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# On 'maximal' poles of zeta functions, roots of $b$-functions, and monodromy Jordan blocks

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*Dedicated with admiration to C. T. C. Wall on the occasion of his seventieth birthday*


## Abstract

The main objects of this study are the poles of several local zeta functions: the Igusa, topological, and motivic zeta function associated to a polynomial or (germ of) holomorphic function in $n$ variables. We are interested in poles of maximal possible order $n$. In all known cases (curves, non-degenerate polynomials) there is at most one pole of maximal order $n$, which is then given by the log canonical threshold of the function at the corresponding singular point. For an isolated singular point we prove that if the log canonical threshold yields a pole of order $n$ of the corresponding (local) zeta function, then it induces a root of the Bernstein–Sato polynomial of the given function of multiplicity $n$ (proving one of the cases of the strongest form of a conjecture of Igusa–Denef–Loeser). For an arbitrary singular point, we show under the same assumption that the monodromy eigenvalue induced by the pole has 'a Jordan block of size $n$ on the (perverse) complex of nearby cycles'.


## *Introduction*

0.1. Let $f : X \to \mathbb{C}$ be a non-constant analytic function on an open part $X$ of $\mathbb{C}^n$. The 'classical' complex zeta function associated to $f$ is an integral

$$Z_\varphi(f;s) := \int_X |f(x)|^{2s}\varphi(x)dx \wedge d\bar{x}$$

for $s \in \mathbb{C}$ with $\Re(s) > 0$, where $\varphi$ is a $C^\infty$ function with compact support on $X$. (Here and further, $x = (x_1, \ldots, x_n)$ and $dx = dx_1 \wedge \ldots \wedge dx_n$.) One verifies that $Z_\varphi(f;s)$ is holomorphic in $s$. Either by resolution of singularities [**3**], or using the Bernstein–Sato polynomial $b_f(s)$ of $f$ [**2**], one can show that it admits a meromorphic continuation to $\mathbb{C}$. The second method also yields that each pole of $Z_\varphi(f;s)$ is a translate by a non-positive integer of a root of $b_f(s)$. And moreover, for a root $s_0$ of $b_f(s)$, the order of $s_0 - m$ as pole of $Z_\varphi(f;s)$ is at most the multiplicity of $s_0$ as root of $b_f(s)$ [**13**]. In particular a pole of (maximal) order $n$ induces a root of multiplicity $n$.

0.2. Let now $f : X \to \mathbb{Q}_p$ be a non-constant ($\mathbb{Q}_p$-)analytic function on a compact open $X \subset \mathbb{Q}_p^n$, where $\mathbb{Q}_p$ denotes the field of $p$-adic numbers. Let $|\cdot|_p$ and $|dx|$ denote the $p$-adic norm and the Haar measure on $\mathbb{Q}_p^n$, normalized in the standard way. The $p$-adic integral

$$Z_p(f;s) := \int_X |f(x)|_p^s |dx|,$$

again defined for $s \in \mathbb{C}$ with $\Re(s) > 0$, is called the ($p$-adic) Igusa zeta function of $f$. Using resolution of singularities, Igusa [**11**, **12**] showed that it is a rational function of $p^{-s}$; hence

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it also admits a meromorphic continuation to $\mathbb{C}$. (Everything can be generalized to finite extensions of $\mathbb{Q}_p$.) There are various 'algebro-geometric' zeta functions, related to the $p$-adic Igusa zeta functions: the motivic, Hodge, and topological zeta functions. We recall the definition of the local and global version of the topological zeta function.

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a non-zero holomorphic function $f$ (resp. $f \in \mathbb{C}[x_1, \ldots, x_n]$ non-zero, $f(0) = 0$). Let $B$ be an open ball centred at the origin. Let $\pi : X \to B$ (resp. $\pi : X \to \mathbb{C}^n$) be an embedded resolution of $(f^{-1}\{0\}, 0)$ (resp. $f^{-1}\{0\}$). We denote by $E_i, i \in J$, the irreducible components of $\pi^{-1}(f^{-1}\{0\})_{\mathrm{red}}$. Let $N_i$ (resp. $\nu_i - 1$) be the multiplicity of $f \circ \pi$ (resp. of $\pi^*(dx_1 \wedge \ldots \wedge dx_n)$) at a generic point of $E_i$. For $I \subset J$, we set $E_I := \cap_{i \in I} E_i$ and $E_I^\circ := E_I \setminus (\cup_{j \notin I} E_j)$.

