MOSCOW MATHEMATICAL JOURNAL
Volume 10, Number 3, July–September 2010, Pages 593–602

ON GENERATING SERIES OF CLASSES OF EQUIVARIANT HILBERT SCHEMES OF FAT POINTS

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Abstract. We discuss different definitions of equivariant (with respect to an action of a finite group on a manifold) Hilbert schemes of zero-dimensional subschemes and compute generating series of classes of equivariant Hilbert schemes for actions of cyclic groups on the plane in some cases.

2000 Math. Subj. Class. 14C05, 14G10.

Key words and phrases. Hilbert schemes of zero-dimensional subschemes, group actions, generating series.

1. Introduction

For a complex d-dimensional quasi-projective variety X, let Hilb^k_X be the Hilbert scheme of zero-dimensional subschemes (sets of "fat points" of length k of X. The Hilbert scheme Hilb^k_X is also a quasi-projective variety: see, e.g., [15, Section 4.3]. For a locally closed subvariety $Y \subset X$, let us denote by $\mathrm{Hilb}^k_{X,Y}$ the Hilbert scheme of zero-dimensional subschemes of length k of the variety X supported at points of the variety Y, and for a point $x \in X$, $\mathrm{Hilb}^k_{X,x} := \mathrm{Hilb}^k_{X,\{x\}}$.

Let $K_0(\mathcal{V}_{\mathbb{C}})$ be the Grothendieck ring of complex quasi-projective varieties. This is the Abelian group generated by the classes $[X]$ of all complex quasi-projective varieties X modulo the relations:

(1) if varieties X and Y are isomorphic, then $[X] = [Y]$;
(2) if Y is a Zariski closed subvariety of X, then $[X] = [Y] + [X \setminus Y]$.

One has to consider all varieties to be reduced. The multiplication in $K_0(\mathcal{V}_{\mathbb{C}})$ is defined by the Cartesian product of varieties: $[X_1] \cdot [X_2] = [X_1 \times X_2]$. The class $[\mathbb{C}] \in K_0(\mathcal{V}_{\mathbb{C}})$ of the complex affine line is denoted by $\mathbb{L}$. For a quasi-projective variety X, the class $[X]$, being an additive invariant of the variety, can be considered as a *generalized Euler characteristic* $\chi_g(X)$ of the variety X. The additivity of $\chi_g(\bullet)$ (property (2) above) permits to use it as a measure for the notion of the integral with respect to the generalized Euler characteristic.

Received February 11, 2010.
The first named author supported in part by the grants RFBR-10-01-00678, NSh-709.2008.1. The last two authors were supported in part by the grant MTM2007-67908-C02-02.

Let

$$\mathbb{H}_X(T) := 1 + \sum_{k=1}^{\infty} [\mathrm{Hilb}_X^k]\, T^k \in 1 + T \cdot K_0(\mathcal{V}_{\mathbb{C}})[[T]] \quad \text{and}$$

$$\mathbb{H}_{X,Y}(T) := 1 + \sum_{k=1}^{\infty} [\mathrm{Hilb}_{X,Y}^k]\, T^k \in 1 + T \cdot K_0(\mathcal{V}_{\mathbb{C}})[[T]]$$

be the generating series of classes of Hilbert schemes.

In [6], there was defined a notion of a *power structure* over a ring and there was described a natural power structure over the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ of complex quasi-projective varieties. This means that for a series $A(T) = 1+a_1T+a_2T^2+\cdots \in 1 + T \cdot K_0(\mathcal{V}_{\mathbb{C}})[[T]]$ and for an element $m \in K_0(\mathcal{V}_{\mathbb{C}})$ one defines a series $(A(T))^m \in 1+T\cdot K_0(\mathcal{V}_{\mathbb{C}})[[T]]$ so that all the usual properties of the exponential function hold. For the natural power structure over the ring $K_0(\mathcal{V}_{\mathbb{C}})$ and for $a_i = [A_i]$, $m = [M]$, where A_i and M are quasi-projective varieties, the series $(A(T))^m$ has the following geometric description. The coefficient at T^k in the series

$$(1 + [A_1]T + [A_2]T^2 + \dots)^{[M]}$$

is represented by the configuration space of pairs $(K,\, \varphi)$ consisting of a finite subset K of the variety M and a map φ from K to the disjoint union $\coprod_{i=1}^{\infty} A_i$ of the varieties A_i, such that $\sum_{x\in K} I(\varphi(x)) = k$, where $I\colon \coprod_{i=1}^{\infty} A_i \to \mathbb{Z}$ is the tautological function sending the component A_i of the disjoint union to i.

To describe the coefficients of this series as elements of the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$, one can write $(A(T))^{[M]}$ as

$$1+\sum_{k=1}^{\infty}\left\{\sum_{\underline{k}\colon \sum ik_i=k}\left[\left(\left(\left(\prod_i M^{k_i}\right)\setminus\Delta\right)\times\prod_i A_i^{k_i}\right)/\prod_i S_{k_i}\right]\right\}\cdot T^k, \qquad (1)$$

where $\underline{k} = \{k_i\colon i \in \mathbb{Z}_{>0},\, k_i \in \mathbb{Z}_{\geqslant 0}\}$ is a partition of k, Δ is the "large diagonal" in $\prod_i M^{k_i} = M^{\Sigma k_i}$ which consists of $(\sum k_i)$-tuples of points of M with at least two coinciding ones, the permutation group S_{k_i} acts by permuting corresponding k_i factors in $\prod_i M^{k_i} \supset (\prod_i M^{k_i})\setminus\Delta$ and the spaces A_i simultaneously. The connection between this formula and the description above is clear.

