Bifurcations and topology of meromorphic germs

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Introduction: polynomial and meromorphic functions

Maps defined by polynomial functions are traditional objects of interest in algebraic geometry and singularity theory. A polynomial P in n complex variables defines a map $P : \mathbb{C}^n \to \mathbb{C}$. The map P is not a locally trivial fibration over critical values of P. However, since the source $\mathbb{C}^n$ is not compact, the map P fails to be a locally trivial fibration over some other values as well. It is well known that a polynomial map defines a locally trivial fibration over the complement to a finite set in $\mathbb{C}$ (the *bifurcation set of* P): [Th],[V1],[Ve].

To describe the atypical values which detect an irregular behaviour of a polynomial at infinity is an important and unsolved problem. There are several known regularity conditions which guarantee that a value is not atypical at infinity or that there are no atypical values at infinity (see, e.g., [Br], [HL], [NZ],[P1],[ST1], [T1], [T2], [T3], [Sa]).

A number of papers are devoted to the study of the topology of generic fibres (e.g. theorems of bouquet type) and their difference from non-generic ones (see, e.g., [D2], [ST1], [ACD], [CD], [ALM1], [ALM2]).

Important invariants of a polynomial map are monodromy operators corresponding to small loops around atypical values and (usually the most important) the monodromy operator at infinity corresponding to a big loop around all atypical values. They are related to a number of properties of the polynomial, including arithmetic ones (see, e.g., [N], [LS], [GN], [G], [ST2], [DN], [D3]).

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A natural way to understand the behaviour of a polynomial at infinity is to consider it as a rational function on the projective space $\mathbb{CP}^n$. In this case one has to study its behaviour at points of the infinite hyperplane $\mathbb{CP}^{n-1}_\infty$. The problem is that at such points a polynomial map defines not a holomorphic germ, but a meromorphic one (of a particular type). Thus in order to use local considerations for a description of global properties of a polynomial map one needs to describe related properties of meromorphic germs.

It is thus natural to reconsider the entire local theory in the more general context of arbitrary meromorphic germs P/Q. A first step in this direction was taken by V.I.Arnold [A], who gave classifications of simple meromorphic germs with respect to certain equivalence relations. One may now seek to establish and study analogues of notions already well-understood for analytic germs and by now also developed in some detail for the behaviour of polynomial maps at infinity, such as Milnor fibres and monodromy, for meromorphic germs in general. We will thus not attempt to survey the by now considerable literature on polynomial maps, and we refer to the paper [D3] by Dimca in this volume for an account of the Hodge-theoretic aspects of this study.

Despite a number of close parallels with the earlier results, certain new features require attention. Their study was begun in [GLM2] and further developed in [GLM4] and [ST3]. Although the problem is very closely related to the study of pencils $sP + tQ = 0$, for which also a number of interesting results are known — for example, the characterisation by Lê and Weber [LW] of atypical fibres — this aspect also is slightly different.

In the first chapter of this article we present the basic definitions, and then study the monodromy by calculating its zeta-function. For results on homology splitting, and for bouquet type theorems, we refer to [ST2]. In the second chapter we give some corresponding results for the global case of meromorphic functions on compact complex manifolds. The traditional case is that of rational functions on $\mathbb{CP}^n$ (see e.g. [GLM3], [GLM5], [ST3]), including in particular polynomial functions on $\mathbb{C}^n$, and we give applications to this in the final section.

1. Local theory

A *meromorphic germ* at the origin in the complex space $\mathbb{C}^n$ is a ratio $f = \frac{P}{Q}$ of two holomorphic germs P and Q on $(\mathbb{C}^n, 0)$.

The following equivalence relation (first introduced by V.I. Arnold) is essential for further considerations. Two meromorphic germs $f = \frac{P}{Q}$ and $f' = \frac{P'}{Q'}$ are *equal* if and only if $P' = P \cdot U$ and $Q' = Q \cdot U$

for a holomorphic germ U not equal to zero at the origin: $U(0) \neq 0$. According to this definition $\dfrac{x}{y} \neq \dfrac{x^2}{xy}$, but $\dfrac{x}{y} = \dfrac{x\exp(x)}{y\exp(x)}$.