The *local topological zeta function* $Z_{\mathrm{top},0}(f, s)$ (resp. *topological zeta function* $Z_{\mathrm{top}}(f, s)$) of $f$ at 0 (resp. of $f$) is the rational function defined by

$$Z_{\mathrm{top},0}(f, s) := \sum_{I \subset J} \chi\big(E_I^\circ \cap \pi^{-1}\{0\}\big) \prod_{i \in I} \frac{1}{\nu_i + N_i s} \in \mathbb{Q}(s), \qquad (*)$$

$$Z_{\mathrm{top}}(f, s) := \sum_{I \subset J} \chi\big(E_I^\circ\big) \prod_{i \in I} \frac{1}{\nu_i + N_i s} \in \mathbb{Q}(s),$$

respectively. In [**8**], Denef and Loeser proved that these rational functions are well defined (they do not depend on the resolution $\pi$), by expressing them as a kind of limit of $p$-adic Igusa zeta functions. We just mention that the motivic zeta function specializes to the topological zeta function and to the various $p$-adic Igusa zeta functions (for almost all $p$).

In this paper we study a piece of a remarkable conjecture of Igusa–Denef–Loeser, relating the poles of these zeta functions to roots of the Bernstein–Sato polynomial, modelled on the result for $Z_\varphi(f; s)$. We will treat poles of (maximal possible) order $n$. For the topological zeta function, it is clear that these occur if and only if there exist $n$ different components $E_i$ with the same quotient $\nu_i/N_i$ and having a non-empty intersection. For the other zeta functions the situation is analogous. For that reason we formulate everything in terms of the 'simplest' zeta function, being the topological one. Our results are, however, also valid for the other zeta functions mentioned.

CONJECTURE 1. The poles of $Z_{\mathrm{top},0}(f, s)$ are roots of the local Bernstein–Sato polynomial $b_{f,0}(s)$.

CONJECTURE 2. The function $b_{f,0}(s) \cdot Z_{\mathrm{top},0}(f, s)$ is a polynomial.

Conjecture 2 is a stronger version of Conjecture 1, saying that the order of a pole $s_0$ of $Z_{\mathrm{top},0}(f, s)$ is at most the multiplicity of $s_0$ as root of $b_{f,0}(s)$. For curves ($n = 2$) Conjecture 1 was proved by Loeser [**17**]. In that paper he also verified Conjecture 2 for *reduced* $f$. For arbitrary $n$ these conjectures are still wide open.

0.3. There is a well-known relation between roots of Bernstein–Sato polynomials and monodromy eigenvalues of $f$. In particular, if $s_0$ is a root of $b_{f,0}(s)$, then $\exp(2\pi i s_0)$ is an eigenvalue of the monodromy acting on some cohomology group of the (local) Milnor fibre of $f$ at some point of the germ of $f^{-1}\{0\}$ at 0 (equivalently; $\exp(2\pi i s_0)$ is a monodromy eigenvalue on the nearby cycle complex $\psi_f \mathbb{C}$). So the following conjecture, relating poles of $Z_{\mathrm{top},0}(f, s)$ to monodromy eigenvalues, is implied by Conjecture 1.

CONJECTURE 3. If $s_0$ is a pole of $Z_{\mathrm{top},0}(f, s)$, then $\exp(2\pi i s_0)$ is an eigenvalue of the local monodromy acting on some cohomology group of the Milnor fibre of $f$ at some point of the germ of $f^{-1}\{0\}$ at 0.

When $(f^{-1}\{0\},0)$ is a germ of an *isolated* singularity, then a result of Varchenko [**25**] relates the multiplicity of a root of $b_{f,0}(s)$ to the size of the monodromy Jordan blocks for the associated monodromy eigenvalue. A root of multiplicity $n$ corresponds essentially to a Jordan block of size $n$ (see Theorem 3 for the precise formulation).

This is certainly not true in general for non-isolated singularities: for any homogeneous $f$ its monodromy is finite and hence all Jordan blocks have size 1. And, for instance, when $f = \prod_{i=1}^{n} x_i^N$, we have that $b_{f,0}(s) = \prod_{j=1}^{N}(s - j/N)^n$. The 'right' generalization of Varchenko's result should be stated in terms of the sub-complex $\psi_{f,\lambda}\mathbb{C}$ of the nearby cycle complex $\psi_f\mathbb{C}$; see Section 1.

0.4. In this paper we investigate for an arbitrary $f$ in $n$ variables, assuming that its topological zeta function has a pole $s_0$ of maximal order $n$, the implications concerning $s_0$ being a root of $b_f(s)$ of multiplicity $n$, and concerning a possible associated monodromy Jordan block of size $n$. In a forthcoming paper we will study the case $n = 2$ more in detail, in particular for non-reduced $f$.

