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# A POWER STRUCTURE OVER THE GROTHENDIECK RING OF VARIETIES

S.M. Gusein-Zade, I. Luengo and A. Melle–Hernández

Abstract. Let $\mathcal{R}$ be either the Grothendieck semiring (semigroup with multiplication) of complex quasi-projective varieties, or the Grothendieck ring of these varieties, or the Grothendieck ring localized by the class $\mathbb{L}$ of the complex affine line. We define a power structure over these (semi)rings. This means that, for a power series $A(t) = 1 + \sum_{i=1}^{\infty} [A_i]t^i$ with the coefficients $[A_i]$ from $\mathcal{R}$ and for $[M] \in \mathcal{R}$, there is defined a series $(A(t))^{[M]}$, also with coefficients from $\mathcal{R}$, so that all the usual properties of the exponential function hold. In the particular case $A(t) = (1-t)^{-1}$, the series $(A(t))^{[M]}$ is the motivic zeta function introduced by M. Kapranov. As an application we express the generating function of the Hilbert scheme of points, 0-dimensional subschemes, on a surface as an exponential of the surface.

By a semiring we mean a semigroup with multiplication. If $\mathcal{R}$ is a semiring, we have the semiring $\mathcal{R}[[t]]$ of formal power series with coefficients in $\mathcal{R}$. Let $1+\mathcal{R}_+[[t]]$ denote the set of series of the form $A(t) = 1 + \sum_{i=1}^{\infty} A_i t^i$ with $A_i \in \mathcal{R}$. If $\mathcal{R}$ is a ring, resp. a semiring, $1+\mathcal{R}_+[[t]]$ is an abelian group, resp. a semigroup, with respect to the multiplication. The Grothendieck semiring $S_0(\mathrm{Var}_{\mathbb{C}})$ of complex quasi-projective varieties is the semigroup generated by isomorphism classes $[X]$ of such varieties modulo the relation $[X] = [X-Y] + [Y]$ for a closed subvariety $Y \subset X$; the multiplication is defined by $[X_1]\cdot[X_2] = [X_1 \times X_2]$. The Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ is the group generated by these classes with the same relation and the same multiplication. Let $\mathbb{L} \in K_0(\mathrm{Var}_{\mathbb{C}})$ be the class of the affine line, which itself also will be denoted by $\mathbb{L}$, and let $\mathcal{M} := K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$ be the localization of Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ with respect to $\mathbb{L}$.

Power series $\sum_{i=0}^{\infty} [A_i]t^i$ with coefficients from one of these (semi)rings are usual objects of study, in particular, in the framework of the theory of motivic integration, see for instance [3],[4], [9]. The main result of the paper shows that there exists a natural notion of the $[M]$-th power of such a series with $[A_0] = 1$ for the exponent $[M]$ from the same ring or semiring.

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**Definition**: A *power structure* over a (semi)ring $\mathcal{R}$ is a map $(1+\mathcal{R}_+[[t]])\times\mathcal{R}\to 1+\mathcal{R}_+[[t]]:\ (A(t),m)\mapsto (A(t))^m$ which possesses the properties:

1. $(A(t))^0 = 1$.
2. $(A(t))^1 = A(t)$.
3. $(A(t)\cdot B(t))^m = (A(t))^m\cdot (B(t))^m$.
4. $(A(t))^{m+n} = (A(t))^m\cdot (A(t))^n$.
5. $(A(t))^{mn} = ((A(t))^n)^m$.

**Remark**. Some examples of power structures can be found in the theory of $\lambda$-rings (see, e.g., [1], [2], [5]). Note also that in the theory of $\lambda$-rings the constructions in [2] and [5] have sense only over rings which are $\mathbb{Q}$-algebras. The main point of our construction is that we use factorization by actions of permutation groups instead of division by integers. Using our approach a natural structure of $\lambda$-ring on the Grothendieck ring of complex algebraic varieties would be given by $\lambda_t([M]) = (1+t)^{[M]}$. But in such a case the coefficient at $t^2$ in the power series $(1+t)^{\mathbb{L}^2}$ computed as $\lambda_t(\mathbb{L}\times\mathbb{L})$ should be equal to $2\mathbb{L}^4-2\mathbb{L}^2$, while in our construction it is equal to $\mathbb{L}^4-\mathbb{L}^2$.