A meromorphic germ $f = \frac{P}{Q}$ defines a map from the complement of the indeterminacy locus $I(f) := \{P = Q = 0\}$ to the complex projective line $\mathbb{CP}^1$. Unfortunately, this map is not a locally trivial fibration over the complement of a finite set in $\mathbb{CP}^1$. Roughly speaking, f fails to be a locally trivial fibration over values c for which the level set $f^{-1}(c)$ is not transversal to the sphere $S_\varepsilon^{2n-1} = \partial B_\varepsilon^{2n}$ (among others). This is a "real condition" and thus it can occur at points of the projective line $\mathbb{CP}^1$ forming a set of real codimension 1. Thus one cannot define a generic fibre of a meromorphic germ in this way.

EXAMPLE 1.1. Let $f = \frac{x^2-y^3}{y^2}$. One can see that $f : B_\varepsilon^4 \setminus \{0\} \to \mathbb{CP}^1$ fails to be a locally trivial fibration over neighbourhoods of 0, ∞, and points c $(=(c:1))$ such that $\|c\| = \frac{3}{2}\varepsilon$.

However if one fixes a value c in $\mathbb{CP}^1$, this does not happen in a neighbourhood of c provided the radius ε is small enough.

THEOREM 1.2. ([GLM2]) *For any value $c \in \mathbb{CP}^1$, there exists $\varepsilon_0 > 0$ $(\varepsilon_0 = \varepsilon_0(c))$ such that for any positive $\varepsilon \leq \varepsilon_0$ the sphere S_ε^{2n-1} is transversal to all strata of the level set $f^{-1}(c)$, and the map $f : B_\varepsilon^{2n} \setminus I(f) \to \mathbb{CP}^1$ is a locally trivial fibration over a punctured neighbourhood of c.*

DEFINITION 1.3. The fibration described is called *the c-Milnor fibration* of the meromorphic germ f. A fibre of the c-Milnor fibration, i.e.,

$$\mathcal{M}_f^c = \{z \in B_\varepsilon^{2n} \setminus I(f) \,:\, f(z) = \frac{P(z)}{Q(z)} = c'\}$$

for ε small enough and for c' close enough to c (in $\mathbb{CP}^1$), is a (non-compact) $(n-1)$-dimensional complex manifold with boundary, and will be called *the c-Milnor fibre* of the meromorphic germ f.

EXAMPLE 1.4. For the f of Example 1.1, $\mathcal{M}_f^0$ is a (2-dimensional) disk minus two points; for $c \neq 0$, $\mathcal{M}_f^c$ is the disjoint union of two punctured disks.

Remark. 1) In particular, the definition means that $\mathcal{M}_f^0$ is equal to

$$\{z \in B_\varepsilon \,:\, P(z) = c' \cdot Q(z), \quad P(z) \neq 0\}$$

(c' close to 0, $c' \neq 0$) and thus, if $R(0) = 0$, the Milnor fibres of the functions $\frac{P}{Q}$ and $\frac{RP}{RQ}$ are, generally speaking, different.

2) For $f = \frac{P}{Q}$, let $f^{-1} = \frac{Q}{P}$. It is not difficult to see that $\mathcal{M}^c_{f^{-1}} = \mathcal{M}^{c^{-1}}_f$, in particular $\mathcal{M}^0_{f^{-1}} = \mathcal{M}^\infty_f$, $\mathcal{M}^\infty_{f^{-1}} = \mathcal{M}^0_f$. Let $f - c = \frac{P-cQ}{Q}$. Then $\mathcal{M}^c_f = \mathcal{M}^0_{f-c}$. The same properties hold for the monodromy transformations and for the zeta-functions discussed below.

A fibration over a punctured neighbourhood of a point in the projective line $\mathbb{CP}^1$ defines a monodromy transformation, which is a diffeomorphism of the fibre (well defined up to isotopy).

DEFINITION 1.5. The monodromy transformation $h^c_f : \mathcal{M}^c_f \to \mathcal{M}^c_f$ of the c-Milnor fibration is called *the c-monodromy transformation* of the meromorphic germ f.

EXAMPLE 1.6. For the f of Example 1.1, h^c_f is trivial (i.e. isotopic to the identity) for all $c \neq 0, \infty$. The 0-monodromy transformation is a self-map of a disk without two points which interchanges these points. The ∞-monodromy transformation interchanges two punctured disks.

One can show that for almost all values c (i.e. for all but a finite number) the c-monodromy transformation h^c_f is trivial, i.e., isotopic to the identity.

