_0-t_m+c_\gamma}$. If $f^{ab}=\sum_m a_m m$ then the equation of the quotient is $f':=\sum_m a_m x^\alpha z^\beta y^{k_0-t_m+c_\gamma}$. Notice that the contribution of u^2 is $z^2y^{k_0}$. Let $d(m)$ be the (w,u,v)-degree $\alpha+\beta+\gamma=\alpha+4t_m+2r_\gamma+2$ of m, respectively, let $d'(m)$ be the (x,y,z)-degree of mv^{4k_0+2}. (In particular, $d(m)\geq 2$ with equality if and only if $m=u^2$.) By on easy verification one gets that $d'(m)\geq 2+k_0=d'(u^2)$, and if $d'(m)=2+k_0$ then y^{k_0} divides the corresponding monomial $x^\alpha z^\beta y^{k_0-t_m+c_\gamma}$. (In fact, the possible monomials are $z^2y^{k_0}, xy^{k_0+1}, zy^{k_0+1}, y^{k_0+2}$.) Hence the tangent cone of f' is not irreducible.
(5) Assume that there is an a_j (say a_1) of type $4s+2$ ($s\in\mathbb{Z}$). Then one has the following possibilities:
(5.1) Set $\epsilon=diag(-1,i,i)$ (acting on (w,u,v)). The principal invariants in $\mathbb{C}[w,u,v]$ are the elements in the set $I=\{v^4,v^3u,v^2u^2,vu^3,u^4,wu^2,wuv,wv^2,w^2\}$. In particular, $embdim(\mathbb{C}^3/\mathbb{Z}_4)=9$.
(5.1.1) Assume that $f^{ab}=w^2+(\deg\geq 3)$. Then f^{ab} is an invariant, which expressed in terms of I has a linear term. Hence $embdim\{f^{ab}=0\}/\mathbb{Z}_4=8$.
(5.1.2) Assume that $f^{ab}=u^2+\ldots$, in particular $\lambda=-1$. Then u^4,u^3v,u^2v^2 and wu^2 can be eliminated using f^{ab}, and one remains with the other five principal invariants $I_r=\{w^2,v^4,uv^3,wv^2,wuv\}$. Write f^{ab} again in the form $f^{ab}=(u+h_2+h_3)^2+(\deg\geq 5)$. Then uh_2 and $h_2^2+2uh_3$ are (-1)-eigenfunctions, hence h_2 and h_3 are linear combination of w^2u and w^2v. Hence $f^{ab}=u^2+au^2w^2+buvw^2+(\deg\geq 5)$. Multiplying such an f^{ab} with any (-1)-eigenfunction h such that $f^{ab}h$ can be expressed in terms of the monomials I_r, the expression of $f^{ab}h$ in terms of these invariants I_r will contain no linear term. Hence, $embdim\{f^{ab}=0\}/\mathbb{Z}_4=5$.
(5.2) Assume that $\epsilon=diag(-1,i,-i)$. The principal invariants are $I=\{v^4,vu,u^4,wu^2,wv^2,w^2\}$.
(5.2.1) If $f^{ab}=w^2+(\deg\geq 3)$ then f^{ab} is an invariant, which expressed in terms of I has a linear term. Hence $embdim\{f^{ab}=0\}/\mathbb{Z}_4=6-1=5$.
(5.2.2) Assume that $f^{ab}=u^2+\ldots$, in particular $\lambda=-1$. Then $I_r=\{w^2,v^4,uv,wv^2\}$. Write $f^{ab}=(u+h_2+h_3)^2+(\deg\geq 5)$. Then uh_2 and uh_3 are (-1)-eigenfunctions. Analyzing the corresponding monomial eigenfunctions, we get that $f^{ab}=u^2+au^2w^2+bv^2w^2+cuvw+(\deg\geq 5)$. Multiplying such an f^{ab} with any (-1)-eigenfunction h such that $f^{ab}h$ can be expressed in terms of the monomials I_r, the expression of $f^{ab}h$ in terms of these invariants I_r will contain no linear term. Hence, $embdim\{f^{ab}=0\}/\mathbb{Z}_4=4$.