This power structure is connected with the pre-λ-structure ([11]) on the ring $K_0(\mathcal{V}_{\mathbb{C}})$ defined by the Kapranov zeta function ([10])

$$\zeta_M(T) := 1 + [S^1M]\cdot T + [S^2M]\cdot T^2 + [S^3M]\cdot T^3 + \cdots,$$

where S^kM is the k-th symmetric power of the variety M: one has $\zeta_M(T) = (1-T)^{-[M]}$.

Remark. One can show (see, e.g., [6]) that $\zeta_{\mathbb{L}^kM}(T) = \zeta_M(\mathbb{L}^kT)$. Therefore, in particular,

$$(1-\mathbb{L}^kT)^{-\mathbb{L}^s} = (\zeta_{\mathbb{L}^k}(T))^{\mathbb{L}^s} = \zeta_{\mathbb{L}^{k+s}}(T) = (1-\mathbb{L}^{k+s}T)^{-1}.$$

This equation also holds if T is substituted by T^j:

$$(1-\mathbb{L}^kT^j)^{-\mathbb{L}^s} = (1-\mathbb{L}^{k+s}T^j)^{-1}, \qquad (2)$$

(see the definition of the power structure in terms of the Kapranov zeta function in [6]).

There are two natural homomorphisms from the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ to the ring $\mathbb{Z}$ of integers and to the ring $\mathbb{Z}[u, v]$ of polynomials in two variables: the Euler characteristic (alternating sum of ranks of cohomology groups with compact support) $\chi\colon K_0(\mathcal{V}_{\mathbb{C}}) \to \mathbb{Z}$ and the Hodge–Deligne polynomial $e\colon K_0(\mathcal{V}_{\mathbb{C}}) \to \mathbb{Z}[u, v]$. These homomorphisms respect the power structures over the corresponding rings (see, e.g., [7]). The power structure over the ring $\mathbb{Z}[u_1, \ldots, u_r]$ of polynomials in r variables with integer coefficients is defined in the following way. Let $P(u_1, \ldots, u_r) = \sum_{\underline{k}\in\mathbb{Z}^r_{\geqslant 0}} p_{\underline{k}}\, \underline{u}^{\underline{k}} \in \mathbb{Z}[u_1, \ldots, u_r]$, where $\underline{k} = (k_1, \ldots, k_r)$, $\underline{u} = (u_1, \ldots, u_r)$, $\underline{u}^{\underline{k}} = u_1^{k_1} \cdots u_r^{k_r}$, $p_{\underline{k}} \in \mathbb{Z}$. Then

$$(1-t)^{-P(u_1,\ldots,u_r)} = \prod_{\underline{k}} (1 - \underline{u}^{\underline{k}} t)^{-p_{\underline{k}}}\,,$$

where, in the right hand side of the equation, the power (with the integer exponent $-p_{\underline{k}}$) means the usual one.

In [7], it was shown that, for a smooth quasi-projective variety X of dimension d, the following equation holds:

$$\mathbb{H}_X(T) = \left(\mathbb{H}_{\mathbb{C}^d,0}(T)\right)^{[X]}\,, \tag{3}$$

where $\mathbb{C}^d$ is the complex affine space of dimension d. Moreover, for a locally closed smooth subvariety $W \subset X$, one has

$$\mathbb{H}_{X,W}(T) = \left(\mathbb{H}_{\mathbb{C}^d,0}(T)\right)^{[W]}\,. \tag{4}$$

For $d = 2$, i.e., for surfaces, in other terms this equation was proved in the Grothendieck ring of motives by L. Göttsche [5]. In this case one has

$$\mathbb{H}_{\mathbb{C}^2,0}(T) = \prod_{i=1}^{\infty} \frac{1}{1 - \mathbb{L}^{i-1} T^i}\,. \tag{5}$$

For an arbitrary dimension d, the reduction of the equation (3) for the Hodge-Deligne polynomial was proved by J. Cheah in [1].

Here we discuss different definitions of equivariant (with respect to an action of a finite group G on a manifold) Hilbert schemes of zero-dimensional subschemes and compute generating series of classes of equivariant Hilbert schemes for actions of cyclic groups on the plane $\mathbb{C}^2$ in some cases.

2. Equivariant Hilbert Schemes of Fat Points

Let G be a finite group (of order $|G|$) acting on a smooth complex d-dimensional quasi-projective complex variety X. For convenience we assume that the action is faithful and the factor space $Y = X/G$ is connected. In particular, this implies that there is a nonempty Zariski open subset of X, where the action is free. The group G also acts on the Hilbert schemes Hilb^k_X of zero-dimensional subschemes of length k on X.

One can say that there are (at least) three natural notions of *equivariant Hilbert schemes* of zero-dimensional subschemes on X (see, e.g., [3], [12], [14]).

First, one can define the equivariant Hilbert scheme ${}^{(1)}\mathrm{Hilb}_X^{G,k}$ as the G-invariant part of the action of the group G on Hilb_X^k. It is quasi-projective being the fixed points scheme of an action of a finite group on a quasi-projective variety.

Second, as the equivariant Hilbert scheme ${}^{(2)}\mathrm{Hilb}_X^{G,k}$ one can take the (unique) component of ${}^{(1)}\mathrm{Hilb}_X^{G,k}$ which maps birationally on the $k/|G|$-th symmetric power of X/G. This is the closure of the set of zero-dimensional subschemes of length k on X supported at k points from $k/|G|$ free orbits and with the usual (nonmultiple) point at each of them (defined by the corresponding maximal ideal).

