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Monodromy Jordan blocks, b-functions and poles of zeta functions for germs of plane curves ☆

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A B S T R A C T

We study the poles of several local zeta functions: the Igusa, topological and motivic zeta function associated to a germ of a holomorphic function in two variables. It was known that there is at most one double pole for (any of) these zeta functions which is then given by the log canonical threshold of the function at the singular point. If the germ is reduced Loeser showed that such a double pole always induces a monodromy eigenvalue with a Jordan block of size 2. Here we settle the non-reduced situation, describing precisely in which case such a Jordan block of maximal size 2 occurs. We also provide detailed information about the Bernstein–Sato polynomial in the relevant non-reduced situation, confirming a conjecture of Igusa, Denef and Loeser.

Introduction

0.1. To a polynomial or analytic function f defined over various fields are associated several (related) zeta functions: the Igusa, topological, motivic and Hodge zeta function. They are essentially invariants

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of the singularities of the associated hypersurface (germ), and occur in particular in fascinating conjectures linking them to monodromy and to Bernstein–Sato polynomials. We first recall the definition of the Igusa and topological zeta function.

Let $f : X \to \mathbb{Q}_p$ be a non-constant ($\mathbb{Q}_p$-)analytic function on a compact open $X \subset \mathbb{Q}_p^n$, where $\mathbb{Q}_p$ denotes the field of p-adic numbers. Let $|\cdot|_p$ and $|dx|$ denote the p-adic norm and the Haar measure on $\mathbb{Q}_p^n$, normalized in the standard way. The p-adic integral

$$Z_p(f;s) := \int_X \left|f(x)\right|_p^s |dx|,$$

defined for $s \in \mathbb{C}$ with $\Re(s) > 0$, is called the (p-adic) Igusa zeta function of f. Using resolution of singularities Igusa [11,12] showed that it is a rational function of p^{-s}; hence it also admits a meromorphic continuation to $\mathbb{C}$. (Everything can be generalized to finite extensions of $\mathbb{Q}_p$.)

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a nonzero holomorphic function f. Let B be an open ball centered at the origin. Let $\pi : X \to B$ be an embedded resolution of $(f^{-1}\{0\}, 0)$. We denote by E_i, $i \in J$, the irreducible components of $\pi^{-1}(f^{-1}\{0\})_{\mathrm{red}}$. Let N_i (resp. $\nu_i - 1$) be the multiplicity of $f \circ \pi$ (resp. of $\pi^*(dx_1 \wedge \cdots \wedge dx_n)$) at a generic point of E_i. For $I \subset J$, we set $E_I := \bigcap_{i \in I} E_i$ and $E_I^\circ := E_I \setminus (\bigcup_{j \notin I} E_j)$.

The (*local*) *topological zeta function* $Z_{\mathrm{top},0}(f,s)$ of f at 0 is the rational function defined by

$$Z_{\mathrm{top},0}(f,s) := \sum_{I \subset J} \chi\left(E_I^\circ \cap \pi^{-1}\{0\}\right) \prod_{i \in I} \frac{1}{\nu_i + N_i s} \in \mathbb{Q}(s).$$

In [8], Denef and Loeser proved that this rational function is well defined (it does not depend on the resolution π), by expressing it as a kind of limit of p-adic Igusa zeta functions. We just mention that the motivic and Hodge zeta functions are other 'algebro-geometric' zeta functions, defined over an arbitrary field of characteristic zero, and that the motivic zeta function specializes to the topological zeta function and to the various p-adic Igusa zeta functions (for almost all p).

0.2. In this paper we mainly study a piece of a remarkable conjecture of Igusa, Denef and Loeser, relating the poles of these zeta functions to roots of the Bernstein–Sato polynomial, modeled on a result for complex integrals, defined similarly as the p-adic integrals defining the Igusa zeta function [4,13]. We will treat poles of (maximal possible) order n. For the topological zeta function it is clear that these occur if and only if there exist n different components E_i with the same quotient ν_i/N_i and having a non-empty intersection. For the other zeta functions, due to similar explicit formulas in terms of an embedded resolution, the situation is analogous. For that reason we formulate everything in terms of the 'simplest' zeta function, being the topological one. Our results are however valid also for the other mentioned zeta functions.

Conjecture 1. *The poles of $Z_{\mathrm{top},0}(f,s)$ are roots of the local Bernstein–Sato polynomial $b_{f,0}(s)$.*

Conjecture 2. *The function $b_{f,0}(s) \cdot Z_{\mathrm{top},0}(f,s)$ is a polynomial.*

Conjecture 2 is a stronger version of Conjecture 1, saying that the order of a pole s_0 of $Z_{\mathrm{top},0}(f,s)$ is at most the multiplicity of s_0 as root of $b_{f,0}(s)$. For curves ($n = 2$) Conjecture 1 was proved by Loeser [19]. In that paper he also verified Conjecture 2 for *reduced* f. For arbitrary n these conjectures are still wide open. (Loeser also proved Conjecture 1 for non-degenerate polynomials satisfying some extra assumptions [20].)

0.3. There is a well-known relation between roots of Bernstein–Sato polynomials and monodromy eigenvalues of f. In particular, if s_0 is a root of $b_{f,0}(s)$, then $\exp(2\pi i s_0)$ is an eigenvalue of the monodromy acting on some cohomology group of the (local) Milnor fibre of f at some point of the germ of $f^{-1}\{0\}$ at 0 (equivalently, $\exp(2\pi i s_0)$ is a monodromy eigenvalue on the nearby cycle complex $\psi_f\mathbb{C}$). So the following conjecture, relating poles of $Z_{\mathrm{top},0}(f,s)$ to monodromy eigenvalues, is implied by Conjecture 1.

Conjecture 3. *If s_0 is a pole of $Z_{\mathrm{top},0}(f,s)$, then $\exp(2\pi i s_0)$ is an eigenvalue of the local monodromy acting on some cohomology group of the Milnor fibre of f at some point of the germ of $f^{-1}\{0\}$ at 0.*

When $(f^{-1}\{0\},0)$ is a germ of an *isolated* singularity, the following result, by Varchenko [33] Theorem 1.4, relates roots of the Bernstein–Sato polynomial $b_{f,0}(s)$ and Jordan blocks of the algebraic monodromy. Let $\tilde{b}_{f,0}(s)$ be the *microlocal* (*or reduced*) Bernstein–Sato polynomial defined by $b_{f,0}(s)=(s+1)\tilde{b}_{f,0}(s)$.

Theorem 1. *(See [33].) Let $M^{n-1}_{f,0}$ be the algebraic monodromy action on the $(n-1)$-th cohomology $H^{n-1}(F_{f,0},\mathbb{C})$ of the Milnor fibre of f at the origin.*

(1) *$\tilde{b}_{f,0}(s)$ is divisible by $(s-\beta)^n$ if and only if $\beta>-1$ and $M^{n-1}_{f,0}$ has a Jordan block of size n for the eigenvalue $\exp(2\pi i(\beta))$.*
(2) *$\tilde{b}_{f,0}(s)$ is divisible by $(s+1-\alpha)^{n-1}$, with $\alpha\in\mathbb{Z}$, if and only if $\alpha=0$ and $M^{n-1}_{f,0}$ has a Jordan block of size $n-1$ for the eigenvalue 1.*

This is certainly not true in general for non-isolated singularities: for any homogeneous f its monodromy is finite and hence all Jordan blocks have size 1. And for instance when $f=\prod_{i=1}^n x_i^N$ we have that $b_{f,0}(s)=\prod_{i=1}^N (s-i/N)^n$. The 'right' generalization of Varchenko's result should be stated in terms of the sub-complex $\psi_{f,\lambda}\mathbb{C}$ of the nearby cycle complex $\psi_f\mathbb{C}$; see [22].

0.4. With the notation of 0.1 the *log canonical threshold* $c_0(f)$ of f at 0 is defined as

$$c_0(f):=\min_{i\in J:\ 0\in\pi(E_i)}\{\nu_i/N_i\},$$

see e.g. Proposition 8.5 in [16]. It does not depend on the resolution π since e.g. $-c_0(f)$ is the root closest to the origin of the Bernstein–Sato polynomial $b_{f,0}(s)$ of f at 0, see Theorem 10.6 in [16] or [18,37]. (In fact by results of Lichtin and Kashiwara every root of $b_f(s)$ is of the form $-\frac{\nu_i+k}{N_i}$, for some $i\in J$ and some integer $k\geqslant 0$, see Theorem 10.7 in [16].) Clearly $-c_0(f)$ is the candidate pole of $Z_{\mathrm{top},0}(f,s)$ closest to the origin.

Using Varchenko's theorem the authors have proved in [22] Theorem 1:

Theorem 2. *Let $f:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ be a germ of a nonzero holomorphic function such that $(f^{-1}\{0\},0)$ is a germ of an isolated hypersurface singularity. If $s_0=-c_0(f)$ is a pole of order n of $Z_{\mathrm{top},0}(f,s)$, then $(s+c_0(f))^n$ divides the Bernstein–Sato polynomial $b_{f,0}(s)$.*

In such a case there exists an integer $N\geqslant 1$ such that $c_0(f)=1/N$ and either

- *$N=1$ and $(s+1)^n$ divides $b_{f,0}(s)$, or*
- *$N>1$ and $(s+1/N)^n(s+2/N)^n\cdots(s+(N-1)/N)^n(s+1)^n$ divides $b_{f,0}(s)$.*

0.5. In this paper we mainly study the case $n=2$, in particular we make the situation concerning monodromy completely clear, answering a question of C.T.C. Wall [35]. By [17] and [34], $Z_{\mathrm{top},0}(f,s)$ has at most one pole of order 2, and if s_0 is such a pole then $s_0=-c_0(f)=-1/N$ for some positive integer N.

In Section 1 we show the following concerning the size of the associated monodromy Jordan block on the first cohomology of the Milnor fibre. Let $f = \prod_{j\in T} f_j^{N_j}$ be the decomposition of f into irreducible germs.

Theorem 5. *Suppose that $s_0 = -c_0(f) = -1/N$ is a pole of order two of $Z_{\mathrm{top},0}(f,s)$. Denote $\lambda := \exp(2\pi i s_0)$.*

(i) *If $N \neq N_j$ for all $j \in T$, then the monodromy eigenvalue λ of f has a Jordan block of size* 2.
(ii) *If $N = N_j$ for some $j \in T$, then λ has only Jordan blocks of size* 1.