(The case $f^{ab}=v^2+\cdots$ is similar.)
(6) Assume that all the integers a_j are odd. Then it is enough (modulo a symmetry) to consider the cases $f^{ab}=w^2+(\deg\geq 3)$ with three different actions for ϵ, namely $diag(i,i,i)$, $diag(i,i,-i)$ and $diag(i,-i,-i)$.

In all of these cases the embedded dimension of the quotient is > 3 (it is the cardinality of I_r, namely 9, 5, resp. 7). The verification is exactly the same as in (5.1.2) or (5.2.2). $\square$

Let us summarize what we have: the superisolated singularity $(X, 0)$ is clearly a Gorenstein singularity with $b_1(M) = 0$. It has only one "splice equation" (cf. 3.2) which defines $(X^{ab}_w, 0)$. The above fact shows that the universal abelian cover $(X^{ab}, 0)$ is not in the equisingular deformation of $(X^{ab}_w, 0)$. Therefore, *even if the construction of the "splicing equations" is not obstructed (cf. 4.3 and 4.4), in general, the $UACC$* [31] *is not valid.*

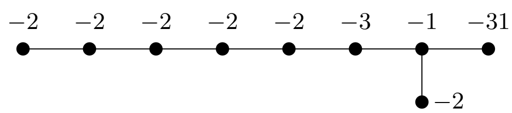
We can go even further: the resolution graphs of $(X^{ab}, 0)$ and $(X^{ab}_w, 0)$ are the same, hence these two singularities have the same topological types. Their links are rational homology spheres (with first homology $\mathbb{Z}_7$). Since the common resolution graph is star-shaped, and $(X^{ab}_w, 0)$ is weighted homogeneous, $(X^{ab}, 0)$ can be equisingularly deformed into $(X^{ab}_w, 0)$ if and only if their geometric genus are the same (cf. with the already mentioned result of Neumann and Wahl [30] (3.3)). Since this is not the case (by the above Fact A), one gets that $p_g(X^{ab}, 0) \neq p_g(X^{ab}_w, 0)$. In particular, *we constructed two Gorenstein singularities (one of them is even a hypersurface Brieskorn singularity) with the same rational homology sphere link, but with different geometric genus.* This provides counterexample for both SWC and GGC.

What is even more striking in the above counterexample, is the fact that the corresponding graphs are star-shaped (and one of the singularity is weighted homogeneous), cf. with the last paragraph of 3.3.

Recall that $p_g(X^{ab}_w, 0) = 10$. Notice also that for *any* normal surface singularity with the same resolution graph as Γ^{ab}, by (9.6) of [23] one has $p_g \leq 10$. In particular, $p_g(X^{ab}, 0) < 10$.

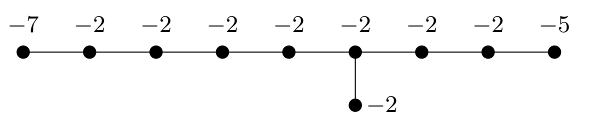
4.6. Counterexample: The case C_2 with $d = 5$ and multiplicity sequence $[2_6]$, and its universal abelian cover. We start with $f = f_5 + z^6$ where $f_5 = z(yz - x^2)^2 - 2xy^2(yz - x^2) + y^5$. The curve C is irreducible with unique singularity at $[0 : 0 : 1]$ (of type A_{12}). The resolution graph Γ of the superisolated singularity $(X, 0)$ is

Γ :



Since the graph is star-shaped, the same resolution graph can be realized by a weighted homogeneous singularity $(X_w, 0)$ as well. In fact, it is much easier to determine the universal abelian cover $(X^{ab}_w, 0)$ of $(X_w, 0)$. By [29], it is the Brieskorn hypersurface singularity $\{u^{13} + v^{31} + w^2 = 0\}$ (and this agrees with the "splice equation" provided by Γ). The corresponding resolution graph Γ^{ab} (of both $(X^{ab}, 0)$ and $(X^{ab}_w, 0)$) is

Γ^{ab} :



which defines the *integral homology sphere* $\Sigma(13, 31, 2)$.