**Definition**: Let $A(t) = 1+[A_1]\,t+[A_2]\,t^2+\dots$ be a formal power series with coefficients from $S_0(\mathrm{Var}_{\mathbb{C}})$ and $[M]\in S_0(\mathrm{Var}_{\mathbb{C}})$. We define

(1)
$$(A(t))^{[M]} := 1+\sum_{k=1}^{\infty}\left\{\sum_{\underline{k}:\Sigma\ ik_i=k}\left[\left((\prod_i M^{k_i})\setminus\Delta\right)\times\prod_i A_i^{k_i}/\prod_i S_{k_i}\right]\right\}\cdot t^k,$$

where $\Delta$ is the "large diagonal" in $M^{\Sigma k_i}$ which consists of $(\sum k_i)$-tuples of points of $M$ with at least two coinciding ones, the group $S_{k_i}$ acts by permuting corresponding $k_i$ factors in $\prod\limits_i M^{k_i}\supset(\prod_i M^{k_i})\setminus\Delta$ and the spaces $A_i$ simultaneously.

**Remarks**. **1.** This definition can also be "read from" the following formula for the usual power $(1+a_1t+a_2t^2+\dots)^m$ of a series with a natural exponent:

$$(1+a_1t+a_2t^2+\dots)^m =$$

$$= 1+\sum_{k=1}^{\infty}\left(\sum_{\underline{k}:\Sigma ik_i=k} m(m-1)\cdot\ldots\cdot(m-\Sigma k_i+1)\cdot\prod_i a_i^{k_i}/\prod_i k_i!\right)\cdot t^k,$$

see, for instance [12], page 40. In this formula one should understand the product $m(m-1)\cdot\ldots\cdot(m-\Sigma k_i+1)$ as $M^{\Sigma k_i}\setminus\Delta$. If $M$ is a set which consists of $m$ points, this product is the number of elements of the indicated space. The division by $\prod\limits_i k_i!$ should be understood as the factorization by the group $\prod\limits_i S_{k_i}$.

**2.** For a complex algebraic variety $M$, its motivic zeta function $\zeta_{[M]}(t)$ was defined by M. Kapranov in [8] as the power series

$$\zeta_{[M]}(t) := \sum_{k=0}^{\infty}[S^kM]\cdot t^k = 1 + [S^1M]\cdot t + [S^2M]\cdot t^2 + [S^3M]\cdot t^3 + \dots ,$$

where $S^kM$ is the $k$-th symmetric power $M^k/S_k$ of the variety $M$, $S_k$ is the symmetric group of permutations on $k$ elements. It has the property $\zeta_{M+N}(t) = \zeta_M(t)\cdot\zeta_N(t)$.

**Theorem 1.** *The equation (1) defines a power structure over the Grothendieck semiring $S_0(\mathrm{Var}_{\mathbb{C}})$ of complex algebraic varieties such that*

$$(1+t+t^2+\dots)^{[M]} = \zeta_{[M]}(t).$$

*Proof.* The fact that $(1+t+t^2+\dots)^{[M]} = \zeta_{[M]}(t)$ and the properties 1 and 2 from the definition of a power structure are obvious. Let us reformulate the definition of the series $(A(t))^{[M]}$ a little bit so that the properties 3 to 5 will be proved by establishing one-to-one correspondences between the sets representing the coefficients of the left hand side (**LHS**) and the right hand side (**RHS**) series.