Let $Z \in {}^{(1)}\mathrm{Hilb}_X^{G,k}$ be a G-invariant subscheme, let Z_x be the connected component of Z supported at a point $x \in X$, so that $Z = \bigcup Z_x$, and let G_x be the isotropy group of x in G. Then the fibre of the tautological bundle over ${}^{(1)}\mathrm{Hilb}_X^{G,k}$ at Z is $H^0(Z, \mathcal{O}_Z) = \bigoplus H^0(Z_x, \mathcal{O}_{Z_x})$. The summand $H^0(Z_x, \mathcal{O}_{Z_x})$ has a representation of the group G_x. The third version ${}^{(3)}\mathrm{Hilb}_X^{G,k}$ of equivariant Hilbert scheme consists of those G-invariant zero-dimensional subschemes (points of ${}^{(1)}\mathrm{Hilb}_X^{G,k}$) for which the described representation of the group G_x in $H^0(Z_x, \mathcal{O}_{Z_x})$ is a multiple of the regular one (i.e., a multiple of $\mathbb{C}[G_x]$) for each point x from the support of the subscheme.

The schemes ${}^{(2)}\mathrm{Hilb}_X^{G,k}$ and ${}^{(3)}\mathrm{Hilb}_X^{G,k}$ are non-empty only if k is a multiple of the order $|G|$ of the group G. Both these schemes are unions of connected components of ${}^{(1)}\mathrm{Hilb}_X^{G,k}$ and therefore they are quasi-projective. One always has

$$ {}^{(1)}\mathrm{Hilb}_X^{G,k} \supset {}^{(3)}\mathrm{Hilb}_X^{G,k} \supset {}^{(2)}\mathrm{Hilb}_X^{G,k}. $$

They are smooth if Hilb_X^k is smooth. In particular this holds if X is a smooth surface ($d = 2$). In many cases, in particular for surfaces, the last two notions coincide. This is not true in general (see, e.g., [3]). The equivariant Hilbert scheme ${}^{(1)}\mathrm{Hilb}_X^{G,k}$ is always larger than the other two if the action of G is not free. In particular ${}^{(1)}\mathrm{Hilb}_{\mathbb{C}^d}^{G,1} \neq \varnothing$ for a rank d representation of a group G. It seems that often the last two notions are more interesting from geometrical point of view. In particular, for a finite group $G \subset \mathrm{SL}(2, \mathbb{C})$ acting on $\mathbb{C}^2$ in the natural way, ${}^{(2)}\mathrm{Hilb}_{\mathbb{C}^2}^{G,|G|}$ ($= {}^{(3)}\mathrm{Hilb}_{\mathbb{C}^2}^{G,|G|}$) is a crepant resolution of the factor space $\mathbb{C}^2/G$. However, the first one could be interesting as well. In particular, it seems that formulae for the generating series of classes of ${}^{(\bullet)}\mathrm{Hilb}_{\mathbb{C}^2}^{G,k}$ (or of ${}^{(\bullet)}\mathrm{Hilb}_{\mathbb{C}^2,0}^{G,k}$; $\bullet = 1$, 2 or 3) are somewhat simpler in this case (i.e., for $\bullet = 1$).

3. Generating Series of Classes of Equivariant Hilbert Schemes for Two-Dimensional Representations of a Cyclic Group

Let the cyclic group $\mathbb{Z}_p$ act on the plane $\mathbb{C}^2$ by $\sigma * (x, y) = (\sigma x, \sigma^q y)$, where $\sigma = \exp\left(\frac{2\pi i}{p}\right)$ is the generator of $\mathbb{Z}_p$. For $q = -1$ (or rather $q \equiv -1 \pmod{p}$) the factor space $\mathbb{C}^2/\mathbb{Z}_p$ has the A_{p-1} singularity.

Let ${}^{(\bullet)}\mathbb{H}_{\mathbb{C}^2}^{p,q}(T)$ be the generating series of classes of ${}^{(\bullet)}\mathrm{Hilb}_{\mathbb{C}^2}^{\mathbb{Z}_p,k}$ of equivariant Hilbert schemes of zero-dimensional subschemes of the plane $\mathbb{C}^2$ with the described

$\mathbb{Z}_p$-action. Let ${}^{(\bullet)}\mathbb{H}^{p;q}_{\mathbb{C}^2,0}(T)$ and ${}^{(\bullet)}\mathbb{H}^{p;q}_{\mathbb{C}^2,\mathbb{C}}(T)$ be defined in the same way (for subschemes support at points of the corresponding subspaces).

Theorem 1. *The following equations hold*:

$$\begin{aligned} {}^{(1)}\mathbb{H}^{p,-1}_{\mathbb{C}^2,0}(T) &= \prod_{i=1}^{\infty}\left(\frac{(1-T^{pi})^p}{1-T^i}\cdot\frac{1}{(1-\mathbb{L}^iT^{pi})^{p-1}\cdot(1-\mathbb{L}^{i-1}T^{pi})}\right),\\ {}^{(2)}\mathbb{H}^{p,-1}_{\mathbb{C}^2,0}(T) &= \prod_{i=1}^{\infty}\frac{1}{(1-\mathbb{L}^iT^{pi})^{p-1}\cdot(1-\mathbb{L}^{i-1}T^{pi})}. \end{aligned} \tag{6}$$