DEFINITION 1.7. A value $c \in \mathbb{CP}^1$ is called a *typical* value of the meromorphic germ f if for ε small enough the map $f : B^{2n}_\varepsilon \setminus I(f) \to \mathbb{CP}^1$ is a locally trivial (and thus trivial) fibration over a neighbourhood (not punctured) of the point c. Otherwise the value c is called *atypical*. The set $B(f)$ of atypical values is called the *bifurcation set* of the germ f.

Note that if a value c is typical, then the corresponding monodromy transformation h^c_f is isotopic to identity. Moreover we have the following.

THEOREM 1.8. ([GLM4]) *There exists a finite set $\Sigma \subset \mathbb{CP}^1$ such that for all $c \in \mathbb{CP}^1 \setminus \Sigma$ the c-Milnor fibres of f are diffeomorphic to each other and the c-monodromy transformations are trivial (i.e., isotopic to identity). In particular, the set of atypical values is finite.*

EXAMPLE 1.9. The meromorphic germ f of Example 1.1 1.1 has two atypical values: 0 and ∞.

To prove Theorems 1.2 and 1.8 we use resolution of singularities. A *resolution of the germ f* is a modification of the space $(\mathbb{C}^n, 0)$ (i.e., a proper analytic map $\pi : \mathcal{X} \to \mathcal{U}$ of a smooth analytic manifold $\mathcal{X}$ onto a neighbourhood $\mathcal{U}$ of the origin in $\mathbb{C}^n$, which is an isomorphism outside of

a proper analytic subspace in $\mathcal{U}$) such that the total transform $\pi^{-1}(H)$ of the hypersurface $H = \{P = 0\} \cup \{Q = 0\}$ is a normal crossing divisor at each point of the manifold $\mathcal{X}$. We assume that the map π is an isomorphism outside of the hypersurface H.

The fact that the preimage $\pi^{-1}(H)$ is a divisor with normal crossings implies that in a neighbourhood of any of its points there exists a local system of coordinates $y_1, \dots, y_n$ such that the liftings $\widetilde{P} = P \circ \pi$ and $\widetilde{Q} = Q \circ \pi$ of the functions P and Q to the total space $\mathcal{X}$ of the modification have the forms $u \cdot y_1^{k_1} \cdots y_n^{k_n}$ and $v \cdot y_1^{\ell_1} \cdots y_n^{\ell_n}$ respectively, where $u(0) \neq 0$, $v(0) \neq 0$, and the k_i and ℓ_i are nonnegative integers.

Note that the values 0 and ∞ in the projective line $\mathbb{CP}^1$ are used as distinguished points for convenience, so as to use the usual notion of a resolution of a function.

Proof (of Theorem 1.8). One can make additional blow-ups along intersections of pairs of irreducible components of the divisor $\pi^{-1}(H)$ so that the lifting $\widetilde{f} = f \circ \pi = \frac{\widetilde{P}}{\widetilde{Q}}$ of the function f defines a holomorphic map from the manifold $\mathcal{X}$ to the complex projective line $\mathbb{CP}^1$. This means that $\widetilde{P} = V \cdot P'$, $\widetilde{Q} = V \cdot Q'$ where V is a section of a line bundle, say $\mathcal{L}$, over $\mathcal{X}$, P' and Q' are sections of the line bundle $\mathcal{L}^{-1}$, and P' and Q' have no common zeroes on $\mathcal{X}$. Let $\widetilde{f}' = \frac{P'}{Q'}$.

On each component of the divisor $\pi^{-1}(H)$ and on each intersection of several of the components, $\widetilde{f}'$ defines a map to the projective line $\mathbb{CP}^1$. These maps have a finite number of critical values, say a_1, a_2, ..., a_s.

If the function $\widetilde{f}'$ is constant on a component of the divisor $\pi^{-1}(H)$, or on an intersection of components, then this constant value is critical. The value of the function $\widetilde{f}'$ on an intersection of n components (this intersection is zero-dimensional) should also be considered as a critical value.

Let $c \in \mathbb{CP}^1$ be different from a_1, a_2, ..., a_s. We shall show that for all c' in a neighbourhood of the point c (including c itself) the c'-Milnor fibres of the meromorphic function f are diffeomorphic to each other and the c'-monodromy transformations are trivial.