The coefficient at the monomial $t^k$ in the series $(A(t))^{[M]}$ is represented by the set whose element is a finite subset $K$ of points of the variety $M$ with positive multiplicities such that the total number of points of the set $K$ counted with multiplicities is equal to $k$ plus a map $\varphi$ from $K$ to $\mathcal{A} = \coprod_{i=0}^{\infty} A_i$ such that a point of multiplicity $s$ goes to $A_s \subset \mathcal{A}$. Here $\coprod$ means the disjoint union which is the sum in the semiring $S_0(\mathrm{Var}_{\mathbb{C}})$. For short, instead of writing that "a coefficient of a series is represented by a set which consists of elements of the form ... " we shall write that "an element of the coefficient is ... ".

*Proof of 3.* Consider another series $B(t) = \sum_{i=0}^{\infty}[B_i]\, t^i$ with coefficients from the same semiring. Let $[C_j] := \sum_{i=0}^{j}[A_i][B_{j-i}]$ be the coefficient at the monomial $t^j$ in the product $A(t)\cdot B(t)$. An element of the coefficient at the monomial $t^k$ in the **LHS** of the equation 3 is a $k$-point subset $K$ of $M$ with a map $\varphi$ from $K$ to $\coprod_{i=0}^{\infty} C_i = \coprod_{i,j\geq 0} A_iB_j$ such that a point of multiplicity $s$ goes to $C_s = \coprod_{i=0}^{s} A_i \times B_{s-i}$. An element of the coefficient at the monomial $t^k$ in the **RHS** of this equation is: an $\ell$-point subset $K_1$ of the variety $M$ with $0 \leq \ell \leq k$, a map $\varphi_1$ from $K_1$ to $\mathcal{A} = \coprod_{i=0}^{\infty} A_i$ such that a point of multiplicity $s$ goes to $A_s$, a $(k-\ell)$-point subset $K_2$ of the variety $M$, a map $\varphi_2$ from $K_2$ to $\mathcal{B} = \coprod_{i=0}^{\infty} B_i$ such that a point of multiplicity $s$ goes to $B_s$. Suppose we have an element of the first set. Let us decompose the subset $K$ into two parts $K_1$ and $K_2$. A point $x$ of $K$

of multiplicity $s = s(x)$ goes to $C_s = \coprod_{i=0}^{s} A_i \times B_{s-i}$ and thus $\varphi(x)$ belongs to one of the summands, say to $A_{i_0} \times B_{s-i_0}$. Let us include the point $x$ into the set $K_1$ with the multiplicity $i_0$ and into the set $K_2$ with the multiplicity $s - i_0$. If $i_0$ and/or $s - i_0$ is positive, define $\varphi_1(x)$ and/or $\varphi_2(x)$ as the corresponding projection, to $A_{i_0}$ and/or to $B_{s-i_0}$ respectively, of the point $\varphi(x) \in A_{i_0} \times B_{s-i_0}$. In the other direction, from an element of the second set one can construct an element of the first one uniting the subsets $K_1$ and $K_2$ so that a multiplicity of a point $x$ in $K_1 \cup K_2$ is equal to the sum $s_1 + s_2$ of its multiplicities in $K_1$ and in $K_2$ and defining $\varphi(x)$ as $(\varphi_1(x), \varphi_2(x)) \in A_{s_1} \times B_{s_2} \subset C_{s_1+s_2}$. One easily sees that these correspondences are inverse to each other and thus are one-to-one. □

*Proof of 4.* An element of the coefficient at the monomial $t^k$ in the **LHS** of the equation is a $k$-point subset $K$ of the variety $M \cup N$ with a map $\varphi$ from $K$ to $\mathcal{A} = \coprod_{i=0}^{\infty} A_i$. An element of the coefficient at the monomial $t^k$ in the **RHS** is a pair: an $\ell$-point subset $K_1$ of the variety $M$ with $0 \leq \ell \leq k$ and a $(k - \ell)$-point subset $K_2$ of the variety $N$ with their maps to $\mathcal{A}$. The union of these two subsets with the corresponding map to $\mathcal{A}$ gives a subset in the union $M \cup N$ with a map to $\mathcal{A}$. This correspondence is obviously one-to-one. □