(For reduced f this was considered in [19]; but then case (ii) can only occur if f is (analytically) of the form $f = xy$.) The dichotomy in Theorem 5 can also be described in terms of the minimal part of the dual resolution graph with respect to the quotient of its numerical data ν_i/N_i, see Section 1. In the course of the proof we show a property of arbitrary chains between two rupture vertices in the dual resolution graph (Proposition 2), that could be of independent interest.

When f is reduced, Loeser [19] actually proved Conjecture 2: the function $b_{f,0}(s) \cdot Z_{\mathrm{top},0}(f,s)$ is a polynomial. Assume below that f is not reduced.

Suppose that $-c_0(f) = -1/N$ is a pole of order two of $Z_{\mathrm{top},0}(f,s)$, and denote $\lambda := \exp(2\pi i s_0)$. In case (i) of Theorem 5 we have that λ has a Jordan block of size 2, and then by [19] one can conclude that $(s+1/N)^2$ divides $b_{f,0}(s)$. However in case (ii) of Theorem 5 we have that λ has only Jordan blocks of size 1 and then the argument of [19] fails. (So the conclusion there should be restricted to our case (i)!)

It turns out (see Proposition 1) that the remaining case to investigate concerning Conjecture 2 is the following. *Let $f = x^N g$ where $N \geqslant 2$, g is not a multiple of x, and the intersection number of $x = 0$ and $g = 0$ in the origin is N. Does $(s+1/N)^2$ divide $b_{f,0}(s)$?*

Studying Bernstein–Sato polynomials for non-reduced f is in general very difficult; in fact it was not treated before – except if f is a monomial. In Sections 2 and 3 we treat such $f = x^N g$ and in particular we answer the question above positively whenever g is weighted homogeneous and reduced; see Proposition 3. For instance if the degree of g as a polynomial in $(\mathbb{C}[x])[y]$ is equal to N, we prove also that $b_{f,0}(s)$ is divisible by $\prod_{\ell=1}^{N}(s+\ell/N)^2$ as in Theorem 2. Moreover we provide much more detailed information about $b_{f,0}(s)$. In particular we obtain in Theorem 6 of Section 3 the first closed formulae in such a non-reduced setting.

For example the local topological zeta function $Z_{\mathrm{top},0}(f,s)$ of the germ of plane curve singularity at the origin defined by $f = x^3(y^3+x^2)$ has a unique pole of order two which is $s_0 = -1/3$ and, as Conjecture 2 predicts, $(s+1/3)^2$ divides $b_{f,0}(s)$, after Proposition 3. Nevertheless, by Theorem 5, $\lambda := \exp(-2\pi i/3)$ has only Jordan blocks of size 1. On the other hand, by [22] Corollary 1, 'λ has a Jordan block of size 2 on the perverse sheaf $\psi_f\mathbb{C}$'.

Finally we treat in Section 4 an instance of the case $n = 3$, more precisely superisolated surface singularities. We make Theorem 2 more precise, showing in particular that there is at most one pole of maximal order 3 (see Theorem 8). This confirms a conjecture of the third author.

1. Monodromy Jordan blocks for curves

1.1. For completeness we first recall the definition of monodromy. Let f be a holomorphic function on an n-dimensional complex manifold X. Denote by X_t the hypersurface $f^{-1}\{t\}$ for $t \in \mathbb{C}$. Let $x \in X_0$ and choose $\varepsilon, \eta > 0$ with $\eta \ll \varepsilon \ll 1$. The restriction of f to $\{z \in X \mid |z-x| \leqslant \varepsilon,\ 0 < |f(z)| < \eta\}$ is a C^∞ fibre bundle, *the Milnor fibration*, whose typical fibre

$$F_{f,x} := \left\{z \in X \;\middle|\; |z-x| \leqslant \varepsilon,\ f(z) = \delta\right\} \quad \text{for } 0 < \delta < \eta$$

is called the *Milnor fibre* of f at $x \in X_0$. The Milnor fibre is endowed with the monodromy automorphism $M_{f,x}$ which induces an automorphism, denoted by $M^q_{f,x}$, on the cohomology groups $H^q(F_{f,x}, \mathbb{C})$.

1.2. Let now $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be an algebraic or analytic function germ which is singular at the origin. Let $f = \prod_i f_i^{N_i}$ be its decomposition into irreducible germs and denote $d := \gcd_i N_i$.

In case f is reduced, Loeser already noticed in [19] that if s_0 ($\neq -1$) is a pole of order 2 of $Z_{\mathrm{top},0}(f, s)$, then $\exp(2\pi i s_0)$ has a Jordan block of size 2 of the corresponding monodromy $M^1_{f,0}$ on $H^1(F_{f,0}, \mathbb{C})$. The main topic in this section is to provide a clear answer in the non-reduced case concerning the existence of a corresponding Jordan block of the monodromy of size 2, see Theorem 5.

1.3. We consider the algebraic monodromy action $M^q_{f,0}$ on the q-th cohomology $H^q(F_{f,0}, \mathbb{C})$ of the Milnor fibre of f at the origin for $q = 0, 1$. It is well known that the characteristic polynomial of $M^0_{f,0}$ is $t^d - 1$.

Denote by $\Delta(t)$ the characteristic polynomial of $M^1_{f,0}$ and by $\Delta_2(t)$ the characteristic polynomial of $M^1_{f,0}$ on the quotient $H^1(F_{f,0}, \mathbb{C})/\mathrm{Ker}((M^1_{f,0})^k - 1)$ where k is sufficiently large. So the roots of Δ are the eigenvalues of $M^1_{f,0}$ and the roots of Δ_2 those belonging to the Jordan blocks of size 2. We recall the determination of Δ and Δ_2 in terms of an embedded resolution, see e.g. [24,31].

Let $\pi : X \to (\mathbb{C}^2, 0)$ be the minimal embedded resolution of $(f^{-1}\{0\}, 0)$. We denote as usual by E_i, $i \in J$, the irreducible components of $\pi^{-1}(f^{-1}\{0\})$, and by N_i their multiplicity in the divisor of $f \circ \pi$ on X. In the sequel we exclude the case where $f^{-1}\{0\}$ is already a normal crossings divisor, i.e. $f(x, y)$ is (analytically) of the form $x^{N_1} y^{N_2}$. (For this case the corresponding results are obvious.)

In the (dual) resolution graph Γ one associates to each E_i a vertex v_i, more precisely an ordinary vertex to each exceptional E_i and an arrowhead vertex to each (analytically) irreducible component of the strict transform of $f^{-1}\{0\}$. Each intersection between E_i and E_j is indicated by an edge connecting v_i and v_j; we sometimes denote this edge by e_{ij}. We put $\mathcal{V} := \{\text{ordinary vertices}\}$, $\mathcal{A} := \{\text{arrowhead vertices}\}$ and $\mathcal{E} := \{\text{edges between vertices in } \mathcal{V}\}$.

For $v \in \mathcal{V} \cup \mathcal{A}$ we put

$$N_v := N_i \quad \text{if } v \text{ corresponds to } E_i,$$
$$\delta_v := \text{number of incident edges to } v,$$
$$m_v := \gcd\{N_w \mid w \text{ is a vertex adjacent or equal to } v\}.$$

Note that $\delta_v = 1$ for $v \in \mathcal{A}$. A vertex v with $\delta_v \geqslant 3$ is classically called a *rupture vertex* (corresponding to a *rupture component*). For $e \in \mathcal{E}$ we put $m_e = m_{ij} := \gcd\{N_i, N_j\}$ if e connects v_i and v_j.

Theorem 3. *(See [1,24,31].) With notation as above we have*

$$\Delta(t) = \left(t^d - 1\right) \prod_{v \in \mathcal{V}} \left(t^{N_v} - 1\right)^{\delta_v - 2} \tag{1}$$

and

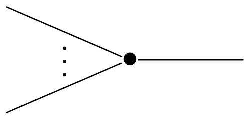
$$\Delta_2(t) = \left(t^d - 1\right) \frac{\prod_{e \in \mathcal{E}} (t^{m_e} - 1)}{\prod_{v \in \mathcal{V}} (t^{m_v} - 1)}. \tag{2}$$

Denote furthermore by $\mathcal{V}'$ the set of separating rupture vertices v, i.e. those with arrowheads in at least two components of $\Gamma \setminus \{v\}$, and by $\mathcal{E}'$ a subset of $\mathcal{E}$ consisting of just one edge from each chain connecting two separating rupture vertices. Then

$$\Delta_2(t) = \left(t^d - 1\right) \frac{\prod_{e \in \mathcal{E}'} (t^{m_e} - 1)}{\prod_{v \in \mathcal{V}'} (t^{m_v} - 1)}. \tag{3}$$

1.4. We denote by $\nu_i - 1$ the multiplicity of E_i in the divisor of $\pi^*(dx \wedge dy)$ on X. (In particular $\nu_i \geqslant 1$, and $\nu_i > 1$ if and only if E_i is an exceptional component.) We recall the ordered tree structure of Γ with respect to the $\frac{\nu_i}{N_i}$, $i \in J$, found by the third author.

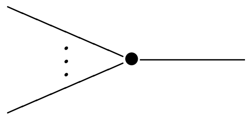
Convention: we will draw a vertex of Γ with at least three edges as



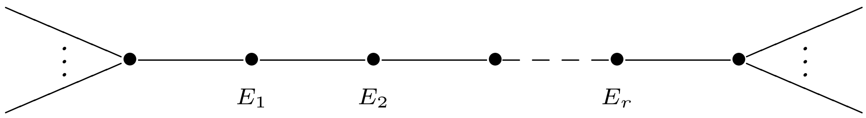
Theorem 4. *(See [34].)*

(i) *The $v_j (= E_j)$, $j \in J$, for which $\frac{\nu_j}{N_j} = \min_{i \in J} \frac{\nu_i}{N_i}$, together with their edges, form a connected part $\mathcal{M}$ of the resolution graph. More precisely $\mathcal{M}$ has one of the following forms* (*with $r \geqslant 0$*):

(1)

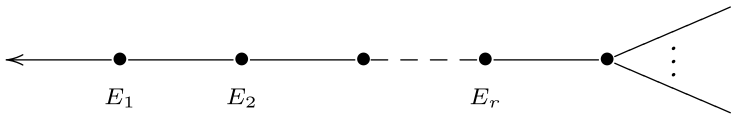


(2)



(3)

(4)



(ii) *Starting from an end vertex of the minimal part $\mathcal{M}$, the numbers $\frac{\nu_i}{N_i}$ strictly increase along any path in the tree* (*away from $\mathcal{M}$*).