0.5. With the notation of 2 the *log canonical threshold* $c_0(f)$ of $f$ at 0 (resp. $c(f)$ of $f$) is defined as

$$c_0(f) := \min_{i\in J:0\in\pi(E_i)}\{\nu_i/N_i\}, \quad c(f) := \min_{i\in J:f^{-1}\{0\}\cap\pi(E_i)\neq\emptyset}\{\nu_i/N_i\};$$

see, for example, Proposition 8.5 in [**15**]. It does not depend on the resolution $\pi$ since, for example, $-c(f)$ (resp. $-c_0(f)$) is the root closest to the origin of the Bernstein–Sato polynomial $b_f(s)$ (resp. $b_{f,0}(s)$) of $f$ (at 0); see, for example, Theorem 10.6 in [**15**]. (In fact by results of Lichtin and Kashiwara every root of $b_f(s)$ is of the form $-(\nu_i + k)/N_i$, for some $i \in J$ and some integer $k \geqslant 0$; see Theorem 10.7 in [**15**].)

Clearly, $-c_0(f)$ is the candidate pole of $Z_{\mathrm{top},0}(f,s)$ closest to the origin. The third author has formulated the following.

CONJECTURE 4. (1) $Z_{\mathrm{top},0}(f,s)$ has at most one pole of order $n$.
(2) If $Z_{\mathrm{top},0}(f,s)$ has in $s_0$ a pole of order $n$, then $s_0$ is the pole closest to the origin of $Z_{\mathrm{top},0}(f,s)$.

This conjecture is proved in case $n = 2$ by Veys [**26**] and with Laeremans [**16**] when $f$ is non-degenerate with respect to its Newton polyhedron and in these cases $s_0 = -c_0(f)$ in (2).

Our main result is roughly as follows. Let $f : (\mathbb{C}^n,0) \to (\mathbb{C},0)$ be a germ of a holomorphic function with $f(0) = 0$ such that $s_0 = -c_0(f)$ is a pole of order $n$ of $Z_{\mathrm{top},0}(f,s)$. Denote $\lambda := \exp(2\pi i s_0)$. Then the $\lambda$-characteristic subspace of the $(n-1)$th cohomology of the Milnor fibre of $f$ at 0 has a non-zero $(2n-2)$-graded part of its weight filtration. Morally, '$\lambda$ has a Jordan block of size $n$ on the perverse sheaf $\psi_f\mathbb{C}$'. See Theorem 2 and Corollary 1 for a precise formulation. The result of Varchenko then implies the following.

THEOREM 1. *Let $f : (\mathbb{C}^n,0) \to (\mathbb{C},0)$ be a germ of a non-zero holomorphic function such that $(f^{-1}\{0\},0)$ is a germ of an isolated hyper-surface singularity. If $s_0 = -c_0(f)$ is a pole of order $n$ of $Z_{\mathrm{top},0}(f,s)$, then $(s + c_0(f))^n$ divides the Bernstein–Sato polynomial $b_{f,0}(s)$.*

*In such a case there exists an integer $N \geqslant 1$ such that $c_0(f) = 1/N$ and either*
*(1) $N = 1$ and $(s+1)^n$ divides $b_{f,0}(s)$, or*
*(2) $N > 1$ and $(s+1/N)^n(s+2/N)^n \dots (s+(N-1)/N)^n(s+1)^n$ divides $b_{f,0}(s)$.*

If Conjecture 4 is true, then Theorems 2 and 1 treat in fact 'all' poles of maximal order $n$. Our proof uses a result of Saito (Proposition 1) and the ideas in the proof of the main result of van Doorn and Steenbrink [**10**]. In Theorem 4 we provide a global version for polynomials $f$ and $Z_{\mathrm{top}}(f, s)$.

## 1. *Preliminaries*

### 1.1. *Monodromy*

Let $f$ be a holomorphic function on an $n$-dimensional complex manifold $X$. Denote by $X_t$ the hyper-surface $f^{-1}\{t\}$ for $t \in \mathbb{C}$. Let $x \in X_0$ and choose $\varepsilon, \eta > 0$ with $\eta \ll \varepsilon \ll 1$. The restriction of $f$ to $\{z \in X \mid |z - x| \leqslant \varepsilon,\ 0 < |f(z)| < \eta\}$ is a $C^\infty$ fibre bundle, *the Milnor fibration*, whose typical fibre

$$F_{f,x} := \{z \in X \mid |z - x| \leqslant \varepsilon, f(z) = \delta\} \quad \text{for } 0 < \delta < \eta$$

is called the *Milnor fibre* of $f$ at $x \in X_0$. The Milnor fibre is endowed with the monodromy automorphism $M_{f,x}$, which induces an automorphism, denoted by $M^q_{f,x}$, on the cohomology groups $H^q(F_{f,x}, \mathbb{C})$.

Following Deligne [**6**] one has a sheaf theoretic version of the previous constructions. Let $D$ be a small disk around the origin in $\mathbb{C}$, $D^* := D \setminus \{0\}$ and $\tilde{D}^*$ the universal covering of $D^*$. Consider the pre-image $X^*$ of $D^*$ in $X$ and denote by $\tilde{X}^*$ the fibre product $X^* \times_{D^*} \tilde{D}^*$. Let $i : X_0 \to X$ be the inclusion morphism and $j : \tilde{X}^* \to X$.