*Proof of 5.* An element of the coefficient at the monomial $t^k$ in the **LHS** of the equation is a $k$-point subset $K$ of the product $M \times N$ with a map $\varphi : K \to \mathcal{A} = \coprod_{i=0}^{\infty} A_i$. An element of the coefficient at the monomial $t^k$ in the **RHS** is a $k$-point subset $K_M$ of the variety $M$ with a map from it which sends a point of multiplicity $s$ to an $s$ point subset of $N$ with a map from it to $\mathcal{A}$. To establish a one-to-one correspondence between these coefficients, rather between the variety representing them, one should define $K_M$ as the projection $\mathrm{pr}_M K$ of the subset $K \subset M \times N$ to the first factor and, for $x \in K_M$, the corresponding subset in $N$ as $\mathrm{pr}_N \mathrm{pr}_M^{-1}(x)$ with the natural map to $\mathcal{A}$. □

□

**Example**.

$$(1+t)^{[M]} = 1 + \sum_{k=1}^{\infty} \left[(M^k \setminus \Delta)/S_k\right] \cdot t^k,$$

where $(M^k \setminus \Delta)/S_k$ is the configuration space of $k$ distinct unlabeled points in $M$.

Let $\chi(X)$ be the Euler characteristic of the space $X$ (the alternating sum of ranks of the cohomology groups with compact support). For a series $A(t) = 1 + [A_1]t + [A_2]t^2 + \ldots$, one defines its Euler characteristic as the series

$$\chi\left(A(t)\right) = 1 + \chi([A_1])t + \chi([A_2])t^2 + \ldots \in \mathbb{Z}[[t]].$$

**Statement 1.**

$$\chi\left((A(t))^{[M]}\right) = (\chi\,(A(t)))^{\chi([M])}\,.$$

The **proof** follows either from direct calculations or from the fact that the coefficients at the monomials $t^k$ in the **LHS** of the equation are polynomials in the Euler characteristics of the varieties $M$ and $A_i$ $(i = 1, 2, \dots)$ and the equation holds for "natural numbers", i.e., for the case when all the varieties $M$ and $A_i$ are finite sets of points. $\square$

**Theorem 2.** *There exists a unique power structure over the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$ of complex algebraic varieties which extends the one defined over the semiring $S_0(\mathrm{Var}_{\mathbb{C}})$.*

*Proof.* To define the operation notice that for any series $A(t) \in 1+K_0(\mathrm{Var}_{\mathbb{C}})_+[[t]]$ there exists a series $B(t) = \sum_{i=0}^{\infty}[B_i]\,t^i \in 1 + K_0(\mathrm{Var}_{\mathbb{C}})_+[[t]]$ with the coefficients from the image of the natural map $S_0(\mathrm{Var}_{\mathbb{C}}) \to K_0(\mathrm{Var}_{\mathbb{C}})$ such that all the coefficients of the product $C(t) = A(t) \cdot B(t)$ are from the same image as well. Then one puts:

$$(A(t))^{[M]} := (C(t))^{[M]} \,/\, (B(t))^{[M]}\,.$$

To define the power of a series with the exponent $[M]$ from the Grothendieck ring $K_0(\mathrm{Var}_{\mathbb{C}})$, one puts

$$(A(t))^{-[M]} := 1/\,(A(t))^{[M]}\,.$$

The properties of the definition of a power structure obviously hold. $\square$

**Theorem 3.** *There exists a unique power structure over the ring $\mathcal{M} = K_0(\mathrm{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$ which extends the one defined over the ring $K_0(\mathrm{Var}_{\mathbb{C}})$.*

*Proof.* First let us define the operation for a series $A(t)$ with the coefficients $[A_i]$ from the ring $\mathcal{M}$ and with the exponent $[M]$ from the "non-localized" ring $K_0(\mathrm{Var}_{\mathbb{C}})$. This is possible because of the following statement.