Let $r^2(z)$ be the square of the distance from the origin in the space $\mathbb{C}^n$ and let $\widetilde{r}^2(x) = r^2(\pi(x))$ be the lifting of this function to the total space $\mathcal{X}$ of the modification. In order to obtain a c'-Milnor fibre one has to choose $\varepsilon_0 > 0$ (a Milnor radius) small enough so that the level manifold $\{\widetilde{r}^2(x) = \varepsilon^2\}$ is transversal to $\{\widetilde{f}'(x) = c'\}$ for all ε with $0 < \varepsilon \leq \varepsilon_0$. Let $\varepsilon_0 = \varepsilon_0(c)$ be a Milnor radius for the value c. Since $\{\widetilde{f}'(x) = c\}$ is transversal to all components of the divisor $\pi^{-1}(H)$ and to all their intersections, ε_0 is also a Milnor radius for all c' in a neighbourhood of the point $c \in \mathbb{CP}^1$ (and the level manifold $\{\widetilde{f}'(x) =$

$c'\}$ is transversal to all components of the divisor $\pi^{-1}(H)$ and to their intersections). This implies that for all such c' the c'-Milnor fibres of the meromorphic germ f are diffeomorphic to each other and the c'-monodromy transformations are trivial.

The c-Milnor fibre for a generic value $c \in \mathbb{CP}^1$ can be called *the generic Milnor fibre* of the meromorphic germ f. One can see that the generic Milnor fibre of a meromorphic germ can be considered as embedded in the c-Milnor fibre for any value $c \in \mathbb{CP}^1$.

THEOREM 1.10. *Let $f = \frac{P}{Q}$ be a meromorphic germ on the space $(\mathbb{C}^n, 0)$ such that the numerator P has an isolated critical point at the origin and, if $n = 2$, assume also that the germs of the curves $\{P = 0\}$ and $\{Q = 0\}$ have no common irreducible components. Then, for a generic $t \in \mathbb{C}$,*

$$\chi(\mathcal{M}_f^0) = (-1)^{(n-1)}(\mu(P, 0) - \mu(P + tQ, 0)).$$

Here $\mu(g, 0)$ stands for the usual Milnor number of the holomorphic germ g at the origin.

Remark. Similar results for polynomials (i.e., for meromorphic germs of the form P/z^d) can be found in [P1] and [ST1].

Proof. The Milnor fibre $\mathcal{M}_f^0$ of the meromorphic germ f has the following description. Let ε be small enough (and thus be a Milnor radius for the holomorphic germ P). Then

$$\mathcal{M}_f^0 = B_\varepsilon(0) \cap (\{P + tQ = 0\} \setminus I(f))$$

for $t \neq 0$ with $|t|$ small enough (and thus t generic). Note that the zero-level set $\{P + tQ = 0\}$ is non-singular outside the origin for $|t|$ small enough. The space $B_\varepsilon(0) \cap \{P = Q = 0\}$ is homeomorphic to a cone and therefore its Euler characteristic is equal to 1. Therefore

$$\chi(\mathcal{M}_f^0) = \chi(B_\varepsilon(0) \cap \{P + tQ = 0\}) - 1.$$

Now Theorem 1.17 is a consequence of the following well known statement (see, e.g., [GZ]).

STATEMENT 1.11. *Let $P : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function with an isolated critical point at the origin and let P_t be a deformation of P (i.e., $P_0 = P$). Let ε be small enough. Then for $|t|$ small enough,*

$$(-1)^{n-1}(\chi(B_\varepsilon(0) \cap \{P_t = 0\}) - 1)$$

is equal to the number of critical points of P (counted with multiplicities) which split from the zero level set, i.e., to

$$\mu(P,0) - \sum_{Q\in\{P_t=0\}\cap B_\varepsilon} \mu(P_t,Q).$$

EXAMPLE 1.12. The example $f = \frac{xy}{x}$ shows the necessity of the condition that, for $n = 2$, the curves $\{P = 0\}$ and $\{Q = 0\}$ have no common components.

EXAMPLE 1.13. In Theorem1.171.17 , for the difference of Milnor numbers we can substitute the difference of Euler characteristics of the corresponding Milnor fibres (of the germs P and $P + tQ$) (up to sign). However the formula obtained this way is not correct if the germ P has a nonisolated critical point at the origin. This is shown by the example $f = \frac{x^2+z^2y}{z^2}$.

THEOREM 1.14. *Let $f = \frac{P}{Q}$ be the germ of a meromorphic function on $(\mathbb{C}^n, 0)$ such that the numerator P has an isolated critical point at the origin. The value 0 is typical for the meromorphic germ f if and only if $\chi(\mathcal{M}_f^0) = 0$.*

Proof. " Only if" follows from the definition and 1.171.17.