Proof. It is somewhat simpler to describe the computation of ${}^{(\bullet)}\mathbb{H}^{p,-1}_{\mathbb{C}^2,\mathbb{C}}(T)$, where $\mathbb{C}$ is the x-coordinate line in the plane $\mathbb{C}^2$ and then to get ${}^{(\bullet)}\mathbb{H}^{p,-1}_{\mathbb{C}^2,0}(T)$. To compute ${}^{(\bullet)}\mathbb{H}^{p,q}_{\mathbb{C}^2}(T)$, ${}^{(\bullet)}\mathbb{H}^{p,q}_{\mathbb{C}^2,\mathbb{C}}(T)$, or ${}^{(\bullet)}\mathbb{H}^{p,q}_{\mathbb{C}^2,0}(T)$, one uses the method of G. Ellingsrud and S.A. Strømme [4] based on a result of A. Bialynicki-Birula. The computations of ${}^{(\bullet)}\mathbb{H}^{p,-1}_{\mathbb{C}^2,\mathbb{C}}(T)$ correspond to the computations of $W(0, d, 0)$ in [4] on page 351. For that, one considers the action of the complex 2-torus $\mathbb{C}^*\times\mathbb{C}^*$ on the projective plane $\mathbb{CP}^2$ induced by the natural action of the torus on $\mathbb{C}^2\subset\mathbb{CP}^2$ and the corresponding action on the Hilbert schemes of zero-dimensional subschemes on it. This action has a finite number of fixed points. For a finite subgroup $G\subset\mathbb{C}^*\times\mathbb{C}^*$ (say, for a cyclic one) the torus acts on the corresponding G-equivariant Hilbert schemes as well. To apply the method of A. Bialynicki-Birula, one has to choose a subgroup of $\mathbb{C}^*\times\mathbb{C}^*$ isomorphic to $\mathbb{C}^*$. In order to benefit from the computations by G. Ellingsrud and S.A. Strømme in [4], we shall fix the subgroup used by them. This means that we take a subgroup consisting elements of the form $(\lambda, \mu) = (t^a, t^b)\in\mathbb{C}^*\times\mathbb{C}^*$ with integers $a<0$ and $b>0$. The action of this subgroup defines cell decompositions of the Hilbert schemes of zero-dimensional subschemes on $\mathbb{CP}^2$ and of the equivariant one(s). Cells (locally closed subvarieties isomorphic to complex affine spaces) correspond to fixed points of the action on the Hilbert schemes. The cell corresponding to a fixed point A consist of points on the Hilbert scheme whose orbits tend to A for $t\to 0$. The dimension of a cell corresponding to a fixed point is equal to the dimension of the subspace of the tangent space to $\mathrm{Hilb}^k_{\mathbb{CP}^2}$ at this point corresponding to representations of $\mathbb{C}^*$ with positive characters. For the described subgroup $\mathbb{C}^*\subset\mathbb{C}^*\times\mathbb{C}^*$, the set of points of the Hilbert scheme $\mathrm{Hilb}^k_{\mathbb{CP}^2}$ whose orbits tend to subschemes supported at the origin in $\mathbb{C}^2\subset\mathbb{CP}^2$ coincides with $\mathrm{Hilb}^k_{\mathbb{C}^2,\mathbb{C}}$ (see [4]). Fixed points of the natural action of the torus $\mathbb{C}^*\times\mathbb{C}^*$ on $\mathrm{Hilb}^k_{\mathbb{C}^2}\subset\mathrm{Hilb}^k_{\mathbb{CP}^2}$ and also on ${}^{(1)}\mathrm{Hilb}^{p,q;k}_{\mathbb{C}^2}\subset{}^{(1)}\mathrm{Hilb}^{p,q;k}_{\mathbb{CP}^2}$ are the monomial ideals in $\mathbb{C}[[x, y]]$ of length (codimension) k. Monomial ideals of length k correspond to partitions of k, i.e., to Young diagrams of size k.

The fibre of the tautological bundle over $\mathrm{Hilb}^k_{\mathbb{C}^2}$ over a fixed point has a natural basis given by the monomials inside the corresponding Young diagram. Moreover this basis is an equivariant one: the one-dimensional subspace generated by a monomial x^iy^j is preserved by the group action. The corresponding representation of the group $\mathbb{Z}_p$ is given by the quasi-homogeneous weight $v = i + qj \mod p$ of the monomial, i.e., it is δ^v, where δ is the natural basic one-dimensional representation of $\mathbb{Z}_p$: the multiplication by σ. The regular representation of the cyclic group $\mathbb{Z}_p$

has the decomposition of the form $1+\delta+\delta^2+\cdots+\delta^{p-1}$. Therefore a monomial ideal belongs to ${}^{(2)}\mathrm{Hilb}^{p,q;k}_{\mathbb{C}^2}$ iff the corresponding Young diagram has the same numbers of boxes with quasi-homogeneous weights 0, 1, ..., $p-1$.

For a fixed point in $\mathrm{Hilb}^k_{\mathbb{C}^2,\mathbb{C}}$ described by the Young diagram of the partition $\{b_0 \geqslant b_1 \geqslant \cdots \geqslant b_{r-1} > 0 = b_r\}$ of the integer k, the tangent space to the corresponding cell of the cell decomposition of $\mathrm{Hilb}^k_{\mathbb{C}^2,\mathbb{C}}$ is the "positive part" T^+ of the tangent space to the Hilbert scheme $\mathrm{Hilb}^k_{\mathbb{C}^2}$ ([4]). The positive part T^+ carries a representation of the group $\mathbb{C}^*\times\mathbb{C}^*$. The equivariant formula for it (i.e., with the decomposition into one-dimensional spaces corresponding to different characters of $\mathbb{C}^*\times\mathbb{C}^*$) is given in [4]:

$$T^+ = \sum_{1\leqslant i\leqslant j\leqslant r}\ \sum_{s=b_j}^{b_{j-1}-1} \lambda^{i-j-1}\mu^{b_{i-1}-s-1}. \tag{7}$$