**Statement 2.** *Let $[A_i]$ and $[M]$ be from the Grothendieck ring of complex algebraic varieties. Then, for any integer $s \geq 0$, $(A(\mathbb{L}^s t))^{[M]} = \left(A(t)^{[M]}\right)|_{t\mapsto\mathbb{L}^s t}$.*

*Proof.* It is sufficient to prove this equation for a series with the coefficients from $S_0(\mathrm{Var}_{\mathbb{C}})$. The coefficient at the monomial $t^k$ in the power series $(A(t))^{[M]}$ is a sum of the classes of varieties of the form

$$V = \left((\prod_i M^{k_i}) \setminus \Delta\right) \times \prod_i A_i^{k_i} / \prod_i S_{k_i}$$

with $\sum ik_i = k$. The corresponding summand $[\widetilde{V}]$ in the coefficient at the monomial $t^k$ in the power series $(A(\mathbb{L}^s t))^{[M]}$ has the form

$$\widetilde{V} = \left((\prod_i M^{k_i}) \setminus \Delta\right) \times \prod_i (\mathbb{L}^{si} \times A_i)^{k_i} / \prod_i S_{k_i}.$$

There is a natural map $\widetilde{V} \to V$ which from the point of view of differential geometry is a complex analytic vector bundle of rank $sk$. It is locally trivial over a neighbourhood of each point in the usual topology. According to [11] this is a vector bundle in the "algebraic sense" and then it is locally trivial over a Zariski open neighbourhood of each point. This implies that $[\widetilde{V}] = \mathbb{L}^{sk} \cdot [V]$. □

For a series $A(t) = \sum_{i=0}^{\infty} [A_i]\, t^i$, let $J^r A(t)$ denote its truncation $\sum_{i=0}^{r} [A_i]\, t^i$ up to terms of degree $r$. Statement 2 implies that, for $[A_i] \in \mathcal{M}$, $i \geq 1$ and $[M] \in K_0(\mathrm{Var}_{\mathbb{C}})$, one can define the power $(A(t))^{[M]}$ by the formula

$$J^r\left((A(t))^{[M]}\right) := (J^r A(\mathbb{L}^s t))^{[M]}\,_{|t \mapsto t/\mathbb{L}^s} \mod t^{r+1}$$

for $s$ large enough so that all the coefficients of $J^r A(\mathbb{L}^s t)$ belong to the image of the map $K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathcal{M}$. One can easily see that the properties 3–5 hold.

Now we have to extend the operation to the exponent $[M]$ from the localized ring $\mathcal{M}$. First let us do it for one particular series, namely for $(1 + t + t^2 + \dots)$. In other words, we define the zeta function $\zeta_{[M]}(t)$ for $[M] \in \mathcal{M}$.

**Statement 3.** *For $M \in K_0(\mathrm{Var}_{\mathbb{C}})$, $s \geq 0$, one has*

$$\zeta_{\mathbb{L}^s[M]}(t) = \zeta_{[M]}(\mathbb{L}^s t).$$

*Proof.* It is sufficient to prove the equation for $s = 1$. One has $\zeta_{\mathbb{L}}(t) = 1 + \mathbb{L}t + \mathbb{L}^2 t^2 + \dots$ and therefore, using Statement 2, gets

$$\zeta_{\mathbb{L}[M]}(t) = (1 + \mathbb{L}t + \mathbb{L}^2 t^2 + \dots)^{[M]} = (1 + t + t^2 + \dots)^{[M]}{}_{|t \mapsto \mathbb{L}t}.$$

□

This statement permits to define $\zeta_{[M]}(t)$ for $[M] \in \mathcal{M}$ by the formula

$$\zeta_{[M]}(t) = \zeta_{\mathbb{L}^s[M]}(\mathbb{L}^{-s} t)$$

for $s$ large enough so that $\mathbb{L}^s[M]$ belongs to the image of the map $K_0(\mathrm{Var}_{\mathbb{C}}) \to \mathcal{M}$.