For the constructible sheaf $\mathbb{C}_X$ on $X$ and for any $q \geqslant 0$, the *nearby cycle sheaf* $R^q\psi_f\mathbb{C}_X := i^*R^qj_*j^*\mathbb{C}_X$ is a constructible sheaf on $X_0$. The deck transformation $(x, u) \mapsto (x, u + 1)$ on $\tilde{X}^*$ induces the action of a canonical monodromy automorphism $T^q$ on $R^q\psi_f\mathbb{C}_X$ such that the vector space $(R^q\psi_f\mathbb{C}_X, T^q)_x$ with automorphism is canonically isomorphic to $(H^q(F_{f,x}, \mathbb{C}), M^q_{f,x})$.

In fact, working on the derived category of complexes with automorphisms and bounded constructible cohomology, the *nearby cycle complex* $\psi_f\mathbb{C}_X$ on $X_0$ is defined by $\psi_f\mathbb{C}_X := i^*Rj_*j^*\mathbb{C}_X$; see [**6**]. Recall that the sheaf $\psi_f\mathbb{C}_X[n-1]$ is a perverse sheaf. The monodromy $T$ on the shifted perverse sheaf $\psi_f\mathbb{C}_X$ admits a decomposition $T = T_sT_u$, where $T_s$ is semi-simple and $T_u$ is unipotent. For $\lambda \in \mathbb{C}$, let

$$\psi_{f,\lambda}\mathbb{C}_X = \mathrm{Ker}\,(T_s - \lambda) \subset \psi_f\mathbb{C}_X.$$

There are also decompositions

$$\psi_f\mathbb{C}_X = \bigoplus\nolimits_\lambda \psi_{f,\lambda}\mathbb{C}_X, \quad H^q(F_{f,x}, \mathbb{C}) = \bigoplus\nolimits_\lambda H^q(F_{f,x}, \mathbb{C})_\lambda,$$

such that the action of $T_s$ on $\psi_{f,\lambda}\mathbb{C}_X$ and on $H^q(F_{f,x}, \mathbb{C})_\lambda$ is the multiplication by $\lambda \in \mathbb{C}^*$. The groups $H^q(F_{f,x}, \mathbb{C})_\lambda \oplus H^q(F_{f,x}, \mathbb{C})_{\bar{\lambda}} = \mathcal{H}^q(\psi_{f,\lambda}\mathbb{C}_X)_x \oplus \mathcal{H}^q(\psi_{f,\bar{\lambda}}\mathbb{C}_X)_x$ have a canonical mixed Hodge structure; see, for example, [**19**, **20**, **24**]. Let $W$ be the weight filtration of this canonical mixed Hodge structure. The following proposition is proved by Saito (for a proof see (1.1.3) and Proposition 1.7 in [**21**]).

Proposition 1 [**21**]. *Let $N$ be the logarithm of the unipotent part $T_u$ of the monodromy $T$. If $\mathrm{Gr}^W_{2n-2}H^{n-1}(F_{f,x}, \mathbb{C})_\lambda \neq 0$, then $N^{n-1} \neq 0$ on $\psi_{f,\lambda}\mathbb{C}_X$ in the category of shifted perverse sheaves.*

Since $X$ is smooth and $n$-dimensional, $N^n = 0$ on the nearby cycle sheaf $\psi_f\mathbb{C}_X[n-1]$. This implies that the Jordan blocks of the monodromy $M^q_{f,x}$ on the cohomology groups $H^q(F_{f,x}, \mathbb{C})$

have size $\leqslant q+1$ (see, for example, [**9**] and references there). In fact, it is also proved there that the support of the perverse sheaf $N^{n-1}\psi_f\mathbb{C}_X[n-1]$ is empty or 0-dimensional (see Proposition 0.5 in [**9**]).

### 1.2. *Bernstein–Sato polynomials*

Let $X$ be a complex $n$-dimensional manifold, resp. smooth algebraic variety, and let $X_0$ be the hyper-surface defined as the zero locus of a holomorphic function, resp. regular function, $f$. Let $\mathcal{D}_X$ be the ring of analytic, resp. algebraic, partial differential operators associated to $X$.

The Bernstein–Sato polynomial (or $b$-function) $b_f(s)$ of $f$ is the unique monic polynomial of lowest degree satisfying

$$b_f(s)f^s = Pf^{s+1} \quad \text{with } P \in \mathcal{D}_X[s].$$

It exists at least locally, and globally if $X$ is an affine algebraic variety [**2**, **4**, **22**]. Moreover, the $b$-function of a regular function $f$ and of its associated analytic function coincide. Restricting to the stalk at a point $x \in X_0$, one can also define the local $b$-function $b_{f,x}(s)$. If $X$ is Stein, resp. affine, then $b_f(s)$ is the least common multiple of these local $b$-functions.