Proposition 1. *The minimal part $\mathcal{M}$ in Theorem* 4 *is as in case* (4) *if and only if f can be written* (*analytically*) *in the form $x^N g(x, y)$, where g is not a multiple of x and the intersection number of $x = 0$ and $g = 0$ in the origin is N.*

Proof. If $\mathcal{M}$ is as in case (4) it is shown in [34] Proposition 3.8 that f can be written in the form $x^N g$, where g is not a multiple of x.

Denote a priori by m the intersection number of $x = 0$ and $g = 0$ in the origin. Say that in the resolution process yielding π the strict transform of $x = 0$ gets separated from the strict transform of $g = 0$ after exactly ℓ blowing-ups. Then it is easy to verify that the numbers N_ℓ and ν_ℓ associated to the at that stage created exceptional curve E_ℓ are $\nu_\ell = \ell + 1$ and $N_\ell = \ell N + m$. So, if $\mathcal{M}$ is as in case (4), then $\frac{1}{N} = \frac{\ell+1}{\ell N + m}$, which is equivalent to $m = N$.

Now the other implication is easy to verify. $\square$

1.5. Using Theorem 3 we now determine exactly when a double pole of the topological zeta function induces a monodromy eigenvalue with a Jordan block of size 2.

Theorem 5. *Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be an analytic function germ that is non-reduced. Suppose that $-1/N$ is a pole of order two of $Z_{\mathrm{top},0}(f, s)$. Denote $\lambda := \exp(2\pi i(-1/N))$.*

(i) *If $N \neq N_i$ for all components E_i of the strict transform of f, i.e. if the minimal part $\mathcal{M}$ in Theorem* 4 *is of the form* (2)*, then the monodromy eigenvalue λ of f has a Jordan block of size* 2.

(ii) *If $N = N_i$ for a component E_i of the strict transform of f, i.e. if the minimal part is of the form* (4)*, then λ has only Jordan blocks of size* 1.

Proof. (i) We will show that in formula (3) of Theorem 3 there are more 'edge contributions' than 'vertex contributions'. The two exterior vertices of the minimal part $\mathcal{M}$ are separating rupture vertices (for example by [34] 3.6). Since N divides N_ℓ for all components E_ℓ in $\mathcal{M}$, clearly $N \mid m_e$ for the chosen edge in $\mathcal{M}$. On the other hand we claim that $N \nmid m_j$ when v_j is an exterior vertex of $\mathcal{M}$. This follows from the fact that $N \nmid N_i$, where v_i is any neighboring vertex of v_j (outside $\mathcal{M}$). Indeed, by Loeser [19] (or Rodrigues in [27]) we have $(0 <)\nu_i - \frac{\nu_j}{N_j} N_i = \nu_i - \frac{1}{N} N_i < 1$; so N cannot divide N_i.

Further, it is easy to see that, whenever $N \mid m_i$ for some other separating rupture vertex v_i, then necessarily also $N \mid m_e$ for the chosen edge in the chain from v_i towards $\mathcal{M}$.

We conclude that indeed λ is a zero of Δ_2. (A similar statement is already proved in the reduced case by Loeser in [19].)

(ii) We know by Proposition 1 that in this case f can be written (analytically) in the form $x^N g(x, y)$, such that the intersection number of $x = 0$ and $g = 0$ in the origin is N, and hence $N \geqslant \mu$, where μ is the multiplicity of g at the origin. We may suppose moreover that, writing $g = \prod_i g_i^{N_i}$ in its factorization in irreducible components, we have $N_i < N$ for all i. Indeed, otherwise g must be of the form g_1^N with g_1 having multiplicity 1 at the origin. Since then also the intersection number of $x = 0$ and $g_1^N = 0$ in the origin must be N, this means that in fact f is (analytically) of the form $x^N y^N$, and then the statement in (ii) is obvious.

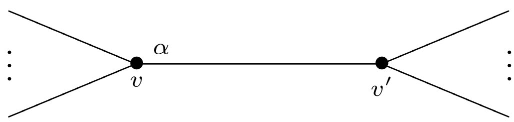
So in particular we may suppose that λ is not a root of the first factor in formula (3) for Δ_2. We will show that moreover there is *no* edge $e \in \mathcal{E}'$ satisfying $N \mid m_e$.

Consider a chain between two separating rupture components. Suppose that $N \mid m_e$ for an edge in such a chain. Then necessarily $N \mid N_\ell$ for all vertices v_ℓ in the chain (including the two exterior ones). Denote here the multiplicities of E_i in the divisor of $\pi^* g$ by N_i'. Since $f = x^N g$ we clearly have that $N \mid N_i$ if and only if $N \mid N_i'$. So $N \mid N_\ell'$ for all vertices v_ℓ in the chain. Proposition 2 below then implies that $N < \mu$, contradicting the fact that $N \geqslant \mu$. □

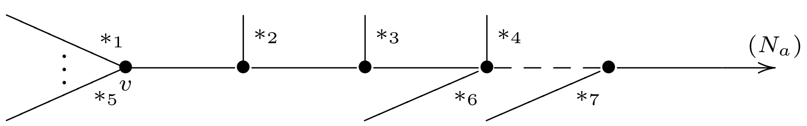
Proposition 2. *Let $g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ determine a plane curve singularity germ at the origin of multiplicity μ. We will use all notations associated to its dual* (*minimal*) *embedded resolution graph that we used before for f. Fix a chain between two rupture vertices, and denote by v_i, $i \in C$, all vertices in the chain* (*including the two rupture vertices*)*. Then* $\gcd_{i \in C} N_i < \mu$.

Proof. We use the language of Eisenbud–Neumann diagrams associated to the (full) dual resolution graph, see [10]. More precisely we will use the following facts.

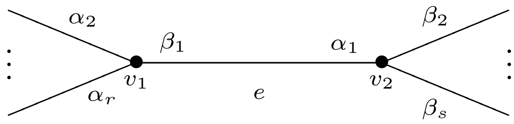
(1) A number α in the position below, i.e. on an edge between v and v', and next to v, indicates that α is the absolute value of the determinant of the intersection matrix for the vertices in the connected part of $\Gamma \setminus \{v\}$ that contains v'.



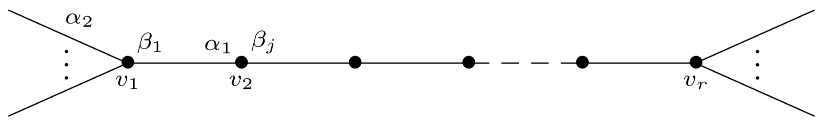
(2) Fix a vertex $v \in \mathcal{V}$. Then we have that $N_v = \sum_{a \in \mathcal{A}} k_a$, where for each arrowhead a the number k_a is the product of all numbers α on the Eisenbud–Neumann diagram *adjacent to* (but not on) the path from v to a. (Here, if a is decorated with its multiplicity N_a, then this N_a has to be considered as factor in the product.)



(3) Fix an edge e between vertices v_1 and v_2 in $\mathcal{V}$. Let α_1 and β_1 be the numbers along e next to v_2 and v_1, respectively. Let also α_i and β_i denote the numbers along *other* edges, next to v_1 and v_2, respectively. For a general Eisenbud–Neumann diagram we have that the edge determinant $\alpha_1\beta_1 - (\prod_{i\geqslant 2}\alpha_i)(\prod_{j\geqslant 2}\beta_j)$ is positive (see [10]). But since we are dealing with Eisenbud–Neumann diagram associated to the (full) dual resolution graph then one has the edge determinant rule $\alpha_1\beta_1 - (\prod_{i\geqslant 2}\alpha_i)(\prod_{j\geqslant 2}\beta_j) = 1$, see [7,25].



Consider now the fixed chain between the two rupture components in the statement of the proposition. We may suppose that the first created exceptional curve E_0, corresponding to the vertex v_0, is 'on the right-hand side of v_1', i.e. belongs to the connected part of $\Gamma \setminus \{v_1\}$ that contains v_2. This implies then that there is at least one arrowhead somewhere 'on the left-hand side of v_1'.



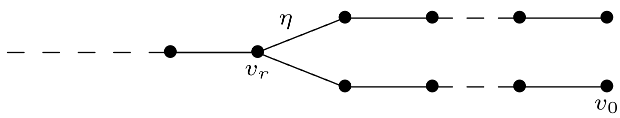
We want to express N_1 and N_2 as in (2) above. Let the numbers α_1, β_1, α_i and β_j be as in (3) above, associated to the edge connecting v_1 and v_2. (Note that there is just one number β_j if $r > 2$.) Consider for each path from v_2 to an arrowhead 'on the right-hand side of v_2' the product of all numbers on the Eisenbud–Neumann diagram adjacent to the path *except* α_1, and denote by b the sum of all these products. Consider analogously for each path from v_1 to an arrowhead 'on the left-hand side of v_1' the product of all numbers adjacent to the path *except* β_1, and let c be the sum of all these products (certainly $c \neq 0$). Then by (2) we have

$$N_1 = \left(\prod_{i\geqslant 2}\alpha_i\right)b + \beta_1 c \quad \text{and} \quad N_2 = \alpha_1 b + \left(\prod_{j\geqslant 2}\beta_j\right)c.$$

Consequently $\alpha_1 N_1 - (\prod_{i\geqslant 2}\alpha_i)N_2 = (\alpha_1\beta_1 - (\prod_{i\geqslant 2}\alpha_i)(\prod_{j\geqslant 2}\beta_j))c = c$, using (3), and so in particular $\gcd_{i\in C} N_i \mid c$.

Recall that the multiplicity N_0 of E_0 is μ. We will finally show that $c < \mu$, which yields the statement of the proposition. We consider two subcases.