For $A(t) \in 1 + K_0(\mathrm{Var}_{\mathbb{C}})_+[[t]]$, $[M] \in K_0(\mathrm{Var}_{\mathbb{C}})$, $s > 0$, one has

$$(A(t^s))^{[M]} = \left(A(t)^{[M]}\right)_{|t \mapsto t^s}.$$

Since $\zeta_{[M]}(t) = 1 + [M]\, t + \dots$, for any series $A(t) \in 1 + \mathcal{M}_+[[t]]$ and for any $r > 0$, the truncated series $J^r A(t)$ can be represented in a unique way as

$$J^r\left(\zeta_{[A_1]}(t) \cdot \zeta_{[A_2']}(t^2) \cdot \zeta_{[A_3']}(t^3) \cdot \ldots \cdot \zeta_{[A_r']}(t^r)\right)$$

with $[A_i'] \in \mathcal{M}$. One defines $(A(t))^{[M]}$ by the formula

$$J^r\left((A(t))^{[M]}\right) := J^r\left(\zeta_{[M][A_1]}(t) \cdot \zeta_{[M][A_2']}(t^2) \cdot \ldots \cdot \zeta_{[M][A_r']}(t^r)\right). \tag{2}$$

Properties 3–5 of the definition of a power structure obviously hold.

Since the formula (2) can be used to define the power structure over the rings $K_0(\mathrm{Var}_{\mathbb{C}})$ and $\mathcal{M}$, such a structure over them is **unique**. □

**Remarks**. **1.** Yu.I.Manin has informed us that his calculations in [10] resemble a particular case of our construction.

**2.** The following construction was inspired by the paper of E.Getzler [6]. Let $\mathcal{R}$ be either the ring $K_0(\mathrm{Var}_{\mathbb{C}})$ or the ring $\mathcal{M}$. Consider $1+\mathcal{R}_+[[t]]$ and $\mathcal{R}_+[[t]] = (t)\mathcal{R}[[t]]$ as groups with respect to the multiplication and addition respectively. The construction described here permits to define group isomorphisms $\mathbf{Exp}$ : $\mathcal{R}_+[[t]] \to 1+\mathcal{R}_+[[t]]$ and $\mathbf{Log} : 1+\mathcal{R}_+[[t]] \to \mathcal{R}_+[[t]]$. Namely,

$$\mathbf{Exp}([A_1]t+[A_2]t^2+..) := \prod_{k\geq 1}(1-t^k)^{-[A_k]} = \prod_{k\geq 1}\zeta_{[A_k]}(t^k);$$

for a series $M(t) = 1+[M_1]t+[M_2]t^2+\ldots \in 1+\mathcal{R}_+[[t]]$ ($\mathcal{R}$ is either $K_0(\mathrm{Var}_{\mathbb{C}})$ or $\mathcal{M}$),

$$\mathbf{Log}(M(t)) := \sum_{k\geq 1}[A'_k]t^k,$$

where $[A'_k] \in \mathcal{R}$ are defined by the equation $M(t) = \zeta_{[A'_1]}(t)\cdot\zeta_{[A'_2]}(t^2)\cdot\zeta_{[A'_3]}(t^3)\cdot\ldots$ (such a representation is unique). Obviously $\mathbf{Exp}$ and $\mathbf{Log}$ are inverse to each other. The properties of the power structure imply that the maps $\mathbf{Exp}$ and $\mathbf{Log}$ posess the usual properties of the log and exp functions: $\mathbf{Exp}(A(t)+B(t)) = \mathbf{Exp}(A(t))\mathbf{Exp}(B(t))$, $\mathbf{Log}(M(t)\cdot N(t)) = \mathbf{Log}(M(t))+\mathbf{Log}(N(t))$, $\mathbf{Exp}([M]t) = \zeta_{[M]}(t)$, ...