" If" is a consequence of a result of A. Parusiński [P2] (or rather of its proof). He proved that, if $\mu(P) = \mu(P + tQ)$ for $|t|$ small enough, then the family of maps $P_t = P+tQ$ is topologically trivial. In particular the family of germs of hypersurfaces $\{P_t = 0\}$ is topologically trivial. For $n \neq 3$ this was proved by Lê D.T. and C.P. Ramanujam [LR]. However in order to apply the result to the present situation, it is necessary to have a topological trivialization of the family $\{P_t = 0\}$ which preserves the subset $\{P = Q = 0\}$ and is smooth outside the origin. For the family $P_t = P + tQ$, such a trivialization was explicitly constructed in [P2] without any restriction on the dimension.

EXAMPLE 1.15. If the germ of the function P has a non-isolated critical point at the origin, then this characterization is no longer true. Take, for example, $P(x, y) = x^2y^2$ and $Q(x, y) = x^4 + y^4$.

1.1. The monodromy and its zeta-function

DEFINITION 1.16. *For a transformation $h : X \to X$ of a topological space X its zeta-function $\zeta_h(t)$ is the rational function defined by*

$$\zeta_h(t) = \prod_{q\geq 0} \{\det[\, id - t\, h_*|_{H_q(X;\mathbb{C})}]\}^{(-1)^q}.$$

This definition coincides with that in [AGV] and differs by a sign in the exponent from that in [A'C].

Let $\zeta_f^c(t)$ be the zeta-function of the c-monodromy transformation h_f^c of the meromorphic germ f. The degree of the rational function $\zeta_f^c(t)$ (i.e., the degree of the numerator minus the degree of the denominator) is equal to the Euler characteristic of the c-Milnor fibre $\mathcal{M}_f^c$. In the general case one has the following statement.

THEOREM 1.17. ([GLM4]) *If the value c is typical then the Euler characteristic of the c-Milnor fibre is equal to* 0 *and its zeta-function $\zeta_f^c(t)$ is equal to* 1.

EXAMPLE 1.18. For the f of Example 1.1, $\zeta_f^0(t) = \frac{1}{1+t}$ and $\zeta_f^c(t) = 1$ for $c \neq 0$.

In the holomorphic case, resolution is an important tool for understanding the Milnor fibration. An excellent example of this fact is the formula of A'Campo, [A'C]. We also have an A'Campo formula in the meromorphic case.

Let $f = \frac{P}{Q}$ be a germ of a meromorphic function on $(\mathbb{C}^n, 0)$ and let $\pi : \mathcal{X} \to \mathcal{U}$ be a resolution of the germ f. The preimage $\mathcal{D} = \pi^{-1}(0)$ of the origin of $\mathbb{C}^n$ is a normal crossing divisor. Let $S_{k,\ell}$ be the set of points of $\mathcal{D}$ in a neighbourhood of which the functions $P \circ \pi$ and $Q \circ \pi$ in some local coordinates have the form $u \cdot y_1^k$ and $v \cdot y_1^\ell$ respectively ($u(0) \neq 0$, $v(0) \neq 0$). A slight modification of the arguments of A'Campo ([A'C]) permits us to obtain the following version of his formula for the zeta-function of the monodromy of a meromorphic function.

THEOREM 1.19. ([GLM2]) *Let the resolution $\pi : \mathcal{X} \to \mathcal{U}$ be an isomorphism outside the hypersurface $H = \{P = 0\} \cup \{Q = 0\}$. Then*

$$\begin{aligned} \zeta_f^0(t) &= \prod_{k>\ell} (1 - t^{k-\ell})^{\chi(S_{k,\ell})}, \\ \zeta_f^\infty(t) &= \prod_{k<\ell} (1 - t^{\ell-k})^{\chi(S_{k,\ell})}. \end{aligned}$$

Remark. A resolution π of the germ $f' = \frac{RP}{RQ}$ is at the same time a resolution of the germ $f = \frac{P}{Q}$. Moreover the multiplicities of any component C of the exceptional divisor in the zero divisors of the liftings $(RP) \circ \pi$ and $(RQ) \circ \pi$ of the germs RP and RQ are obtained from those of the germs P and Q by adding one and the same integer, the multiplicity $m = m(C)$ of R. Nevertheless the meromorphic functions f and f' can have different zeta-functions. The reason why the formulae in the previous theorem give different results for f and f' consists in

the fact that if an open part of the component C lies in $S_{k,\ell}(f)$ then, generally speaking, its part which lies in $S_{k+m,\ell+m}(f')$ is smaller.