Case I. Suppose that there is no arrowhead 'on the right-hand side of v_r'. This can only happen if the part of the diagram on that side has the form below.



Then the expression in (2) for $\mu = N_0$ is $\mu = \eta c$, where it is well known that $\eta > 1$, hence in particular $c < \mu$.

Case II. Suppose that there is at least one arrowhead 'on the right-hand side of v_r'. Now (2) yields that μ is of the form $\mu = (\geqslant 1)c + (\geqslant 1)$, and so again $c < \mu$. $\square$

2. Maximal roots of *b*-functions for curves

2.1. We first recall the definition of the Bernstein–Sato polynomial. Let X be a complex n-dimensional manifold, resp. smooth algebraic variety, and let X_0 be the hypersurface defined as the zero locus of a holomorphic function, resp. regular function, f. Let $\mathcal{D}_X$ be the ring of analytic, resp. algebraic, partial differential operators associated to X.

The Bernstein–Sato polynomial (or b-function) $b_f(s)$ of f is the unique monic polynomial of lowest degree satisfying

$$b_f(s)f^s = Pf^{s+1} \quad \text{with } P \in \mathcal{D}_X[s].$$

It exists at least locally, and globally if X is an affine algebraic variety [4,5,29]. Moreover the b-function of a regular function f and of its associated analytic function coincide. Restricting to the stalk at a point $x \in X_0$, one can also define the local b-function $b_{f,x}(s)$. If X is Stein, resp. affine, then $b_f(s)$ is the least common multiple of these local b-functions. (In fact the b-function of f is locally the minimal polynomial of the action of s on the left holonomic $\mathcal{D}_X[s]$-module $\mathcal{D}_X[s]f^s/\mathcal{D}_X[s]f^{s+1}$ [29].)

Let R_f be the set of the roots of $b_f(-s)$, and m_α the multiplicity of $\alpha \in R_f$. Then $R_f \subset \mathbb{Q}_{>0}$, and $m_\alpha \leqslant n$ because $b_f(s)$ is closely related to the monodromy on the nearby cycle sheaf $\psi_f\mathbb{C}_X$, see [15,23]. Moreover $\min R_f$ coincides with the log canonical threshold, see [16,28].

The determination of $b_f(s)$ is difficult in general, even if f defines an isolated singularity; we mention the algorithm due to Briançon et al. [6] for a non-degenerate convenient germ with respect to its Newton polyhedron, which allows to construct the functional equation step by step. There exist also effective algorithms using Gröbner bases, see [26] for instance.

2.2. Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be an algebraic or analytic function germ which is singular at the origin. When f is *reduced*, Loeser [19] proved Conjecture 2: the function $b_{f,0}(s) \cdot Z_{\mathrm{top},0}(f,s)$ is a polynomial. Assume from now on that f is not reduced.

Suppose that $-c_0(f) = -1/N$ is a pole of order two of $Z_{\mathrm{top},0}(f,s)$, and denote $\lambda := \exp(2\pi i s_0)$. In case (i) of Theorem 5 we have that λ has a Jordan block of size 2, and then the argument in the proof of [19], Théorème III.3.3.b, indeed yields that $(s+1/N)^2$ divides $b_{f,0}(s)$. However in case (ii) of Theorem 5 we have that λ has only Jordan blocks of size 1 and then this argument fails. (So the conclusion in [19] should be restricted to our case (i)!)

So, after Proposition 1 and Theorem 5, the remaining case to investigate concerning Conjecture 2 is the following. *Let $f = x^N g$ where $N \geqslant 2$, g is not a multiple of x, and the intersection number of $x = 0$ and $g = 0$ in the origin is N. Does $(s+1/N)^2$ divide $b_{f,0}(s)$?*

Studying Bernstein–Sato polynomials for non-reduced f is in general very difficult. In this section we treat such $f = x^N g$ and in particular we answer the question above positively whenever g is a weighted homogeneous and reduced polynomial.

2.3. Let us observe some simple facts. First, the polynomial $b_{x^N,0}(s) = \prod_{\ell=1}^{N}(s+\ell/N)$ divides $b_{f,0}(s)$ since g is a unit at any point $(0,a) \neq (0,0)$ close enough to the origin. Moreover, $(s+1)^2$ divides always $b_{f,0}(s)$ if f is reducible (see [32]).

Henceforth, we assume that g is a weighted homogeneous polynomial. Our first result deals with the multiplicity of the factors $(s+\ell/N)$, $1 \leqslant \ell \leqslant N-1$, in $b_{f,0}(s)$.

Proposition 3. *Let $f \in \mathbb{C}[x,y]$ be a polynomial of the form $x^N g$ where $N \geqslant 2$ and $g \in \mathbb{C}[x,y]$ is neither a constant nor a multiple of x. Assume that g is weighted homogeneous. Let m (resp. $\overline{m}$) denote the degree of g (resp. of the reduced polynomial of g) as a polynomial in $(\mathbb{C}[x])[y]$. Let ℓ be an integer such that $1 \leqslant \ell \leqslant N-1$.*

If N divides $m\ell$ and $m\ell \leqslant (\overline{m}-1)N$ then $(s+\ell/N)^2$ divides $b_{f,0}(s)$.

In particular when g is reduced and $m = N \geqslant 2$ we have indeed that $(s+1/N)^2$ divides $b_{f,0}(s)$ and more precisely that $\prod_{\ell=1}^{N-1}(s+\ell/N)^2$ divides $b_{f,0}(s)$.

2.4. The rest of this section is devoted to the proof of Proposition 3. Further we denote by $\mathcal{O}$ the ring of germs of holomorphic functions $\mathbb{C}\{x, y\}$, by $\mathcal{D}$ the ring of differential operators $\mathcal{O}\langle\partial/\partial x, \partial/\partial y\rangle$, and by $\mathcal{D}[s]$ the ring $\mathcal{D} \otimes_{\mathbb{C}} \mathbb{C}[s]$.

Now we introduce $\tilde{b}'(s)$ and $b_\ell(s)$, $1 \leqslant \ell \leqslant N-1$, the monic polynomials of smallest degree which verify the identities:

$$\tilde{b}'(s)x^{N-1}f^s \in \mathcal{D}[s]\big(f'_x, f'_y\big)f^s,$$
$$b_\ell(s)x^{\ell-1}f^s \in \mathcal{D}[s]x^\ell f^s, \quad 1 \leqslant \ell \leqslant N-1.$$

The existence of such nontrivial equations is a consequence of the existence of a nontrivial Bernstein equation for f. More precisely, we have the following result.

Lemma 1. *Let $f = x^N g \in \mathcal{O}$ be a nonzero germ such that $N \geqslant 2$, and g is neither a unit nor a multiple of x. Then the polynomials $\tilde{b}'(s), b_1(s), \ldots, b_{N-1}(s)$ divide $\tilde{b}_{f,0}(s)$, and $\tilde{b}_{f,0}(s)$ divides the product $\tilde{b}'(s) \times b_1(s) \times \cdots \times b_{N-1}(s)$.*

Indeed, we have the following short exact sequences of $\mathcal{D}[s]$-modules:

$$0 \to \widetilde{\mathcal{N}}' = \frac{\mathcal{D}[s]x^{N-1}f^s}{\mathcal{D}[s](f'_x, f'_y)f^s} \hookrightarrow \widetilde{\mathcal{N}} = \frac{\mathcal{D}[s]f^s}{\mathcal{D}[s](f'_x, f'_y)f^s} \twoheadrightarrow \frac{\mathcal{D}[s]f^s}{\mathcal{D}[s]x^{N-1}f^s} \to 0, \tag{4}$$

$$0 \to \mathcal{N}_\ell = \frac{\mathcal{D}[s]x^{\ell-1}f^s}{\mathcal{D}[s]x^\ell f^s} \hookrightarrow \frac{\mathcal{D}[s]f^s}{\mathcal{D}[s]x^\ell f^s} \twoheadrightarrow \frac{\mathcal{D}[s]f^s}{\mathcal{D}[s]x^{\ell-1}f^s} \to 0, \quad 2 \leqslant \ell \leqslant N-1, \tag{5}$$

and $\tilde{b}_{f,0}(s)$ (resp. $\tilde{b}'(s)$, $b_1(s), \ldots, b_{N-1}(s)$) is the minimal polynomial of the action of s on the holonomic $\mathcal{D}$-module $\widetilde{\mathcal{N}}$ (resp. $\widetilde{\mathcal{N}}'$, $\mathcal{N}_1, \ldots, \mathcal{N}_{N-1}$). In particular, the knowledge of the roots of $b_{f,0}(s)$ is equivalent to the one of $\tilde{b}'(s), b_1(s), \ldots, b_{N-1}(s)$. On the other hand, this result is in general not enough for the full determination of $b_{f,0}(s)$ (since we do not know if $\tilde{b}_{f,0}(s)$ coincides – or not – with the lcm of $\tilde{b}'(s), b_1(s), \ldots, b_{N-1}(s)$).

The proof of Proposition 3 relies on the explicit determination of the polynomials $b_\ell(s)$ when g is weighted homogeneous (Proposition 4, Remark 1). Let us remark that the very last technical assumption is a consequence of the other ones when g is reduced.

2.5. The polynomials $b_\ell(s)$. This part is devoted to the determination of the polynomials $b_\ell(s)$ when g is weighted homogeneous. To this end, we need to know the annihilator in $\mathcal{D}$ of $x^\ell f^s$, $\ell \geqslant 0$.

Lemma 2. *Let $g \in \mathcal{O}$ be a nonzero germ which is neither a unit nor a multiple of x. Let f denote the germ $x^N g$ where $N \geqslant 2$. For all integers $\ell \geqslant 0$, the annihilator $\mathrm{Ann}_{\mathcal{D}}\, x^\ell f^s$ in $\mathcal{D}$ of $x^\ell f^s$ is generated by the operator*

$$\frac{Ng + xg'_x}{h}\frac{\partial}{\partial y} - \frac{xg'_y}{h}\frac{\partial}{\partial x} + \ell\frac{g'_y}{h}$$

where h is a greatest common divisor of g'_y and $Ng + xg'_x$.