Let $R_f$ be the set of the roots of $b_f(-s)$, and $m_\alpha$ the multiplicity of $\alpha \in R_f$. Then $R_f \subset \mathbb{Q}_{>0}$, and $m_\alpha \leqslant n$ because $b_f(s)$ is closely related to the monodromy on the nearby cycle sheaf $\psi_f\mathbb{C}_X$; see, for example, [**14**]. Set $\alpha_f = \min R_f$; this number coincides with the log canonical threshold; see [**15**, **19**].

## 2. *Monodromy on $\psi_f\mathbb{C}_X$ and poles of zeta functions*

2.1. We are interested in poles of maximal order $n$ of $Z_{\mathrm{top},0}(f,s)$. Laeremans and the third author [**16**] proved that every pole of maximal order of $Z_{\mathrm{top},0}(f,s)$ (or of $Z_{\mathrm{top}}(f,s)$) must be of the form $-1/N$, for a positive integer $N \geqslant 1$.

THEOREM 2. *Let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be a germ of a non-zero holomorphic function. If $s_0=-c_0(f)$ is a pole of order $n$ of $Z_{\mathrm{top},0}(f,s)$, then*

$$\mathrm{Gr}^W_{2n-2}H^{n-1}(F_{f,0},\mathbb{C})_\lambda \neq 0 \quad \text{for } \lambda := \exp(2\pi i s_0).$$

*In such a case, there exists an integer $N \geqslant 1$, such that $c_0(f)=1/N$ and either*

(1) *$N=1$ and $\mathrm{Gr}^W_{2n-2}H^{n-1}(F_{f,0},\mathbb{C})_1 \neq 0$ or*
(2) *$N>1$ and $\mathrm{Gr}^W_{2n-2}H^{n-1}(F_{f,0},\mathbb{C})_{\exp(2\pi i(-j/N))} \neq 0$ for all $j$ with $1 \leqslant j \leqslant N$.*

*Proof.* Assume $s_0$ is a pole of (maximal) order $n$ of $Z_{\mathrm{top},0}(f,s)$, then write $s_0 = -c_0(f) = -1/N$ for some integer $N \geqslant 1$ and set $\lambda := \exp(2\pi i(-1/N))$. To show that $\mathrm{Gr}^W_{2n-2}H^{n-1}(F_{f,0},\mathbb{C})_\lambda \neq 0$, we will adapt the proof of the main result of van Doorn and Steenbrink in [**10**]; see also Varchenko [**25**].

Let $B$ be an open ball centred at the origin. Let $\pi: X \to B$ be an embedded resolution of the germ $(f^{-1}\{0\},0)$, which is an isomorphism outside of the pre-image of $f^{-1}\{0\}$. Set $E := \pi^{-1}(f^{-1}\{0\})$ and $E_x := \pi^{-1}(0)$ and denote by $E_i, i \in J$, the irreducible components of $E$. For $I \subset J$, put also $E_I := \bigcap_{j\in I} E_j$.

By the definition of the local topological zeta function, see $(*)$, since $s_0$ is a pole of order $n$ of $Z_{\mathrm{top},0}(f,s)$, there exist $n$ irreducible components $E_0,\ldots,E_{n-1}$ of $E$ and there is a point $\tilde{x}_0 \in \cap_{i=0}^{n-1} E_i$ such that $\pi(\tilde{x}_0)=0$ and $\nu_i/N_i = 1/N$ for all $0 \leqslant i \leqslant n-1$.

We may assume that one of these irreducible components, called, for example, $E_0$, is a Kähler compact non-singular variety. Otherwise we blow up $X$ at $\tilde{x}_0$ and get a new configuration of exceptional divisors where the new exceptional divisor is a Kähler compact non-singular variety $E_0$ with $\nu_0/N_0 = 1/N$, and we can choose a 'new' $\tilde{x}_0$ on this $E_0$ satisfying the requirements above.

To describe the quotient $\mathrm{Gr}^W_{2n-2}H^{n-1}(F_{f,0},\mathbb{C})_\lambda$, one uses the fact that it is pure of type $(n-1,n-1)$ and, therefore, a quotient of the piece $F^{n-1}$ of the Hodge filtration. These terms can be computed using the relative logarithmic de Rham complex.