If the discussed fixed point belongs to ${}^{(1)}\mathrm{Hilb}^{p,q;k}_{\mathbb{C}^2}$, the tangent space to ${}^{(1)}\mathrm{Hilb}^{p,q;k}_{\mathbb{C}^2}$ at this point is the part of the tangent space to $\mathrm{Hilb}^k_{\mathbb{C}^2}$ corresponding to the trivial representation of the group $\mathbb{Z}_p$. Therefore the tangent space to the corresponding cell in the cell decomposition of ${}^{(1)}\mathrm{Hilb}^{p,q;k}_{\mathbb{C}^2,\mathbb{C}}$ is the part of T^+ corresponding to the trivial representation. This means that the dimension of this cell is equal to the number of the monomials in the right hand side of (7) with the weights $(i-j-1)+q(b_{i-1}-s-1)\equiv 0 \pmod p$.

For $q=-1$ these weights are just the hook lengths of the corresponding Young diagram. Thus one has to count the number of Young diagrams of size k with w hook lengths divisible by p. The necessary information on this subject can be found, e.g., in [9, Section 2.7]. A Young diagram is called a p-core diagram if it has no hook lenghts divisible by p. There is an algorithm to remove the so called rim p-hooks (containing p boxes each) from a Young diagram $[\alpha]$ (one by one in any order) so that finally one gets the (well defined) p-core diagram $[\tilde\alpha]$ of $[\alpha]$ and p diagrams $[\alpha]_0$, $[\alpha]_1$, ..., $[\alpha]_{p-1}$ which constitute the so called star p-diagram (or the p-quotient: Theorem 2.7.37) of $[\alpha]$. The number of removed rim p-hooks $w=w([\alpha])$ is called the p-weight of the diagram $[\alpha]$ and it is equal to the total number of boxes in the star p-diagram of $[\alpha]$ and also to the number of hook lengths divisible by p (Statement 2.7.40). Moreover any collection consisting of a p-core diagram of size $k'=k-pw$ and p (arbitrary) diagrams of sizes k_0, k_1, ..., k_{p-1} with $k_0+k_1+\cdots+k_{p-1}=w$ gives rise to a well defined Young diagram (Theorem 2.7.30). The p-core diagram $[\tilde\alpha]$ of a Young diagram $[\alpha]$ is defined by the p-content of $[\alpha]$, that is, by the numbers of boxes (i,j) in $[\alpha]$ with the p-residues $i-j \bmod p$ equal to 0, 1, ..., $p-1$ respectively (Theorem 2.7.41). In particular, the p-core diagram $[\tilde\alpha]$ of a Young diagram $[\alpha]$ of size k is empty iff it has the equal numbers (k/p; in this case k has to be divisible by p) of boxes with different p-residues.

This means that monomial ideals belonging to ${}^{(2)}\mathrm{Hilb}^{p,-1;k}_{\mathbb{C}^2,\mathbb{C}}$ are in one-to-one correspondence with collections of Young diagrams (or of partitions) of sizes k_0, k_1, ..., k_{p-1} with $k_0+k_1+\cdots+k_{p-1}=k/p$. The generating series for the numbers of Young diagrams of different sizes is $\prod_{i=1}^{\infty}\frac{1}{1-T^i}$ (see, e.g., [13, Section 6.1]). Therefore, the generating series for collections consisting of p Young diagrams is

$\left(\prod_{i=1}^{\infty}\frac{1}{1-T^i}\right)^p=\prod_{i=1}^{\infty}\frac{1}{(1-T^i)^p}$. The dimension of the corresponding cell in the cell decomposition of ${}^{(2)}\mathrm{Hilb}^{p,-1;k}_{\mathbb{C}^2,\mathbb{C}}$ is equal to $k_0+k_1+\cdots+k_{p-1}$. Therefore

$$ {}^{(2)}\mathbb{H}^{p,-1}_{\mathbb{C}^2,\mathbb{C}}(T)=\prod_{i=1}^{\infty}\frac{1}{(1-\mathbb{L}^iT^{pi})^p}\,. $$

The generating series for the numbers of p-cores of different sizes is

$$ \prod_{i=1}^{\infty}\frac{(1-T^{pi})^p}{1-T^i} $$

(see, e.g., [2]). Therefore

$$ {}^{(1)}\mathbb{H}^{p,-1}_{\mathbb{C}^2,\mathbb{C}}(T)=\prod_{i=1}^{\infty}\frac{(1-T^{pi})^p}{1-T^i}\cdot\prod_{i=1}^{\infty}\frac{1}{(1-\mathbb{L}^iT^{pi})^p}. $$

A $\mathbb{Z}_p$-invariant zero-dimensional subschemes of $\mathbb{C}^2$ supported at points of the line $\mathbb{C}$ is the union of its corresponding parts supported at points of $\mathbb{C}^*=\mathbb{C}\setminus\{0\}$ and at the origin respectively. Therefore ${}^{(\bullet)}\mathrm{Hilb}^{\mathbb{Z}_p,k}_{\mathbb{C}^2,\mathbb{C}}=\bigsqcup_{l=0}^{k}{}^{(\bullet)}\mathrm{Hilb}^{\mathbb{Z}_p,l}_{\mathbb{C}^2,0}\times{}^{(\bullet)}\mathrm{Hilb}^{\mathbb{Z}_p,k-l}_{\mathbb{C}^2,\mathbb{C}^*}$ and thus

$$ {}^{(\bullet)}\mathbb{H}^{p,-1}_{\mathbb{C}^2,\mathbb{C}}(T)={}^{(\bullet)}\mathbb{H}^{p,-1}_{\mathbb{C}^2,0}(T)\,{}^{(\bullet)}\mathbb{H}^{p,-1}_{\mathbb{C}^2,\mathbb{C}^*}(T). $$