Proof. From Kashiwara [14], the characteristic variety of the $\mathcal{D}$-module $\mathcal{D}x^\ell f^s = \mathcal{D}/\mathrm{Ann}_{\mathcal{D}}\, x^\ell f^s$ is $W_f = \overline{\{(x, \lambda df) \mid \lambda \in \mathbb{C}\}} \subset T^*\mathbb{C}^2$, the relative conormal space of f. In our particular case, this space is a hypersurface in $T^*\mathbb{C}^2$. As the irreducible polynomial $\Upsilon = ((Ng + xg'_x)/h)\xi_y - x(g'_y/h)\xi_x \in \mathcal{O}[\xi_x, \xi_y]$ is zero on W_f, this is a reduced equation of W_f. In particular, the principal symbol of any operator in $\mathcal{D}$ annihilating $x^\ell f^s$ is a multiple of this polynomial. Since the given operator relieves Υ and belongs to $\mathrm{Ann}_{\mathcal{D}}\, x^\ell f^s$, we conclude by an easy induction on the degree of operators. $\square$

Proposition 4. *Let $f = x^N g \in \mathbb{C}[x, y]$ be a weighted homogeneous polynomial of degree 1 for a system $\alpha = (\alpha_x, \alpha_y) \in (\mathbb{Q}_{>0})^2$ where $N \geqslant 2$ and g is neither a constant nor a multiple of x. Let $\overline{m}$ denote the degree in y of the reduced polynomial of g.*

(i) *For $1 \leqslant \ell \leqslant N-1$, the polynomial $b_\ell(s)$ is equal to*:

$$\left(s+\frac{\ell}{N}\right)\prod_{i=1}^{\overline{m}-1}(s+\ell\alpha_x+i\alpha_y).$$

(ii) *If g is reduced, then the polynomial $\tilde{b}'(s)$ is equal to*:

$$\prod_{q\in\Pi_2}(s+\alpha_x+\alpha_y+q),$$

where $\Pi_2 \subset \mathbb{Q}_{\geqslant 0}$ is the set of the degrees of the elements of a weighted homogeneous basis of $\mathbb{C}[x, y]x^{N-1}/((Ng + xg'_x)x^{N-1}, g'_y x^N)\mathbb{C}[x, y]$.

Proof. We denote by χ the Euler vector field $\alpha_x x(\partial/\partial x) + \alpha_y y(\partial/\partial y) \in \mathcal{D}$ associated with α. Let us prove the first formula. In the particular case $\overline{m} = 1$, g is equal to y^m up to a change of coordinates; thus $b_\ell(s) = (s + \ell/N)$ by an easy computation. Henceforth, we assume that $\overline{m} \geqslant 2$.

We recall that $b_\ell(s)$ is the minimal polynomial of the action of s on the holonomic $\mathcal{D}$-module $\mathcal{N}_\ell = \mathcal{D}[s]x^{\ell-1} f^s/\mathcal{D}[s]x^\ell f^s$. Let us consider the following morphism of $\mathcal{D}[s]$-modules:

$$\begin{aligned}\widetilde{\mathcal{N}}_\ell = \frac{\mathcal{D}[s]x^{\ell-1}f^s}{\mathcal{D}[s](x, g'_y/h)x^{\ell-1}f^s} &\xrightarrow{\pi} (s+\ell/N)\mathcal{N}_\ell\\ \overline{Px^{\ell-1}f^s} &\longmapsto \overline{(s+\ell/N)Px^{\ell-1}f^s}\end{aligned}$$

where $(s + \ell/N)\mathcal{N}_\ell = ((s + \ell/N)\mathcal{D}[s]x^{\ell-1} f^s + \mathcal{D}[s]x^\ell f^s)/\mathcal{D}[s]x^\ell f^s$ is the image of the endomorphism of $\mathcal{N}_\ell$ of multiplication by $(s+\ell/N)$, and h is a greatest common divisor of g'_x and g'_y. The morphism π is well defined since

$$\left(s+\frac{\ell}{N}\right)\frac{g'_y}{h}x^{\ell-1}f^s = \left[\frac{g'_y}{h}\frac{\partial}{\partial x} - \frac{g'_x}{h}\frac{\partial}{\partial y}\right]\cdot\frac{x^\ell}{N}f^s \in \mathcal{D}[s]x^\ell f^s.$$

In order to get the expected formula, let us prove that π is an isomorphism. Since π is obviously an epimorphism, we just have to check the injectivity of π.

Let $P \in \mathcal{D}[s]$ be an operator such that $\overline{Px^{\ell-1} f^s} \in \ker\pi$. By an Euclidian division by the operator $s + (\ell-1)\alpha_x - \chi \in \mathrm{Ann}_{\mathcal{D}[s]}\, x^{\ell-1} f^s$, we can assume that $P \in \mathcal{D}$. Moreover, since $x \cdot \overline{x^{\ell-1} f^s} = 0$ in $\widetilde{\mathcal{N}}_\ell$, we will also assume that $P \in \mathbb{C}\{y\}\langle\partial/\partial x, \partial/\partial y\rangle$. By definition of π, we have

$$\begin{aligned}\left(s+\frac{\ell}{N}\right)P &\in \mathcal{D}[s]x + \mathrm{Ann}_{\mathcal{D}[s]}\, x^{\ell-1}f^s\\ &= \mathcal{D}[s]\big(x, s+(\ell-1)\alpha_x-\chi\big) + \mathcal{D}[s]\,\mathrm{Ann}_{\mathcal{D}}\, x^{\ell-1}f^s\\ &= \mathcal{D}[s]\left(x, s+(\ell-1)\alpha_x-\chi, \frac{(Ng+xg'_x)}{h}\frac{\partial}{\partial y} - \frac{xg'_y}{h}\frac{\partial}{\partial x} + (\ell-1)\frac{g'_y}{h}\right)\end{aligned}$$

(by Lemma 2, using that h is also a greatest common divisor of g'_y and $Ng + xg'_x$). By division, we can eliminate the variable s:

$$P\left[\chi-(\ell-1)\alpha_x+\frac{\ell}{N}\right]\in\mathcal{D}\left(x,\frac{Ng}{h}\frac{\partial}{\partial y}+\ell\frac{g'_y}{h}\right)=\mathcal{D}\left(x,Ny^{\overline{m}}\frac{\partial}{\partial y}+\ell m y^{\overline{m}-1}\right),$$

and the variable x in the left-hand side member too:

$$P\left[\alpha_y y\frac{\partial}{\partial y}-\ell\alpha_x+\frac{\ell}{N}\right]\in\mathcal{D}\left(x,y^{\overline{m}-1}\left[Ny\frac{\partial}{\partial y}+\ell m\right]\right),$$

where $m\in\mathbb{Z}_{>0}$ is the degree of g as a polynomial in $(\mathbb{C}[x])[y]$. By using that $\chi(f)=1$ with $f=x^N g$, we obtain that $1=N\alpha_x+m\alpha_y$, and then we have $\ell/N=\ell\alpha_x+(\ell m/N)\alpha_y$. Thus, the identity becomes

$$\alpha_y P\left[y\frac{\partial}{\partial y}+\frac{\ell m}{N}\right]\in\mathcal{D}y^{\overline{m}-1}\left[y\frac{\partial}{\partial y}+\frac{\ell m}{N}\right]$$

and $P\in\mathcal{D}y^{\overline{m}-1}$ necessarily, i.e. $\overline{Px^{\ell-1}f^s}=0$ in $\widetilde{\mathcal{N}}_\ell$ – since $(g'_y/f,x)\mathcal{O}=(y^{\overline{m}-1},x)\mathcal{O}$. In other words, π is injective.

Now, we remark that $\widetilde{\mathcal{N}}_\ell$ is supported by the origin. In fact, $\widetilde{\mathcal{N}}_\ell$ is isomorphic to the $\mathcal{D}[s]$-module $\mathcal{D}[s]/\mathcal{D}[s](s+(\ell-1)\alpha_x-\chi,x,y^{\overline{m}-1})$; thus it is not hard to compute the minimal polynomial $\tilde{b}_\ell(s)$ of the action of s on $\widetilde{\mathcal{N}}_\ell$:

$$\tilde{b}_\ell(s)=\prod_{i=1}^{\overline{m}-1}(s+\ell\alpha_x+i\alpha_y).\tag{6}$$

This is analogous to the (classical) computation of the Bernstein polynomial of a weighted homogeneous polynomial with an isolated singularity at the origin (see [36] for instance). The assertion (i) follows. The proof of (ii) is similar to the computation of $\tilde{b}_\ell(s)$, since $(f'_x,f'_y)\mathcal{O}=((Ng+xg'_x)x^{N-1},g'_y x^N)\mathcal{O}$ where the ideal $(Ng+xg'_x,xg'_y)\mathcal{O}$ defines the origin when g is reduced. □

Proposition 3 is a direct consequence of the following remark and Lemma 1.

Remark 1. Under the assumptions of Proposition 4, we have that $(s+\ell/N)^2$ divides $b_\ell(s)$ if and only if $m\ell\leqslant(\overline{m}-1)N$ and N divides $m\ell$. Indeed, from the identities (6) and $N\alpha_x+m\alpha_y=1$, these conditions mean that $i=m\ell/N$ belongs to $\{1,\dots,\overline{m}-1\}$.

3. Closed formulae of *b*-functions for non-reduced curves

3.1. Still using the notation of 2.2, we investigate in this section closed formulae for $b_{f,0}(s)$ when g is reduced. This is a result in a wider context than the main topic of the paper, but it fits well with it.

Theorem 6. *Let $f=x^N g\in\mathbb{C}[x,y]$ be a weighted homogeneous polynomial of degree 1 for a system $\alpha=(\alpha_x,\alpha_y)\in(\mathbb{Q}_{>0})^2$ where $N\geqslant 2$ and g is reduced, non-constant, and not a multiple of x. Let $|\alpha|$ denote the sum $\alpha_x+\alpha_y$. Let m denote the degree of g as a polynomial in $(\mathbb{C}[x])[y]$. Assume that m is greater than or equal to 2.*

(i) *The reduced Bernstein polynomial $\tilde{b}_{f,0}(s)$ of f divides*

$$\operatorname{lcm}\left\{\left[\prod_{\ell=1}^{N-1}\left(s+\frac{\ell}{N}\right)\right]\prod_{q\in\Pi_1}\left(s+|\alpha|+q\right),\ \prod_{q\in\Pi_2}\left(s+|\alpha|+q\right)\right\},$$

where $\Pi_1 \subset \mathbb{Q}_{\geqslant 0}$ (resp. $\Pi_2 \subset \mathbb{Q}_{\geqslant 0}$) is the set of the degrees of the elements of a weighted homogeneous basis of $\mathbb{C}[x,y]/(x^{N-1}, y^{m-1})\mathbb{C}[x,y]$ (resp. $\mathbb{C}[x,y]x^{N-1}/((Ng+xg'_x)x^{N-1}, g'_y x^N)\mathbb{C}[x,y]$).