**An application.** For a smooth quasi–projective surface $M$, let $M^{[n]} = \mathrm{Hilb}^n M$ be the Hilbert scheme of 0-dimensional subschemes of length $n$ on $M$.

**Statement 4.** *In the Grothendieck ring* $K_0(\mathrm{Var}_{\mathbb{C}})$ *one has:*

$$\begin{aligned} 1+\sum_{n\geq 1}[M^{[n]}]t^n &= \prod_{k\geq 1}\zeta_{\mathbb{L}^{-1}[M]}((\mathbb{L}t)^k) = \left(\prod_{k\geq 1}\frac{1}{1-\mathbb{L}^{k-1}t^k}\right)^{[M]} \\ &= \prod_{k\geq 1}\left(\frac{1}{1-t^k}\right)^{\mathbb{L}^{k-1}[M]} = \left(\mathbf{Exp}(\frac{\mathbb{L}t}{1-\mathbb{L}t})\right)^{\mathbb{L}^{-1}[M]}. \end{aligned}$$

The **proof** follows from the following result of L. Göttsche [7]:

$$[M^{[n]}] = \sum_{\underline{k}:\Sigma ik_i=n}[S^{\underline{k}}M]\cdot\mathbb{L}^{n-|\underline{k}|},$$

where $|\underline{k}| = \sum k_i$, $S^{\underline{k}}M = S^{k_1}M \times \cdots \times S^{k_n}M$. Therefore

$$1 + \sum_{n\geq 1} [M^{[n]}]t^n = \sum_{n\geq 0} \left( \sum_{\underline{k}:\Sigma i k_i = n} [S^{\underline{k}}M] \cdot \mathbb{L}^{n-|\underline{k}|} \right) \cdot t^n$$

$$= \sum_{n\geq 0} \left( \sum_{\underline{k}:\Sigma i k_i = n} [S^{k_1}M] \cdot \ldots \cdot [S^{k_n}M] \cdot \mathbb{L}^{n-(k_1+\ldots+k_n)} \right) \cdot t^{\Sigma i k_i}$$

$$= \sum_{n\geq 0} \left( \sum_{\underline{k}:\Sigma i k_i = n} ([S^{k_1}M] \cdot \mathbb{L}^{-k_1}) \cdot \ldots \cdot ([S^{k_n}M] \cdot \mathbb{L}^{-k_n}) \right) \cdot (\mathbb{L}t)^{\Sigma i k_i}$$

$$= \prod_{k\geq 1} \left( \sum_{r\geq 0} [S^r M] \cdot \mathbb{L}^{-r} (\mathbb{L}t)^{kr} \right) = \prod_{k\geq 1} \zeta_{\mathbb{L}^{-1}[M]} \left( (\mathbb{L}t)^k \right). \quad \square$$

**Remarks**. **1.** D. van Straten told us that the fact that the generating function of the Hilbert scheme of 0-dimensional subschemes of a surface is in some sense an exponent was conjectured by several people.

**2.** All the results of this paper can be extended to the Grothendieck ring of algebraic varieties over an algebraically closed field of characteristic zero. In this case Statement 2 follows from a version of Lemma 4.4 in [7]. In Statement 1, the Euler characteristic should be replaced by the $\ell$-adic Euler characteristic with compact support.

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Moscow State University, Faculty of Mathematics and Mechanics, Moscow, 119902, Russia

*E-mail address*: `sabir@mccme.ru`

University Complutense de Madrid, Dept. of Algebra, Madrid, 28040, Spain.

*E-mail address*: `iluengo@mat.ucm.es, amelle@mat.ucm.es`