The space of $\mathbb{Z}_p$-invariant zero-dimensional subschemes of $\mathbb{C}^2$ supported at points of the line $\mathbb{C}^*$ is in one-to-one correspondence with the space of (usual) zero-dimensional subschemes of $\mathbb{C}^2\setminus\{0\}/\mathbb{Z}_p$ supported at points of $\mathbb{C}^*/\mathbb{Z}_p\cong\mathbb{C}^*$. The length of the usual subscheme is equal to the length of the corresponding invariant one divided by p. Therefore

$$ {}^{(\bullet)}\mathbb{H}^{p,-1}_{\mathbb{C}^2,\mathbb{C}^*}(T)={}^{(\bullet)}\mathbb{H}_{\mathbb{C}^2\setminus\{0\}/\mathbb{Z}_p,\mathbb{C}^*/\mathbb{Z}_p}(T^p)=\left(\mathbb{H}_{\mathbb{C}^2,0}(T^p)\right)^{[\mathbb{C}^*/\mathbb{Z}_p]}=\left(\mathbb{H}_{\mathbb{C}^2,0}(T^p)\right)^{\mathbb{L}-1} $$

(see (4)) and

$$ {}^{(\bullet)}\mathbb{H}^{p,-1}_{\mathbb{C}^2,\mathbb{C}}(T)={}^{(\bullet)}\mathbb{H}^{p,-1}_{\mathbb{C}^2,0}(T)\cdot\left(\mathbb{H}_{\mathbb{C}^2,0}(T^p)\right)^{\mathbb{L}-1}\,. $$

Equation (5) and the equation $(1-\mathbb{L}^iT^j)^{-\mathbb{L}}=(1-\mathbb{L}^{i+1}T^j)^{-1}$ (see (2)) give

$$ {}^{(\bullet)}\mathbb{H}^{p,-1}_{\mathbb{C}^2,0}(T)={}^{(\bullet)}\mathbb{H}^{p,-1}_{\mathbb{C}^2,\mathbb{C}}(T)\cdot\left(\prod_{i=1}^{\infty}\frac{1}{1-\mathbb{L}^{i-1}T^{pi}}\right)^{1-\mathbb{L}}={}^{(\bullet)}\mathbb{H}^{p,-1}_{\mathbb{C}^2,\mathbb{C}}(T)\cdot\prod_{i=1}^{\infty}\frac{1-\mathbb{L}^iT^{pi}}{1-\mathbb{L}^{i-1}T^{pi}}\,. $$

Therefore

$$ {}^{(1)}\mathbb{H}^{p,-1}_{\mathbb{C}^2,0}(T)=\prod_{i=1}^{\infty}\frac{(1-T^{pi})^p}{1-T^i}\cdot\prod_{i=1}^{\infty}\frac{1}{(1-\mathbb{L}^iT^{pi})^{p-1}(1-\mathbb{L}^{i-1}T^{pi})}, $$

$$ {}^{(2)}\mathbb{H}^{p,-1}_{\mathbb{C}^2,0}(T)=\prod_{i=1}^{\infty}\frac{1}{(1-\mathbb{L}^iT^{pi})^{p-1}(1-\mathbb{L}^{i-1}T^{pi})}. \qquad\square $$

Repeating the same arguments as above, one gets the following equations.

Corollary.

$$
\begin{aligned}
{}^{(1)}\mathbb{H}^{p,-1}_{\mathbb{C}^2}(T) &= \prod_{i=1}^{\infty}\left(\frac{(1-T^{pi})^p}{1-T^i}\cdot\frac{1}{(1-\mathbb{L}^iT^{pi})^{p-1}\cdot(1-\mathbb{L}^{i+1}T^{pi})}\right),\\
{}^{(2)}\mathbb{H}^{p,-1}_{\mathbb{C}^2}(T) &= \prod_{i=1}^{\infty}\frac{1}{(1-\mathbb{L}^iT^{pi})^{p-1}\cdot(1-\mathbb{L}^{i+1}T^{pi})}.
\end{aligned}
\tag{8}
$$

Remark. Equation (8) can be deduced, e.g., from [12] or [8] using the fact that the coefficients of the series ${}^{(2)}\mathbb{H}^{p,-1}_{\mathbb{C}^2}(T)$ are polynomials in $\mathbb{L}$.

Corollary. *Let the cyclic group $\mathbb{Z}_p$ act on a smooth surface S in such a way, that the factor space $S/\mathbb{Z}_p$ has only A_{p-1} singularities (i.e., at each of d fixed points P_1, $\ldots$, P_d one has the representation corresponding to $q=-1$). Then*

$$
\begin{aligned}
{}^{(1)}\mathbb{H}^{\mathbb{Z}_p}_{S}(T) = \left(\prod_{i=1}^{\infty}\frac{(1-T^{pi})^p}{(1-T^i)(1-\mathbb{L}^iT^{pi})^{p-1}(1-\mathbb{L}^{i-1}T^{pi})}\right)^d\\
\cdot\left(\prod_{i=1}^{\infty}\frac{1}{1-\mathbb{L}^{i-1}T^{pi}}\right)^{[(S\setminus\{P_i\})/Z_p]},
\end{aligned}
$$

$$
{}^{(2)}\mathbb{H}^{\mathbb{Z}_p}_{S}(T) = \left(\prod_{i=1}^{\infty}\frac{1}{(1-\mathbb{L}^iT^{pi})^{p-1}(1-\mathbb{L}^{i-1}T^{pi})}\right)^d\cdot\left(\prod_{i=1}^{\infty}\frac{1}{1-\mathbb{L}^{i-1}T^{pi}}\right)^{[(S\setminus\{P_i\})/Z_p]}.
$$