(ii) *The polynomial $\tilde{b}_{f,0}(s)$ is a multiple of the least common multiple of the polynomials $(s+\ell/N)\prod_{i=1}^{m-1}(s+\ell\alpha_x+i\alpha_y)$, $1 \leqslant \ell \leqslant N-1$, and $\prod_{q\in\Pi_2}(s+|\alpha|+q)$.*

(iii) *The factor $(s+1/N)$ of $\tilde{b}_{f,0}(s)$ has multiplicity 2 if and only if N divides m.*

(iv) *Assume that g is homogeneous. Then*:

$$b_{f,0}(s) = \left[\prod_{\ell=1}^{N}\left(s+\frac{\ell}{N}\right)\right] \times \left[\prod_{i=2}^{2m+N-1}\left(s+\frac{i}{N+m}\right)\right].$$

Our proof uses the so-called method of 'increasing the weights' (see [6] for instance). We remark that our method does not allow us to get a closed formula for $b_{f,0}(s)$ if f is not homogeneous.

Example 1. Let $N=4$ and $g=y^6+x^{12}$. Thus $m=6$ and $f=x^4(y^6+x^{12})$ is weighted homogeneous of degree 1 for the system $(1/16, 1/8)$. From Theorem 6, parts (i) and (ii), the polynomial

$$\left(s+\frac{1}{4}\right)\left(s+\frac{1}{2}\right)\left(s+\frac{3}{4}\right)\prod_{i=3}^{24}\left(s+\frac{i}{16}\right)$$

is a multiple of $\tilde{b}_{f,0}(s)$, and $\prod_{i=3}^{24}(s+i/16)$ divides $\tilde{b}_{f,0}(s)$. Since N does not divide m and $3m$ but does divide $2m$, the multiplicity of $(s+1/4)$ (resp. $(s+1/2)$) in $\tilde{b}_{f,0}(s)$ is equal to 1 (resp. 2, using Proposition 3), and our results do not determine the one of $(s+3/4)$.

3.2. Proof of Theorem 6. The proof of part (i) of Theorem 6 requires several steps. First, we will obtain a 'big' multiple of $\tilde{b}_{f,0}(s)$; then we will add some refinements in the method in order to get the expected formula. Without lost of generality, we can assume that the degree m of g as a polynomial in $(\mathbb{C}[x])[y]$ is greater than or equal to 2 – since the case of normal crossings does not present any difficulty.

We need some preliminary notation. Let $\rho : \mathcal{O} \to \mathbb{Q}_{\geqslant 0} \cup \{+\infty\}$ be the weight function associated with α, defined by $\rho(0)=+\infty$ and

$$\rho(u) = \min\{\alpha_x\beta_x + \alpha_y\beta_y \mid u_\beta \neq 0\}$$

if $u=\sum_\beta u_\beta x^{\beta_x} y^{\beta_y} \in \mathcal{O}$ is not zero. For all $q\in\mathbb{Q}$, let $\mathcal{O}_{>q}$ (resp. $\mathcal{O}_{\geqslant q}$) denote the ideal of $\mathcal{O}$ of germs which have a weight strictly greater than (resp. greater than or equal to) q.

The following result provides a first multiple of $\tilde{b}_{f,0}(s)$ when g is reduced.

Lemma 3. *Let $f=x^N g \in \mathbb{C}[x,y]$ be a weighted homogeneous polynomial of degree 1 for a system $\alpha = (\alpha_x,\alpha_y) \in (\mathbb{Q}_{>0})^2$ where $N \geqslant 2$ and g is reduced, non-constant and not a multiple of x. Let $|\alpha|$ denote the sum $\alpha_x+\alpha_y$. Let m denote the degree of g as a polynomial in $(\mathbb{C}[x])[y]$. Assume that $m \geqslant 2$. Then the following identities are verified*:

$$\left[\prod_{\ell=1}^{N-1}\left(s+\frac{\ell}{N}\right)\right] \times \left[\prod_{q\in\Pi_3}\left(s+|\alpha|+q\right)\right] f^s \in \mathcal{D}[s]\mathcal{O}_{>(m-2)\alpha_y-\alpha_x} x^{N-1} f^s, \tag{7}$$

$$\left[\prod_{q\in\Pi'_2}\left(s+|\alpha|+q\right)\right]\mathcal{O}_{>(m-2)\alpha_y-\alpha_x} x^{N-1} f^s \subset \mathcal{D}[s]\left(f'_x, f'_y\right) f^s \tag{8}$$

where $\Pi_3 \subset \mathbb{Q}^+$ is the set of the degrees of the monomials $x^i y^j$ with $j \leqslant m-2$ and such that $\rho(x^i y^j) \leqslant (N-2)\alpha_x + (m-2)\alpha_y$, and $\Pi'_2 \subset \mathbb{Q}^+$ is the set of the degrees strictly greater than $(N-2)\alpha_x + (m-2)\alpha_y$ among the degrees of the elements of a weighted homogeneous basis of $\mathbb{C}[x,y]x^{N-1}/((Ng + xg'_x)x^{N-1}, g'_y x^N)\mathbb{C}[x,y]$.

Proof. In order to get the first formula, we just have to prove that, for all $p \in \Pi_3$,

$$\left[\prod_{i=1}^{N-1}\left(s+\frac{i}{N}\right)\right] \times \left[\prod_{q\in\Pi_3,\, q\leqslant p}\big(s+|\alpha|+q\big)\right] f^s = \sum_{\beta\in B(p)} Q_\beta c_\beta(s) x^{\beta_x} y^{\beta_y} \cdot f^s + R_p \tag{9}$$

where $B(p) \subseteq (\mathbb{Z}_{\geqslant 0})^2$ is the set of indexes $\beta = (\beta_x, \beta_y)$ such that $0 \leqslant \beta_y \leqslant m-2$ and $p < \rho(x^{\beta_x} y^{\beta_y}) \leqslant (N-2)\alpha_x + (m-2)\alpha_y$, $Q_\beta \in \mathbb{C}[s, \partial/\partial x, \partial/\partial y]$ is a differential operator, $c_\beta(s) \in \mathbb{C}[s]$ belongs to the ideal generated by $\prod_{i=\beta_x+1}^{N-1}(s+i/N)$ and R_p belongs to $\mathcal{D}[s]\mathcal{O}_{>(m-2)\alpha_y-\alpha_x} x^{N-1} f^s$. Indeed, the identities (7) and (9) coincide for $p = (N-2)\alpha_x + (m-2)\alpha_y \in \Pi_3$ (since $B(p)$ is also empty).

Let us prove (9) by an increasing induction on $p \in \Pi_3$. If $p = 0$, we have

$$\left[\prod_{i=1}^{N-1}\left(s+\frac{i}{N}\right)\right] \times \big(s+|\alpha|\big) f^s = \left[\prod_{i=1}^{N-1}\left(s+\frac{i}{N}\right)\right]\left[\frac{\partial}{\partial x}\alpha_x x + \frac{\partial}{\partial y}\alpha_y y\right] \cdot f^s \tag{10}$$

since $\chi(f^s) = sf^s$ where χ is the Euler vector field $\alpha_x x(\partial/\partial x) + \alpha_y y(\partial/\partial y)$ associated with α. Thus we get the expected decomposition when $m \geqslant 3$. In the particular case $m = 2$, we have the identity $y = g'_y/(2g(0,1)) + xv(x,y)$ where $v \in \mathbb{C}[x,y]$ is zero or a weighted homogeneous polynomial of degree $\alpha_y - \alpha_x$, and $g(0,1) \neq 0$ under our assumptions. Moreover, we have the following fact.

Lemma 4. *Let $f = x^N g \in \mathcal{O}$ be a nonzero germ where $N \geqslant 2$ and g is reduced, non-constant and not a multiple of x. For $1 \leqslant \ell \leqslant N-1$, we have*

$$\begin{aligned}\left[\prod_{i=\ell}^{N-1}\left(s+\frac{i}{N}\right)\right] g'_y x^{\ell-1} f^s &= \frac{\partial}{\partial x}^{N-\ell} \cdot \frac{g'_y}{N^{N-\ell}} x^{N-1} f^s \\ &\quad - \sum_{j=\ell}^{N-1}\left[\prod_{i=j+1}^{N-1}\left(s+\frac{i}{N}\right)\right]\frac{\partial}{\partial y}\frac{\partial}{\partial x}^{j-\ell} \cdot \frac{g'_x x^j}{N^{j-\ell+1}} f^s.\end{aligned} \tag{11}$$

Proof. This identity is obtained by using the following one:

$$\left(s+\frac{\ell}{N}\right) x^{\ell-1} g'_y f^s = \left[\frac{\partial}{\partial x} g'_y - \frac{\partial}{\partial y} g'_x\right] \cdot \frac{x^\ell}{N} f^s, \quad 1 \leqslant \ell \leqslant N-1. \qquad \square$$

On the other hand, let us observe that $g'_y x^{N-1} f^s \in \mathcal{O}_{>(m-2)\alpha_y-\alpha_x} x^{N-1} f^s$, and if $\rho(x^{\beta_x} y^{\beta_y}) > (N-2)\alpha_x + (m-2)\alpha_y$ with $\beta_y \leqslant m-1$, then necessarily $\beta_x \geqslant N-1$. Thus, by using (11) with $\ell = 1$ and the division of y by g'_y, we deduce from (10) the identity (9) for $p = 0$, $m = 2$.