The action of $H = \mathbb{Z}_5$ on $(X^{ab}_w, 0)$ is $(u, v, w) \mapsto (\zeta^4 u, \zeta^2 v, \zeta w)$, where ζ denotes a 5-root of unity. This action has a lot of principal invariants, but one can eliminate those ones which are multiples of w^2 using the equation $u^{13} + v^{31} + w^2$. Therefore, we have to consider only the following ones: $a := u^5$, $b := v^5$, $c := u^2 v$, $d := uv^3$, $e := uw$ and $f := wv^2$. If one wants to get the equations of $(X_w, 0)$ in $\mathbb{C}^6$ (in variables $a, \cdots, f$), one has to eliminate from the equations $u^{13} + v^{31} + w^2, u^5 - a, v^5 - b, u^2 v - c, uv^3 - d, uw - e, wv^2 - f$ the variables (u, v, w). This can be done by Singular [36], and we get the following set of equations for X_w in $\mathbb{C}^6$:

$$X_w = \begin{cases} ab - c^2 d = 0 \\ bc - d^2 = 0 \\ ad - c^3 = 0 \\ be - df = 0 \\ de - cf = 0 \\ af - c^2 e = 0 \\ e^2 + a^3 + b^6 c = 0 \\ ef + a^2 c^2 + b^6 d = 0 \\ f^2 + ac^4 + b^7 = 0 \end{cases}$$

In fact, these equations can also be obtained without Singular: the first six equations are the principal relations connecting the principal invariants $a, \dots, f$, while the last three equations are obtained (see the recipe in the proof of Fact B, step (3)) by multiplying the ζ^2-eigenfunction $u^{13}+v^{31}+w^2$ by the ζ^3-eigenfunctions u^2, uv^2, v^4.

As a curiosity, separating the first six equations one gets that $(X_w,0)$ is a subgerm of the determinantal singularity defined by the (2×2)-minors of

$$\begin{pmatrix} b & d & f & c^2 \\ d & c & e & a \end{pmatrix}.$$

The weights of the variables $(a,\dots,f)$ are (62,26,30,28,93,91).

Notice also that $(X_w,0)$ is Gorenstein, but it is not a complete intersection. Moreover, *the two singularities* $(X,0)$ *and* $(X_w,0)$ *have the same topological types (the same graphs* Γ*), but their embedded dimensions are not the same*: they are 3 and 6 respectively. It is even more surprising that *their multiplicities are also different*: $mult(X,0)=5$ and $mult(X_w,0)=6$ (the second computed by SINGULAR [36]).

On the other hand, their geometric genera are the same: $p_g(X,0)=10$ by the formula of (4), $p_g(X_w,0)=10$ by Pinkham's formula [35]. In particular, using again [30] (3.3), $(X,0)$ is in the equisingular deformation of $(X_w,0)$.

This deformation can be described as follows. (Again, the weight of λ is -3.) The authors are grateful to J. Stevens for his help in finding these deformation.

$$X(\lambda)=\begin{cases} ab-c^2d=\lambda f\\ bc-d^2=\lambda^2 a\\ ad-c^3=\lambda e\\ be-df=-\lambda ac^2\\ de-cf=-\lambda a^2\\ af-c^2e=-\lambda b^6\\ e^2+a^3+b^6c=0\\ ef+a^2c^2+b^6d=0\\ f^2+ac^4+b^7=0\end{cases}$$