The A'Campo theorem for germs of holomorphic functions has been generalized to the case when the modification $\pi : (\mathcal{X}, \mathcal{D}) \to (\mathbb{C}^n, 0)$ is not a resolution of the singularity, see [GLM1]. This can also be done in the present set-up.

In order to have somewhat more attractive and unified formulae here and below it will be convenient to use the notion of the integral with respect to the Euler characteristic ([Vi]). The main property of a traditional (say, Lebesgue) measure, which, together with positivity, permits one to define a notion of integral, is the property $\sigma(X \cup Y) = \sigma(X) + \sigma(Y) - \sigma(X \cap Y)$. For spaces that can be represented as finite unions of cells, say semialgebraic spaces, the Euler characteristic defined as the alternating sum of numbers of cells of different dimensions also possesses this property. In this sense it can be considered as a measure, though nonpositive. Nonpositivity of the Euler characteristic imposes restrictions on the class of functions for which the integral with respect to the Euler characteristic can be defined.

Let A be an Abelian group with group operation $*$, and let X be a semianalytic subset of a complex manifold. Let $\Psi : X \to A$ be a function on X with values in A for which there exists a finite partition $\mathcal{S}$ of X into semianalytic sets (strata) Ξ such that Ψ is constant on each stratum Ξ (and equal to ψ_Ξ). Then by definition the integral with respect to the Euler characteristic of Ψ over X is equal to

$$\int_X \Psi(x)\, d\chi = \sum_{\Xi \in \mathcal{S}} \chi(\Xi)\psi_\Xi,$$

where $\chi(\Xi)$ is the Euler characteristic of the stratum Ξ. In the above formula we use additive notation for the operation $*$. In what follows, this definition will be used for integer valued functions and also for local zeta-functions $\zeta_x(t)$, which are elements of the Abelian group of non-zero rational functions of the variable t with respect to multiplication. In the latter case, in multiplicative notation, the above formula becomes

$$\int_X \zeta_x(t)\, d\chi = \prod_{\Xi \in \mathcal{S}} (\zeta_\Xi(t))^{\chi(\Xi)}.$$

Let $f = \frac{P}{Q}$ be the germ of a meromorphic function on $(\mathbb{C}^n, 0)$, and let $\pi : (\mathcal{X}, \mathcal{D}) \to (\mathbb{C}^n, 0)$ be an arbitrary modification of $(\mathbb{C}^n, 0)$, which is an isomorphism outside the hypersurface $H = \{P = 0\} \cup \{Q = 0\}$ (i.e. π is not necessarily a resolution). Let $\widetilde{f} = f \circ \pi$ be the lifting of f to the space of the modification, i.e., the meromorphic function $\frac{P \circ \pi}{Q \circ \pi}$. For a

point $x \in \pi^{-1}(H)$, let $\zeta^0_{\widetilde{f},x}(t)$ be the zeta-function of the 0-monodromy of the germ of the function $\widetilde{f}$ at x. Let $\mathcal{S} = \{\Xi\}$ be a pre-stratification of $\mathcal{D} = \pi^{-1}(0)$ (that is, a partitioning into semianalytic subspaces without any regularity conditions) such that, for each stratum Ξ of $\mathcal{S}$, the zeta-functions $\zeta^0_{\widetilde{f},x}(t)$ do not depend on x, for $x \in \Xi$. We denote these zeta-functions by ζ^0_Ξ.

THEOREM 1.20.

$$\zeta^0_f(t) = \int_{\mathcal{D}} \zeta^0_{\widetilde{f},x}(t)\, d\chi, \qquad (1).$$

In the holomorphic case, the Newton diagram of a function gives a lot of information about the singularity (for a singularity non-degenerate with respect to its Newton diagram). It determines an embedded resolution of the singularity and one can read the zeta-function of the singularity from this resolution (by a formula of Varchenko [V2]). There is a version of this formula also for meromorphic germs.