Example. Let the group $\mathbb{Z}_3$ act on the projective plane $\mathbb{CP}^2$ by $\sigma*(x_0:x_1:x_2)=(x_0:\sigma x_1:\sigma^2x_2)$. Then

$$
\begin{aligned}
{}^{(2)}\mathbb{H}^{\mathbb{Z}_3}_{\mathbb{CP}^2}(T) = 1+(1+7\mathbb{L}+\mathbb{L}^2)T^3+(1+8\mathbb{L}+36\mathbb{L}^2+8\mathbb{L}^3+\mathbb{L}^4)T^6\\
+(1+8\mathbb{L}+44\mathbb{L}^2+149\mathbb{L}^3+44\mathbb{L}^4+8\mathbb{L}^5+\mathbb{L}^6)T^9\\
+(1+8\mathbb{L}+45\mathbb{L}^2+192\mathbb{L}^3+543\mathbb{L}^4+192\mathbb{L}^5+45\mathbb{L}^6+8\mathbb{L}^7+\mathbb{L}^8)T^{12}+\cdots.
\end{aligned}
$$

Remarks. 1. One can easily see that if $q_1q_2\equiv 1 \pmod p$, one has ${}^{(\bullet)}\mathbb{H}^{p,q_1}_{\mathbb{C}^2,0}(T) = {}^{(\bullet)}\mathbb{H}^{p,q_2}_{\mathbb{C}^2,0}(T)$. Let the group $\mathbb{Z}_p$ act on the plane $\mathbb{C}^2$ by $\sigma*(x,y)=(\sigma^ax,\sigma^by)$, where $\gcd(a,p)=1$. Then one has ${}^{(\bullet)}\mathbb{H}^{\mathbb{Z}_p}_{\mathbb{C}^2,0}(T) = {}^{(\bullet)}\mathbb{H}^{p,q}_{\mathbb{C}^2,0}(T)$ with $q\equiv b/a \pmod p$.

2. It seems that, for $q\neq -1$, one has somewhat better (less complicated) formulae for the series ${}^{(1)}\mathbb{H}^{p,q}_{\mathbb{C}^2,0}(T)$ than for the series ${}^{(2)}\mathbb{H}^{p,q}_{\mathbb{C}^2,0}(T)$ (at least in the form similar to (5) and (6)). To show that, it is convenient to write down the logarithms $\mathrm{Log}\,{}^{(\bullet)}\mathbb{H}^{p,q}_{\mathbb{C}^2,0}(T)$ of the generating series ${}^{(\bullet)}\mathbb{H}^{p,q}_{\mathbb{C}^2,0}(T)$ in the sense of [6]: if $A(T)=\prod_{i,j}(1-\mathbb{L}^jT^i)^{-k_{ij}}$, with $k_{ij}\in\mathbb{Z}$, then by definition $\mathrm{Log}\,A(T)=\sum_{i,j}k_{ij}\mathbb{L}^jT^i$. In particular, the equation (6) means that

$$
\mathrm{Log}\,{}^{(2)}\mathbb{H}^{p,-1}_{\mathbb{C}^2,0}(T) = \sum_{i=1}^{\infty}\left((p-1)\mathbb{L}^i+\mathbb{L}^{i-1}\right)T^{pi}.
$$

Computations made with Maple gave:

$$\begin{aligned}\operatorname{Log}{}^{(1)}\mathbb{H}^{3,1}_{\mathbb{C}^2,0}(T) = {} & T + \mathbb{L}T^2 + T^3 + \mathbb{L}T^4 + \mathbb{L}^2T^5 + \mathbb{L}T^6 + \mathbb{L}^2T^7 + \mathbb{L}^3T^8 + \mathbb{L}^2T^9 \\ & + \mathbb{L}^3T^{10} + \mathbb{L}^4T^{11} + \mathbb{L}^3T^{12} + \mathbb{L}^4T^{13} + \mathbb{L}^5T^{14} + \mathbb{L}^4T^{15} + \mathbb{L}^5T^{16} \\ & + \mathbb{L}^6T^{17} + \mathbb{L}^5T^{18} + \mathbb{L}^6T^{19} + \mathbb{L}^7T^{20} + \mathbb{L}^6T^{21} + \cdots,\end{aligned}$$

$$\begin{aligned}\operatorname{Log}{}^{(2)}\mathbb{H}^{3,1}_{\mathbb{C}^2,0}(T) = {} & (1+\mathbb{L})T^3 + (2\mathbb{L} + 2\mathbb{L}^2 + \mathbb{L}^3)T^6 + (2\mathbb{L}^2 + 2\mathbb{L}^3 + \mathbb{L}^4)T^9 \\ & + (-\mathbb{L}^2 + \mathbb{L}^3 - \mathbb{L}^6)T^{12} + (-\mathbb{L}^3 - \mathbb{L}^5 - \mathbb{L}^6 - \mathbb{L}^7)T^{15} + (2\mathbb{L}^5 + \mathbb{L}^7)T^{18} \\ & + (2\mathbb{L}^4 + 3\mathbb{L}^5 + 7\mathbb{L}^6 + 6\mathbb{L}^7 + 6\mathbb{L}^8 + 3\mathbb{L}^9 + 2\mathbb{L}^{10})T^{21} + \cdots.\end{aligned}$$

One can make the following

Conjecture.

$$^{(1)}\mathbb{H}^{3,1}_{\mathbb{C}^2,0}(T) = \prod_{i=1}^{\infty} \frac{1}{(1-\mathbb{L}^{i-1}T^{3i-2})(1-\mathbb{L}^{i}T^{3i-1})(1-\mathbb{L}^{i-1}T^{3i})}.$$

A conjectural equation for $^{(2)}\mathbb{H}^{3,1}_{\mathbb{C}^2,0}(T)$ is not clear.