Let $e$ be a common multiple of all multiplicities $N_i, i \in J$, and let $\tilde{\mathbb{C}}$ be another copy of $\mathbb{C}$. Let $\tilde{Y}$ be the normalization of the space $Y$ obtained from $X$ by the base change $\sigma : \tilde{\mathbb{C}} \to \mathbb{C} : \sigma(\tilde{t}) = \tilde{t}^e$. Let $\rho : \tilde{Y} \to X$ and $\tilde{f} : \tilde{Y} \to \tilde{\mathbb{C}}$ be the natural projection maps. Let $D_i := \rho^{-1}(E_i), i \in J$, and set $D := \rho^{-1}(E)$; this is nothing but $D = \cup_{i\in J} D_i$. Let $D_x := \rho^{-1}(E_x)$. For every $I \subset J$, let $D_I := \rho^{-1}(E_I)$. The map $D_I \to E_I$ is a cyclic cover of degree $\gcd(N_i,\, i \in I)$.

$$
\begin{tikzcd}
\tilde{Y} \arrow[dr, dotted] \arrow[drr, bend left, "\rho"] \arrow[ddr, bend right, "\tilde{f}"'] & & \\
& Y = \tilde{\mathbb{C}} \times_{\mathbb{C}} X \arrow[r] \arrow[d] & X \arrow[d, "f\circ\pi"] \\
& \tilde{\mathbb{C}} \arrow[r, "\sigma"] & \mathbb{C}
\end{tikzcd}
$$

By the semi-stable reduction theorem, $\pi$ and $e$ can be chosen in such a way that $\tilde{Y}$ is smooth. The divisor $D = \tilde{f}^{-1}(0)$ is a reduced normal crossing divisor; see [**24**]. From [**23**], see also [**24**], there is an isomorphism

$$H^q(F_{f,0},\mathbb{C}) \simeq \mathbb{H}^q\big(D_x, \Omega^\bullet_{\tilde{Y}/\tilde{\mathbb{C}}}(\log D) \otimes \mathcal{O}_{D_x}\big),$$

so in particular $\mathrm{Gr}^p_F H^q(F_{f,0},\mathbb{C}) \simeq H^{q-p}(D_x, \Omega^p_{\tilde{Y}/\tilde{\mathbb{C}}}(\log D) \otimes \mathcal{O}_{D_x})$. Then

$$
\begin{aligned}
F^{n-1}H^{n-1}(F_{f,0},\mathbb{C}) &\simeq \mathrm{Gr}^{n-1}_F H^{n-1}(F_{f,0},\mathbb{C}) \simeq H^0\big(D_x, \Omega^{n-1}_{\tilde{Y}/\tilde{\mathbb{C}}}(\log\, D) \otimes \mathcal{O}_{D_x}\big) \\
&\simeq H^0\big(D_x, \Omega^n_{\tilde{Y}}(\log\, D) \otimes \mathcal{O}_{D_x}\big).
\end{aligned}
$$

The following results can be deduced from Section 4 in [**25**]. For every $\omega \in H^0(B,\Omega^n)$, define the *geometrical weight* $g(\omega)$ with respect to the resolution $\pi$ as

$$g(\omega) := \min_{i\in J}\left\{\frac{\mathrm{ord}_{E_i}(\omega)+1}{N_i}\right\}.$$

For every $\omega \in H^0(B,\Omega^n)$ with *geometrical weight* $g(\omega) \leqslant 1$, define $R(\omega) := \tilde{f}^{-e/N}(\pi\rho)^*(\omega)$. Then $R(\omega) \in H^0(\tilde{Y}, \Omega^n_{\tilde{Y}}(\log D))$. Let $\sigma(\omega)$ be its Poincaré residue along $D_x$, that is, $\sigma(\omega)$ is the restriction to $D_x$ of $\tilde{f}^{(-e/N)+1}(\pi\rho)^*(\omega)/d\tilde{f}$. Then $\sigma(\omega)$ is an element in $F^{n-1}H^{n-1}(F_{f,0},\mathbb{C})$ and the semi-simple part of the monodromy acts on $\sigma(\omega)$ as

$$T_s(\sigma(\omega)) = \exp(-2\pi i g(\omega))\sigma(\omega).$$

The form $R(\omega)$ has a first-order pole along $D_i$ if and only if $g(\omega) = (\mathrm{ord}_{E_i}(\omega)+1)/N_i$, and else $R(\omega)$ is regular along $D_i$.

Consider the differential form $\eta = dx_1 \wedge \ldots \wedge dx_n$. Since $s_0 = -c_0(f) = -1/N$ does not depend on the resolution, $g(\eta) = 1/N \leqslant 1$ does not depend on $\pi$. In fact $1/N$ is the minimum;

so $R(\eta)$ has a first-order pole along $D_i$ if and only if $1/N = \nu_i/N_i$, and else $R(\eta)$ is regular along $D_i$.