Now we assume that (9) is verified for $p \in \Pi_3$, and let us prove this identity for $p' = \min\{q \in \Pi_3 \mid q > p\}$. Since

$$\big(s+|\alpha|+q\big) u f^s = \left[\frac{\partial}{\partial x}\alpha_x x + \frac{\partial}{\partial y}\alpha_y y + q - \rho(u)\right] \cdot u f^s \tag{12}$$

for any weighted homogeneous polynomial u, we have

$$\left[\prod_{i=1}^{N-1}\left(s+\frac{i}{N}\right)\right] \times \left[\prod_{q\in\Pi_3,\, q\leqslant p'}\big(s+|\alpha|+q\big)\right] f^s$$
$$= \sum_{\beta\in B(p)} Q_\beta c_\beta(s)\left[\frac{\partial}{\partial x}\alpha_x x+\frac{\partial}{\partial y}\alpha_y y+\big(p'-\rho\big(x^{\beta_x}y^{\beta_y}\big)\big)\right]\cdot x^{\beta_x}y^{\beta_y}f^s+\big(s+|\alpha|+p'\big)R_p$$

where $\rho(x^{\beta_x}y^{\beta_y})\geqslant p'$ by definition of p' and $B(p)$. By expanding the products, we get monomials $u=x^{\beta'_x}y^{\beta'_y}$ of degree strictly greater than p'. In view to get the expected decomposition, let us consider the possible cases:

- if (β'_x,β'_y) belongs to $B(p)$, then it belongs to $B(p')$ too;
- if $\beta'_x\geqslant N-1$ with $\rho(u)>(N-2)\alpha_x+(m-2)\alpha_y$, then $u\in\mathcal{O}_{>(m-2)\alpha_y-\alpha_x}x^{N-1}$;
- if $\beta'_y\leqslant m-1$ with $\rho(u)>(N-2)\alpha_x+(m-2)\alpha_y$, then necessarily $\beta'_x\geqslant N-1$;
- if $\beta'_y=m-1$ with $\beta'_x\leqslant N-2$, then u is necessarily a 'successor' of $x^{\beta_x}y^{\beta_y}$ where $\beta=(\beta'_x,\beta'_y-1)$ belongs to $B(p)$. In that case, we divide u by g'_y by using the identity $y^{m-1}=g'_y/(mg(0,1))+v(x,y)x$. The term $v(x,y)x^{\beta'_x+1}$ provides monomials with the same degree as u and a degree in y less than or equal to $m-2$; in particular, we are also in one of the previous cases.

The term $(1/mg(0,1))c_\beta(s)x^{\beta'_x}g'_yf^s$ may be rewritten by using the identity (11), and we obtain an element in $\mathcal{D}[s]g'_yx^{N-1}f^s\subset\mathcal{D}[s]\mathcal{O}_{>(m-2)\alpha_y-\alpha_x}x^{N-1}f^s$ and terms which provide monomials of degree strictly greater than $\rho(u)$, with a degree in y less than or equal to $m-1$ and a degree in x strictly greater than β'_x. Up to some iterations of this last case, we are again in one of the previous cases.

Hence we get the identity (9) for p'. The proof of the identity (8) is easier. Indeed, for any $q\in\mathbb{Q}_{\geqslant 0}$, we have

$$\big(s+|\alpha|+q\big)\mathcal{O}_{\geqslant q-(N-1)\alpha_x}x^{N-1}f^s\subset\mathcal{D}[s]\mathcal{O}_{>q-(N-1)\alpha_x}x^{N-1}f^s$$

by using (12). Thus

$$\prod_{q\in\Pi'_2}\big(s+|\alpha|+q\big)\mathcal{O}_{>(m-2)\alpha_y-\alpha_x}x^{N-1}f^s\subset\mathcal{D}[s]\big(Ng+xg'_x,xg'_y\big)x^{N-1}f^s$$

where $((Ng+xg'_x)x^{N-1},g'_yx^N)\mathcal{O}=(f'_x,f'_y)\mathcal{O}$. $\square$

The first part of Theorem 6 is a refinement of this result.

Proposition 5. *Let $f=x^Ng\in\mathbb{C}[x,y]$ be a weighted homogeneous polynomial of degree 1 for a system $\alpha=(\alpha_x,\alpha_y)\in(\mathbb{Q}_{>0})^2$ where $N\geqslant 2$ and g is reduced, non-constant and not a multiple of x. Let $|\alpha|$ denote the sum $\alpha_x+\alpha_y$. Let m denote the degree of g as a polynomial in $(\mathbb{C}[x])[y]$. Assume that $m\geqslant 2$. Then $\tilde{b}_{f,0}(s)$ divides the polynomial*

$$\operatorname{lcm}\left\{\left[\prod_{\ell=1}^{N-1}\left(s+\frac{\ell}{N}\right)\right]\prod_{q\in\Pi_1}\big(s+|\alpha|+q\big);\ \prod_{q\in\Pi_2}\big(s+|\alpha|+q\big)\right\}$$

where $\Pi_1\subset\mathbb{Q}_{\geqslant 0}$ (resp. $\Pi_2\subset\mathbb{Q}_{\geqslant 0}$) is the set of the degrees of the elements of a weighted homogeneous basis of $\mathbb{C}[x,y]/(x^{N-1},y^{m-1})\mathbb{C}[x,y]$ (resp. $\mathbb{C}[x,y]x^{N-1}/((Ng+xg'_x)x^{N-1},g'_yx^N)\mathbb{C}[x,y]$).

Proof. First of all, we prove that the sets Π_3 and Π_2' may be exchanged with

$$\widetilde{\Pi}_3 = \Pi_1 \cup \left\{ q \in \Pi_3 \;\middle|\; q \notin \Pi_1 \text{ and } q \neq \frac{\ell}{n} - |\alpha|,\ 1 \leqslant \ell \leqslant n-1 \right\} \subset \Pi_3$$

and $\widetilde{\Pi}_2' = \{q \in \Pi_2' \mid q \neq (\ell/N) - |\alpha|,\ 1 \leqslant \ell \leqslant N-1\}$. In other words, the existence of a monomial $x^{\beta_x} y^{\beta_y}$ not in $((Ng + xg_x')x^{N-1}, g_y' x^N)\mathbb{C}[x,y]$, and such that $\beta_x \geqslant N-1$ and $\rho(x^{\beta_x} y^{\beta_y}) = (\ell/N) - |\alpha|$ with $1 \leqslant \ell \leqslant N-1$, does not require to increase the multiplicity of $(s + \ell/N)$ in the algorithm of the previous result.

Indeed, if such a monomial appears, then the associated polynomial $c_\beta(s)$ is a multiple of $(s+\ell/N)$. If not, the monomial $x^{\beta_x} y^{\beta_y}$ would come from one of the terms $x^\ell g_x', \ldots, x^{N-1} g_x'$ or $g_y' x^{N-1}$ appearing in (11), whose degrees are greater than or equal to $\inf\{(\ell-1)\alpha_x + m\alpha_y,\ 1 - |\alpha|\}$; in particular $\inf\{\ell\alpha_x + (m+1)\alpha_y,\ 1\} \leqslant \ell/N$. But this is not possible since $\ell \leqslant N-1$ and $N\alpha_x + m\alpha_y = 1$.

Moreover, this factor $(s + \ell/N)$ of $c_\beta(s)$ cannot be useful for a successor $x^{\beta_x'} y^{\beta_y'}$ of $x^{\beta_x} y^{\beta_y}$ – since necessarily $\beta_x' \geqslant \beta_x$, where $\beta_x \geqslant N-1 > \ell - 1$. Hence, we can use the factor $(s + \ell/N)$ of $c_\beta(s)$ for $x^{\beta_x} y^{\beta_y}$ with the identity (12).

Finally, let us notice that our multiple $[\prod_{\ell=1}^{N-1}(s + \ell/N)] \prod_{q \in \widetilde{\Pi}_3 \cup \widetilde{\Pi}_2'} (s + |\alpha| + q)$ of $\tilde{b}_{f,0}(s)$ coincides with the expected polynomial

$$\operatorname{lcm}\left\{ \left[\prod_{\ell=1}^{N-1} \left(s + \frac{\ell}{N} \right) \right] \prod_{q \in \Pi_1} \big(s + |\alpha| + q\big);\ \prod_{q \in \Pi_2} \big(s + |\alpha| + q\big) \right\}.$$

Indeed, we have the identities $\Pi_2' = \{q \in \Pi_2 \mid q > (N-2)\alpha_x + (m-2)\alpha_y\}$ and $\Pi_3 = \Pi_1 \cup \{q \in \Pi_2 \mid q \leqslant (N-2)\alpha_x + (m-2)\alpha_y\}$ (since the degree of an element in $((xg_x' + Ng)x^{N-1}, g_y' x^N)\mathbb{C}[x,y]$ is strictly greater than $(N-1)\alpha_x + (m-1)\alpha_y$); in particular, the two polynomials have the same roots. Moreover, the multiplicity of a root $-\ell/N$, $1 \leqslant \ell \leqslant N-1$, is the same: in the two cases, it is equal to 2 if and only if $(\ell/N) - |\alpha|$ belongs to Π_1. □

Finally, let us show parts (ii) to (iv) of Theorem 6.

Proof. The second point is a direct consequence of Proposition 4 and Lemma 1. Let us prove (iii). If N divides m, then $(s + 1/N)^2$ divides $\tilde{b}_{f,0}(s)$ (Proposition 3). Conversely, let us prove that the root $-1/N$ is simple when N does not divide m. We repeat an argument which has been used in the proof of Proposition 5.

If $1/N - |\alpha| \notin \Pi_1$, we conclude with (i). Now, let us assume that there exists a monomial $x^{\beta_x} y^{\beta_y}$ of degree $1/N - |\alpha|$ with $0 \leqslant \beta_x \leqslant N-2$ and $0 \leqslant \beta_y \leqslant m-2$. Since $N\alpha_x + m\alpha_y = 1$ and N does not divide m, it is easy to verify that necessarily $\beta_x \neq 0$. Moreover, when such a monomial appears in the algorithm described in the proof of Lemma 3, the associated polynomial $c_\beta(s)$ is necessarily a multiple of $(s + 1/N)$. On the other hand, this factor is not useful for any successor $x^{\beta_x'} y^{\beta_y'}$ of $x^{\beta_x} y^{\beta_y}$ since $\rho(x^{\beta_x'} y^{\beta_y'}) > (1/N) - |\alpha|$ and $\beta_x' \geqslant \beta_x \geqslant 1$ (consider the identity (11)). Hence, we get a multiple of $\tilde{b}_{f,0}(s)$ which has a multiplicity 1 for $(s + 1/N)$, thus so has $\tilde{b}_{f,0}(s)$.