Some other examples:

$$\begin{aligned}\operatorname{Log}{}^{(1)}\mathbb{H}^{4,1}_{\mathbb{C}^2,0}(T) = {} & T + \mathbb{L}T^2 + T^3 + \mathbb{L}T^4 + T^5 + (-1+\mathbb{L}+\mathbb{L}^2)T^6 + T^7 \\ & + (-1+\mathbb{L}+\mathbb{L}^2)T^8 + T^9 + (-1+\mathbb{L}^2+\mathbb{L}^3)T^{10} + T^{11} + (-1+\mathbb{L}^2+\mathbb{L}^3)T^{12} \\ & + T^{13} + (-1+\mathbb{L}^3+\mathbb{L}^4)T^{14} + T^{15} + (-1+\mathbb{L}^3+\mathbb{L}^4)T^{16} + T^{17} \\ & + (-1+\mathbb{L}^4+\mathbb{L}^5)T^{18} + T^{19} + (-1+\mathbb{L}^4+\mathbb{L}^5)T^{20} + \cdots,\end{aligned}$$

$$\begin{aligned}\operatorname{Log}{}^{(2)}\mathbb{H}^{4,1}_{\mathbb{C}^2,0}(T) = {} & (1+\mathbb{L})T^4 + (2\mathbb{L} + 2\mathbb{L}^2 + \mathbb{L}^3)T^8 \\ & + (\mathbb{L} + 4\mathbb{L}^2 + 5\mathbb{L}^3 + 3\mathbb{L}^4 + \mathbb{L}^5)T^{12} + (4\mathbb{L}^3 + 5\mathbb{L}^4 + 3\mathbb{L}^5)T^{16} \\ & + (-\mathbb{L}^2 - 3\mathbb{L}^3 - 2\mathbb{L}^4 - \mathbb{L}^5 - 3\mathbb{L}^6 - 3\mathbb{L}^7 - \mathbb{L}^8)T^{20} + \cdots,\end{aligned}$$

$$\begin{aligned}\operatorname{Log}{}^{(1)}\mathbb{H}^{5,2}_{\mathbb{C}^2,0}(T) = {} & T + T^2 + \mathbb{L}T^3 + \mathbb{L}T^4 + T^5 + \mathbb{L}T^6 + \mathbb{L}T^7 + \mathbb{L}^2T^8 + \mathbb{L}^2T^9 + \mathbb{L}T^{10} \\ & + \mathbb{L}^2T^{11} + \mathbb{L}^2T^{12} + \mathbb{L}^3T^{13} + \mathbb{L}^3T^{14} + \mathbb{L}^2T^{15} + \mathbb{L}^3T^{16} + \mathbb{L}^3T^{17} + \mathbb{L}^4T^{18} \\ & + \mathbb{L}^4T^{19} + \mathbb{L}^3T^{20} + \mathbb{L}^4T^{21} + \mathbb{L}^4T^{22} + \mathbb{L}^5T^{23} + \mathbb{L}^5T^{24} + \mathbb{L}^4T^{25} + \cdots,\end{aligned}$$

$$\begin{aligned}\operatorname{Log}{}^{(2)}\mathbb{H}^{5,2}_{\mathbb{C}^2,0}(T) = {} & (1+2\mathbb{L})T^5 + (3\mathbb{L} + 5\mathbb{L}^2 + 2\mathbb{L}^3)T^{10} + (3\mathbb{L}^2 + 5\mathbb{L}^3 + 2\mathbb{L}^4)T^{15} \\ & + (-3\mathbb{L}^2 - 2\mathbb{L}^3 - 4\mathbb{L}^4 - 3\mathbb{L}^5 - 3\mathbb{L}^6)T^{20} \\ & + (-3\mathbb{L}^3 - 6\mathbb{L}^4 - 9\mathbb{L}^5 - 7\mathbb{L}^6 - 3\mathbb{L}^7)T^{25} + \cdots.\end{aligned}$$

3. Though a conjectural formula for $^{(2)}\mathbb{H}^{M,1}_{\mathbb{C}^2,0}(T)$ (or for $\operatorname{Log}{}^{(2)}\mathbb{H}^{M,1}_{\mathbb{C}^2,0}(T)$) is not clear even for small $M > 2$, computations show that one could have the following stabilization. Let $\operatorname{Log}{}^{(2)}\mathbb{H}^{M,1}_{\mathbb{C}^2,0}(T) = \sum_{i=1}^{\infty} p_i^{M,1}(\mathbb{L})\cdot T^{Mi}$, where $p_i^{M,1}(\mathbb{L})$ are polynomials in $\mathbb{L}$. The computations predict that $p_i^{M',1}(\mathbb{L}) = p_i^{M'',1}(\mathbb{L})$ for $M'' > M' > i$.

The authors are very thankful to A. Kuznetsov for careful reading the text, making a number of useful remarks and, in particular, finding a mistake in the initial version of the paper.

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