Let $D_{00}$ be one of the irreducible components of $D_0$. On the open subspace $D_{00}^\circ = D_{00} \setminus (\cup_{j\neq 0} D_j)$, the restriction of the map $\rho$ from $D_{00}^\circ$ to $E_0^\circ$ is an étale cover. The form $\sigma(\eta)$ is not equal to zero on $D_{00}^\circ$ and is in fact a meromorphic $(n-1)$-form with logarithmic poles on $D_{00} \setminus D_{00}^\circ$. Thus $R(\eta)$ defines a non-zero element in $H^0(\tilde{Y}, \Omega^n_{\tilde{Y}}(\log D))$. Notice that, according to Deligne's theorem [**5**, **23**, **25**], the class of $\sigma(\eta)$ in $H^{n-1}(D_{00}^\circ, \mathbb{C})$ is a *non-zero element* since $D_{00}$ is a projective manifold and $D_{00} \setminus D_{00}^\circ \subset D_{00}$ is a divisor with normal crossings.

On an adequate chart on $\tilde{Y}$, with local coordinates $y_1, \dots, y_n$, the function $\tilde{f}$ is given by

$$\tilde{f}(y_1, \dots, y_n) = y_1 y_2 \dots y_k.$$

Moreover, if $y_1 = 0$ is the equation of the divisor $D_{00}$ (and hence of $D_0$) and $y_j = 0$ gives the divisor $D_{00} \cap D_j$, then

$$\sigma(\eta) = q(y_2, \dots, y_n) \cdot y_2^{a_2} \dots y_k^{a_k} \frac{dy_2}{y_2} \wedge \dots \wedge \frac{dy_k}{y_k} \wedge dy_{k+1} \wedge \dots \wedge dy_n,$$

where $q(y_2, \dots, y_n)$ is holomorphic, $q(0) \neq 0$ and $a_j = e(\nu_{j(i)}/N_{j(i)} - 1/N)$.

In particular, at a pre-image $P_0$ of $\tilde{x}_0$ in $\tilde{Y}$, $R(\eta)$ can be written locally as $u(dy_1/y_1) \wedge \dots \wedge (dy_n/y_n)$, with $u(0) \neq 0$, because of the minimum. Considering, for each $n$-fold point $P$ on $D_x$, the multiple residue map $\mathrm{Res}_P : H^0(\tilde{Y}, \Omega^n_{\tilde{Y}}(\log D)) \to \mathbb{C}$, we have in particular that $\mathrm{Res}_{P_0}$ is surjective.

Let $V_\lambda$ be the set of the $n$-fold points of $D$ that are pre-images of those $n$-fold points in $\cup_I E_I$ for which $|I| = n$, $\lambda^{N_i} = 1$ for all $i \in I$, and at least one $E_i, i \in I$, is an irreducible component of $E_x$. Then

$$\mathrm{Gr}^W_{2n-2} H^{n-1}(F_{f,0}, \mathbb{C})_\lambda \cong \mathrm{Image}\big(\textstyle\bigoplus_{P \in V_\lambda} \mathrm{Res}_P : H^0\big(\tilde{Y}, \Omega^n_{\tilde{Y}}(\log D)\big) \to \mathbb{C}^{V_\lambda}\big);$$

see [**23**] and [**10**].

Since $P_0 \in V_\lambda$ and $\mathrm{Res}_{P_0}$ is surjective then $\mathrm{Gr}^W_{2n-2} H^{n-1}(F_{f,0}, \mathbb{C})_\lambda \neq 0$, which concludes the proof.

To show that $\mathrm{Gr}^W_{2n-2} H^{n-1}(F_{f,0}, \mathbb{C})_{\exp(2\pi i(-j/N))} \neq 0$ also for $2 \leqslant j \leqslant N$, we argue as follows; see [**10**]. Since the weight filtration is defined over $\mathbb{Q}$, it has a complex conjugation compatible with that of $\mathbb{C}^{V_\lambda}$. Let $A_{P_0} = \{g(\omega) : \omega \in H^0(B, \Omega^n),\ g(\omega) \leqslant 1 \text{ and } \mathrm{Res}_{P_0}(R(\omega)) \neq 0\}$. Then $g(\eta) = 1/N \in A_{P_0}$. If $1/N < 1$, then the complex conjugate of $\mathrm{Res}_{P_0}(R(\eta))$ in $\mathbb{C}^{V_\lambda}$ is an eigenvector of the semi-simple part of the monodromy $T_s$ for the eigenvalue $\bar{\lambda} = \exp(2\pi i(1/N))$. In particular there exists $\tilde{\eta} \in H^0(B, \Omega^n)$ such that $g(\tilde{\eta}) = (N-1)/N \in A_{P_0}$. After the remark on page 230 in [**10**], one can prove that $\{1/N, 2/N, \dots, (N-1)/N, 1\} \subset A_{P_0}$. □