In order to understand the deformation, consider the equation (for $\lambda\neq 0$):

$$E:=\lambda^{-2}(a^2b-2ac^2d+c^5)+b^6.$$

Notice that for $\lambda\neq 0$, using the first three equations one can eliminate the variables a,e,f. The last four equations transform into $E\lambda$, Ec, Ed and Eb (where in E we substitute a). Hence their vanishing is equivalent with the vanishing of E. The forth and fifth equations are automatically satisfied. Hence, for $\lambda\neq 0$, the system of equation is equivalent with a hypersurface singularity in variables (b,c,d) given by $E=0$ with the substitution $a=\lambda^{-2}(bc-d^2)$. Taking $\lambda=1$, $b=z$, $c=y$ and $d=x$, one gets exactly the superisolated singularity $f=f_5+z^6$.

On the other hand, similarly as in the case of 4.5, *there is no equisingular deformation at the level of universal abelian covers.* Both $(X^{ab},0)$ and $(X_w^{ab},0)$ have the same graph Γ^{ab} – which is a *unimodular star-shaped graph*, but $(X^{ab},0)$ is not in the equisingular deformation of $(X_w^{ab},0)$. In particular (by the same argument as in 4.5), $p_g(X^{ab},0)<p_g(X_w^{ab},0)$. In particular, *all the conjectures UACC, SWC and GGC fail.* (For the first case notice that the "splice equation" of $(X,0)$ is exactly the equation of $(X_w^{ab},0)$.)

This example shows (cf. with 4.5) that even with the assumption $H=0$ counterexamples for these conjectures exist.

The non-existence of the deformation follows by a similar statement as in the case of 4.5: *Assume that* $\mathbb{Z}_5$ *acts freely in codimension 1 on a hypersurface germ which in some coordinates has the form* $w^2+(\deg\geq 6)$. *Then if the quotient is a hypersurface with multiplicity greater than* 2*, then the tangent cone of the quotient is reducible.*

This has a completely similar proof as the similar statement in 4.5, and we will not give it here.

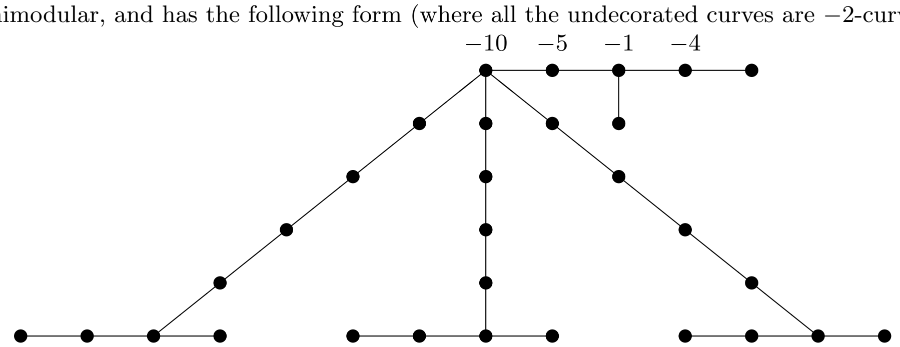
5. Integral homology sphere links

5.1. Recall that the first homology of the link of a hypersurface superisolated singularity $f=f_d+l^{d+1}$ is $\mathbb{Z}_d$ $(d\geq 2)$; in particular, it is never trivial. Nevertheless, we would like to emphasize that even with integral homology sphere links, counterexamples exist (although it is harder to find them); this additional requirement does not change the picture. But, in order to find such examples, we have to enlarge our family. We exemplify here two possibilities.