Let us prove the last part. In that case, we have $\alpha_x = \alpha_y = 1/(N+m)$; moreover, the set Π_2 is $\{(N-1)/(N+m), \ldots, (2m+N-3)/(N+m)\}$, using that the maximal weight of a nonzero element in the artinian algebra $\mathbb{C}[x,y]/(Ng + xg_x', xg_y')$ is equal to $(2m-2)/(N+m)$ (see [30]). Hence we notice that the proposed formula is nothing else but the multiple of $b_{f,0}(s)$ obtained in (i). In other words, we just have to check that

$$\left[\prod_{\ell=1}^{N-1} \left(s + \frac{\ell}{N} \right) \right] \prod_{i=2}^{N+m-2} \left(s + \frac{i}{N+m} \right) = \operatorname{lcm}\left\{ \left(s + \frac{\ell}{N} \right) \prod_{j=1}^{m-1} \left(s + \frac{\ell + j}{N+m} \right);\ 1 \leqslant \ell \leqslant N-1 \right\}$$

according to (ii). The 'unobvious' thing to do is to prove the following fact: if a factor $(s+\ell/N)$, with $1 \leqslant \ell \leqslant N-1$, has its multiplicity equal to 2 in the polynomial on the left-hand side, then it appears also in the product $\prod_{j=1}^{m-1}(s+(\ell+j)/(N+m))$. Indeed, if there exists an integer i such that $i/(N+m) = \ell/N$ then i is equal to $(\ell/N) \times (N+m) = \ell + m \times (\ell/N)$; in particular $1 \leqslant i-\ell \leqslant m-1$ since $0 < \ell/N < 1$. Thus the index $j = i-\ell$ provides the expected factor $(s+\ell/N)$. □

Remark 2. Let $c(s) \in \mathbb{C}[s]$ denote the multiple of $b_{f,0}(s)$ obtained in Proposition 5. Our method allows to construct a functional equation $c(s) f^s = P \cdot f^{s+1}$ where $P \in \mathcal{D}[s]$ has a total degree less than or equal to the degree of $c(s)$.

4. Superisolated surface singularities

4.1. A hypersurface surface singularity $(V,0) \subset (\mathbb{C}^3, 0)$ defined as the zero locus of a holomorphic function $f = f_d + f_{d+1} + \cdots \in \mathbb{C}\{x,y,z\}$ (where f_j is homogeneous of degree j) is a *superisolated surface singularity*, SIS for short, if the complex projective plane curve $C_d := \{f_d = 0\} \subset \mathbb{P}^2$ is reduced with isolated singularities $\{P_i\}_i$, and these points are not situated on the projective curve $\{f_{d+1} = 0\}$, that is $\operatorname{Sing}(C_d) \cap \{f_{d+1} = 0\} = \emptyset$. Notice that this condition implies that C_d is a reduced projective curve in $\mathbb{P}^2$.

The class of SIS singularities was introduced by Luengo in [21] to study the smoothness of the μ-constant stratum. In [2], Artal Bartolo has studied the mixed Hodge structure of the cohomology of the Milnor fibre of a SIS singularity. For that he constructed in an effective way an embedded resolution of a SIS singularity. He proved that the eigenvalues with Jordan blocks of size 3 in the monodromy of a SIS singularity depend only on singularities of the projective plane curve C_d.

More precisely, for each $P \in C_d$, let $\Delta_2^P(t)$ be the first Jordan polynomial of the local singularity (C_d, P), see Theorem 3. It contains the information about Jordan blocks of size 2 of the monodromy of the local singularity $(C_d, P) \subset (\mathbb{C}^2, P)$. We factorise $\Delta_2^P(t)$ in irreducible factors (which are cyclotomic polynomials):

$$\Delta_2^P(t) = \prod_k \phi_k^{n_k(P)}(t).$$

Theorem 7. *(See [2].) The roots of the polynomial*

$$\Delta_3^{\mathrm{SIS}}(t) := \prod_{P \in \operatorname{Sing}(C_d)} \prod_{k|d} \phi_k^{n_k(P)}(t)$$

correspond exactly with the eigenvalues with a Jordan block of size 3 *in the monodromy of any SIS singularity whose tangent cone is given by* f_d.

Conjecture 3 for the local topological zeta function $Z_{\mathrm{top},0}(f,s)$ of a SIS singularity has been proved by Artal Bartolo, Cassou-Noguès, Luengo and the first author in [3].

4.2. We now give a more precise version of Theorem 2 for SIS singularities, in the spirit of the following conjecture of the third author [17].

Conjecture 4.

(1) $Z_{\mathrm{top},0}(f,s)$ *has at most one pole of order n.*
(2) *If* $Z_{\mathrm{top},0}(f,s)$ *has in* s_0 *a pole of order n, then* s_0 *is the pole closest to the origin of* $Z_{\mathrm{top},0}(f,s)$.

This conjecture is proved in case $n=2$ by himself [34] and with Laeremans [17] when f is non-degenerate with respect to its Newton polyhedron. Moreover in these cases we have that then $s_0 = -c_0(f)$ in (2).

Theorem 8. *Let $f = f_d + f_{d+1} + \cdots \in \mathbb{C}\{x, y, z\}$ be a germ of a holomorphic function defining a SIS singularity. Then $Z_{\mathrm{top},0}(f, s)$ has at most one pole of maximal order 3. If $Z_{\mathrm{top},0}(f, s)$ has in s_0 a pole of order 3, then there exists N such that $d = 3N$, $s_0 = -1/N = -3/d$ and $((s+1/N)(s+2/N)\cdots(s+(N-1)/N)(s+1))^3$ divides $b_{f,0}(s)$.*

Proof. We may assume $d > 3$, otherwise one can check the statement of the theorem by simple computations, considering all possible configurations of plane curves of degree at most 3.

Since we have a pole of maximal order, there exists a positive integer N such that $s_0 = -1/N$. By Theorems 1 and 7, to prove that $(s+1/N)^3$ divides $b_{f,0}(s)$, it is enough to prove that the cyclotomic polynomial ϕ_N divides $\Delta_3^{\mathrm{SIS}}(t)$. To prove from this fact that $b_{f,0}(s)$ is divided by $((s+1/N)(s+2/N)\cdots(s+(N-1)/N)(s+1))^3$, we either follow the remark on page 230 at the end of the proof of the main theorem in [9] or we prove that the cyclotomic polynomial ϕ_b divides $\Delta_3^{\mathrm{SIS}}(t)$ for all divisors b of N, by proving that ϕ_b divides $\Delta_2^P(t)$ following the same arguments as below.

Let us describe the candidate poles of maximal order. The local topological zeta function of the SIS singularity satisfies the following equality, see Corollary 1.12 in [3]:

$$
\begin{aligned}
Z_{\mathrm{top},0}(f,s) = {} & \frac{\chi(\mathbb{P}^2 \setminus C_d)}{t-s} + \frac{\chi(\check{C}_d)}{(t-s)(s+1)} \\
& + \sum_{P\in \mathrm{Sing}(C_d)} \left(\frac{1}{t} + (t+1)\left(\frac{1}{(t-s)(s+1)} - \frac{1}{t}\right) Z_{\mathrm{top},P}\big(g^P, t\big)\right),
\end{aligned}
$$

where $t := 3 + (d+1)s$, g^P is a local equation of C_d at P and $\check{C}_d := C_d \setminus \mathrm{Sing}(C_d)$. The set of poles of $Z_{\mathrm{top},0}(f,s)$ is contained in the union of the sets $\{-1, -\frac{3}{d}\}$ and $\{-\frac{\nu_i+3N_i}{(d+1)N_i}\}$, whenever $-\nu_i/N_i$ is a pole of the local topological zeta function of g^P (a local equation of the germ of C_d) at some point $P \in \mathrm{Sing}(C_d)$.

(1) The candidate pole $t = 0$ is not a pole since $Z_{\mathrm{top},P}(g^P, 0) = 1$, see [8].

(2) Assume $s_0 = -1/N$ is a pole of order three. Then, from the above description, $s_0 \in \{-1, -3/d\}$. Moreover, s_0 is also as double pole $-\frac{(\nu_i+3N_i)}{(d+1)N_i}$ induced by a double pole $-\nu_i/N_i$ of $Z_{\mathrm{top},P}(g^P, s)$ of C_d at some singular point P.

(2.1) The case $s_0 = -1$ is excluded because $-1 = -\frac{(\nu_i+3N_i)}{(d+1)N_i}$ if and only if $\nu_i - N_i = N_i(d-3)$. The last equality is impossible because by assumption $d > 3$ and in the curve case $0 < \frac{\nu_i}{N_i} \leqslant 1$.

(2.2) Thus $s_0 = -3/d$ and then we have $d = 3N$ and $\frac{(\nu_i+3N_i)}{(d+1)N_i} = \frac{\nu_i}{N_i} = \frac{1}{N}$. Therefore $-1/N$ is a double pole of $Z_{\mathrm{top},P}(g^P, s)$ of C_d at the isolated singular point P (it is isolated because $\mathrm{Sing}(C_d) \cap \{f_{d+1} = 0\} = \emptyset$, this also implies that g^P is analytically reduced). We are done since $d = 3N$ and, by 1.2, ϕ_N divides $\Delta_2^P(t)$. □

Remark 3. If Conjecture 4 is true, then the pole $-3/d$ of order 3 should be the pole closest to the origin. This gives rise to the following question for plane curves, which we pose as an open problem.

Let $C = \{f_d = 0\}$ be a reduced projective plane curve of degree $d \geqslant 3$. Suppose that C has a singular point P such that $c_P(f) = 3/d$ and the minimal part $\mathcal{M}$ associated to P in Theorem 4 is as in case (2) (i.e. $-3/d$ is a pole of order 2 of $Z_{\mathrm{top},P}(f_d, s)$). Are the log canonical thresholds of f_d at all other singular points of C then at least $3/d$?

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