2.2. Using Proposition 1 one 'morally' obtains a Jordan block of size $n$ in the category of shifted perverse sheaves.

COROLLARY 1. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a non-zero holomorphic function. If $s_0 = -c_0(f)$ is a pole of order $n$ of $Z_{\mathrm{top},0}(f, s)$ and $\lambda := \exp(2\pi i s_0)$, then $N^{n-1} \neq 0$ on $\psi_{f,\lambda}\mathbb{C}_X$ in the category of shifted perverse sheaves.*

## 3. *Applications for isolated hyper-surface singularities*

3.1. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function such that $(f^{-1}\{0\}, 0)$ is a germ of an isolated hyper-surface singularity. The following result, by Varchenko [**25**] Theorem 1.4, relates roots of the Bernstein–Sato polynomial $b_{f,0}(s)$ and Jordan blocks of the algebraic monodromy. Let $\tilde{b}_{f,0}(s)$ be the *microlocal (or reduced)* Bernstein–Sato polynomial defined by $b_{f,0}(s) = (s+1)\tilde{b}_{f,0}(s)$.

THEOREM 3 [**25**]. *Let $M_{f,0}^{n-1}$ be the algebraic monodromy action on the $(n-1)$th cohomology $H^{n-1}(F_{f,0}, \mathbb{C})$ of the Milnor fibre of $f$ at the origin.*

(i) *$\tilde{b}_{f,0}(s)$ is divisible by $(s-\beta)^n$ if and only if $\beta \in (-1, 0)$ and $M_{f,0}^{n-1}$ has a Jordan block of size $n$ for the eigenvalue* $\exp(2\pi i(\beta))$.

(ii) *$\tilde{b}_{f,0}(s)$ is divisible by $(s+1+\alpha)^{n-1}$, with $\alpha \in \mathbb{Z}_+$, if and only if $\alpha = 0$ and $M_{f,0}^{n-1}$ has a Jordan block of size $n-1$ for the eigenvalue* 1.

*Proof of Theorem* 1 (see Introduction). The proof follows from Theorems 2 and 3, together with the fact that for the eigenvalue $\lambda = 1$ (resp. $\lambda \neq 1$), $\mathrm{Gr}_{2n-2}^W H^{n-1}(F_{f,0}, \mathbb{C})_\lambda \neq 0$ if and only if $M_{f,0}^{n-1}$ has a Jordan block of size $n-1$ (resp. of size $n$); see [**10**] and [**23**]. □

THEOREM 4. *Let $f \in \mathbb{C}[z_1, \ldots, z_n]$ be a polynomial such that $f^{-1}\{0\}$ has only isolated singularities. If $s_0 = -c(f)$ is a pole of order $n$ of $Z_{\mathrm{top}}(f, s)$, then $(s + c(f))^n$ divides the Bernstein–Sato polynomial $b_f(s)$.*

*In such a case there exists an integer $N \geqslant 1$ such that $c(f) = 1/N$ and either*

(1) *$N = 1$ and $(s+1)^n$ divides $b_f(s)$, or*

(2) *$N > 1$ and $(s+1/N)^n(s+2/N)^n \ldots (s+(N-1)/N)^n(s+1)^n$ divides $b_f(s)$.*

*Proof.* The proof follows from the following two facts. First, a pole of order $n$ of the local topological zeta function is also a pole of the global $Z_{\mathrm{top}}(f, s)$, and conversely, a pole of order $n$ of $Z_{\mathrm{top}}(f, s)$ is a pole of some $Z_{\mathrm{top},x}(f, s)$ at some point $x \in f^{-1}\{0\}$. Second, $b_f(s)$ is the least common multiple of all local Bernstein–Sato polynomials $b_{f,x}(s)$. □

### 3.2. *Non-degenerate Newton polyhedron*

For the notion of a function that is non-degenerate with respect to its Newton polyhedron, we refer, for instance, to [**16**] or [**1**]. Remark that almost all polynomials are non-degenerate with respect to their (either local or global) Newton polyhedron (see [**1**, p. 151]).

For such functions, Denef proved that a set of candidate poles of the corresponding zeta functions is obtained from the $(n-1)$-dimensional faces of the corresponding polyhedron; for example, see [**7**]. Loeser [**18**] proved that under some additional conditions these candidate poles are roots of the Bernstein–Sato polynomial $b_f(s)$.

COROLLARY 2. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function that defines a germ of an isolated hyper-surface singularity that is non-degenerate with respect to its Newton polyhedron at the origin. If $s_0$ is a pole of order $n$ of $Z_{\mathrm{top},0}(f, s)$ then $(s + s_0)^n$ divides $b_{f,0}(s)$.*

*Proof.* Under the hypothesis, Laeremans and the third author proved in Theorem 2.4 in [**16**] that $Z_{\text{top},0}(f,s)$ has at most one pole of order $n$. Moreover, if such a pole exists, then it is the pole closest to the origin that coincides with $-c_0(f)$. Thus, after Theorem 1 we get the result. □

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