5.2. The universal abelian cover revisited. In the first case we consider the universal abelian cover $(X^{ab},0)$ of a hypersurface superisolated singularity. Notice that, in general, it is hard to give the equations (or identify the analytic structure) of $(X^{ab},0)$. But its topological type can be described completely. Recall that the minimal resolution $\tilde X$ of $(X,0)$ contains only one exceptional divisor C with (plane curve) singularities (C,p_i) and self-intersection $-d$. If one considers the $\mathbb{Z}_d$-cyclic cover $q:\tilde X^{ab}\to\tilde X$ of $\tilde X$, branched along C, one gets a partial resolution of $(X^{ab},0)$. In general $\tilde X^{ab}$ is not smooth, its singularities are the d-suspensions of the plane curve singularities (C,p_i) (in other words, if the local equation of (C,p_i) is $g_i(u,v)=0$, then $Sing\tilde X^{ab}=q^{-1}(\cup_i p_i)$, $q^{-1}(p_i)$ contains only one point, and $(\tilde X^{ab},q^{-1}(p_i))$ is a hypersurface singularity of type $g_i(u,v)+w^d=0$). Moreover, the self-intersection of $\tilde C:=q^{-1}(C)$ is -1. (Indeed, $d\tilde C\cdot\tilde C=q^*C\cdot\tilde C=C\cdot q_*\tilde C=C^2=-d$.) In particular, the minimal good resolution graph of $(X^{ab},0)$ can be obtained in a similar way as the graphs of hypersurface suspension singularities (if one replaces the embedded resolution graphs of the plane curve singularities with the graphs of their d-suspensions, and the self-intersection $-d$ with -1).

This construction also shows that the link of $(X^{ab},0)$ is an integral homology sphere if and only if all the links of the d suspension singularities $g_i(u,v)+w^d$ are integral homology spheres. This fact can be realized, as it is shown by the example 4.6. But even with $N\geq 3$ one can find many examples.

Take for example C_8 in the table 4.1 with $d=5$ and $N=4$. Then the local equations of the plane curve singularities are u^7+v^2 and three times u^3+v^2. Hence the minimal good resolution graph of $(X^{ab},0)$ is unimodular, and has the following form (where all the undecorated curves are -2-curves):



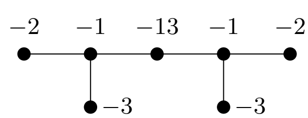
This shows that, even if we deal with integral homology sphere links, in general, the *semigroup conjecture fails* (cf. 3.2).

5.3. Non-hypersurfaces. Another way to extend our class of examples is to consider all the singularities (not only the hypersurfaces) which have the property that one of their resolution graphs has a "central vertex", and all the graph-components of the complement of the central vertex are embedded resolution graphs of plane curve singularities.

Probably the simplest (non-hypersurface) example is the following complete intersection in $(\mathbb{C}^4,0)$, given by the equations

$$(X,0)=\{x^2=u^3+v^2y,\ y^2=v^3+u^2x\}.$$

Its resolution graph is unimodular and has the form:



This example appears in [30] as a "positive" example satisfying the Casson invariant conjecture.

In the spirit of the this section, we present one of its "negative" properties: *its minimal (Artin) cycle does not agree with its maximal cycle (i.e. the minimal cycle cannot be cut out by a holomorphic function-germ).* Notice that, in general, if one wishes a topological characterization of the multiplicity, the first test is exactly the identity of the minimal and maximal cycles.

In order to see that in this case they are not the same, notice two facts. First, the strict transforms of the four coordinate functions are supported by the four leaves (degree one vertices). Second, the intersection of the minimal cycle with C (the -13-curve) is -1, and with all the other irreducible exceptional divisors is zero. In particular, analyzing the graph (e.g. the corresponding linking numbers), one gets that if the divisor of a holomorphic germ f would be the sum of the minimal cycle and the strict transform, then the local intersection multiplicity $i_{(X,0)}(f,z)$ would be 2 for one (in fact, for two) of the coordinate functions z. This would imply that the multiplicity of $(X,0)$ is 2 (or less), in particular $(X,0)$ would be a hypersurface singularity. But this is not the case. (Nevertheless, the topological type of $(X,0)$ supports at least one analytic structure for which the maximal cycle is the minimal cycle.)

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