For a germ $R = \sum a_k x^k : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ of a holomorphic function ($k = (k_1, \dots, k_n)$, $x^k = x_1^{k_1} \cdot \ldots \cdot x_n^{k_n}$), its Newton diagram $\Gamma = \Gamma(R)$ is the union of the compact faces of the polytope $\Gamma_+ = \Gamma_+(R)$ which is the convex hull of $\bigcup_{k:\, a_k \neq 0}(k + \mathbb{R}^n_+) \subset \mathbb{R}^n_+$.

Let $f = \frac{P}{Q}$ be a germ of a meromorphic function on $(\mathbb{C}^n, 0)$ and let $\Gamma_1 = \Gamma(P)$ and $\Gamma_2 = \Gamma(Q)$ be the Newton diagrams of P and Q. We call the pair $\Lambda = (\Gamma_1, \Gamma_2)$ of Newton diagrams Γ_1 and Γ_2 *the Newton pair of* f. We say that the meromorphic germ f is *non-degenerate with respect to its Newton pair* $\Lambda = (\Gamma_1, \Gamma_2)$ if the pair of germs (P, Q) is non-degenerate with respect to the pair $\Lambda = (\Gamma_1, \Gamma_2)$ in the sense of [O] (which is an adaptation for *germs* of complete intersections of the definition of A.G. Khovanskii [Kh]). Almost all meromorphic germs with Newton pair Λ are non-degenerate with respect to it.

Let us define zeta-functions $\zeta^0_\Lambda(t)$ and $\zeta^\infty_\Lambda(t)$ for a Newton pair $\Lambda = (\Gamma_1, \Gamma_2)$. Let $1 \leq \ell \leq n$ and let $\mathcal{I}$ be a subset of $\{1, \dots, n\}$ with cardinality $\#\mathcal{I} = \ell$. Let $L_\mathcal{I}$ be the coordinate subspace $L_\mathcal{I} = \{k \in \mathbb{R}^n : k_i = 0 \text{ for } i \notin \mathcal{I}\}$ and $\Gamma_{i,\mathcal{I}} = \Gamma_i \cap L_\mathcal{I} \subset L_\mathcal{I}$. Let $L^*_\mathcal{I}$ be the dual of $L_\mathcal{I}$ and $L^*_{\mathcal{I}+}$ its positive orthant (the set of covectors with positive values on $L_{\mathcal{I}\geq 0} = \{k \in L_\mathcal{I} : k_i \geq 0 \text{ for } i \in \mathcal{I}\}$). For a primitive integer covector $a \in (\mathbb{R}^*)^n_+$, let $m(a, \Gamma) = \min_{x \in \Gamma}(a, x)$ and let $\Delta(a, \Gamma) = \{x \in \Gamma : (a, x) = m(a, \Gamma)\}$. We denote by $m_\mathcal{I}$ and $\Delta_\mathcal{I}$ the corresponding objects for the diagram $\Gamma_\mathcal{I}$ and a primitive integer covector $a \in L^*_{\mathcal{I}+}$. Let $E_\mathcal{I}$ be the set of primitive integer covectors $a \in L^*_{\mathcal{I}+}$ such that $\dim(\Delta(a, \Gamma_1) + \Delta(a, \Gamma_2)) = \ell - 1$ (the Minkowski sum $\Delta_1 + \Delta_2$ of two polytopes Δ_1 and Δ_2 is the polytope $\{x = x_1 + x_2 : x_1 \in \Delta_1, \quad x_2 \in \Delta_2\}$). There exist only a finite number

of such covectors. For $a \in E_{\mathcal{I}}$, let $\Delta_1 = \Delta(a, \Gamma_1)$, $\Delta_2 = \Delta(a, \Gamma_2)$ and

$$V_a = \sum_{s=0}^{\ell-1} V_{\ell-1}(\underbrace{\Delta_1, \ldots, \Delta_1}_{s \text{ terms}}, \underbrace{\Delta_2, \ldots, \Delta_2}_{\ell-1-s \text{ terms}}),$$

where the definition of the (Minkowski) mixed volume $V(\Delta_1, \ldots, \Delta_m)$ can be found, e.g., in [B] or [O]. The $(\ell-1)$-dimensional volume in a rational $(\ell-1)$-dimensional affine subspace of $L_{\mathcal{I}}$ has to be normalized in such a way that the volume of the unit cube spanned by any integer basis of the corresponding linear subspace is equal to 1. Recall that $V_m(\underbrace{\Delta, \ldots, \Delta}_{m \text{ terms}})$ is simply the m-dimensional volume of Δ. We have to define $V_0(\text{nothing}) = 1$ (this is necessary to define V_a for $\ell = 1$). Let:

$$\begin{aligned}
\zeta^0_{\mathcal{I}}(t) &= \prod_{a \in E_{\mathcal{I}} : m(a,\Gamma_1) > m(a,\Gamma_2)} (1 - t^{m(a,\Gamma_1) - m(a,\Gamma_2)})^{(\ell-1)! V_a}, \\
\zeta^\infty_{\mathcal{I}}(t) &= \prod_{a \in E_{\mathcal{I}} : m(a,\Gamma_1) < m(a,\Gamma_2)} (1 - t^{m(a,\Gamma_2) - m(a,\Gamma_1)})^{(\ell-1)! V_a}, \\
\zeta^\bullet_\ell(t) &= \prod_{\mathcal{I} :\, \#(\mathcal{I}) = \ell} \zeta^\bullet_{\mathcal{I}}(t), \\
\zeta^\bullet_\Lambda(t) &= \prod_{\ell=1}^{n} (\zeta^\bullet_\ell(t))^{(-1)^{\ell-1}},
\end{aligned}$$

where $\bullet = 0$ or ∞.

THEOREM 1.21. ([GLM2]) *Let $f = \frac{P}{Q}$ be a germ of a meromorphic function on $(\mathbb{C}^n, 0)$ non-degenerated with respect to its Newton pair $\Lambda = (\Gamma_1, \Gamma_2)$. Then*

$$\zeta^0_f(t) = \zeta^0_\Lambda(t) \quad \textit{and} \quad \zeta^\infty_f(t) = \zeta^\infty_\Lambda(t).$$

1.2. Examples

For the germ of a meromorphic function of two variables, a resolution can be obtained by a sequence of blow-ups at points.

EXAMPLE 1.22. The minimal resolution of the germ f of Example 1.1 can be described by Fig. 1.

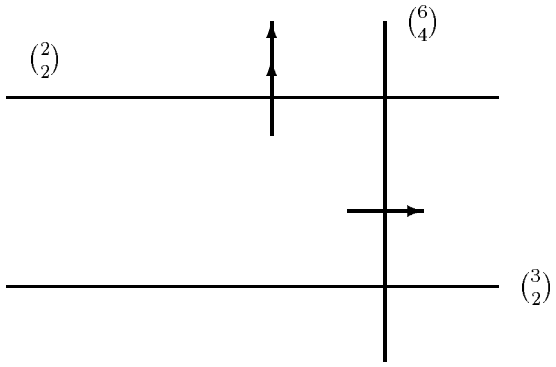

Figure 1

Here lines correspond to components of the exceptional divisor $\mathcal{D}$. Each component is isomorphic to the complex projective line $\mathbb{CP}^1$. The pairs of numbers near them are the multiplicities of the liftings of the numerator P and of the denominator Q along these components. The arrow (respectively the double arrow) corresponds to the strict transform of the curve $\{P = 0\}$ (respectively, of the curve $\{Q = 0\}$). Then $S_{2,2}$ (respectively $S_{3,2}$ and $S_{6,4}$ is the complex projective line minus two points (minus one and three points respectively). Thus

$$\begin{aligned} \zeta_f^0(t) &= (1-t)(1-t^2)^{-1} = \frac{1}{1+t}, \\ \zeta_f^\infty(t) &= 1. \end{aligned}$$

EXAMPLE 1.23. Let $f = \frac{x^3-xy}{y}$. The Milnor fibre $\mathcal{M}_f^0$ (respectively $\mathcal{M}_f^\infty$) is $\{(x,y) : \|(x,y)\| < \varepsilon,\ x^3 - xy = cy\} \setminus \{(0,0)\}$, where $\|c\|$ is small (respectively large). From the equation $x^3 - xy = cy$ one has $y = \frac{x^3}{x+c}$ and thus $\mathcal{M}_f^0$ is diffeomorphic to the disk $\mathcal{D}$ in the x-plane with two points removed: $-c$ and the origin. In the same way $\mathcal{M}_f^\infty$ is diffeomorphic to the punctured disk $\mathcal{D}^*$. It is not difficult to see that the action of the monodromy transformation on the homology groups is trivial in both cases. Thus

$$\zeta_f^0(t) = (1-t)^{-1} \quad \text{and} \quad \zeta_f^\infty(t) = 1.$$

Now let us calculate these zeta-functions from their Newton diagrams.

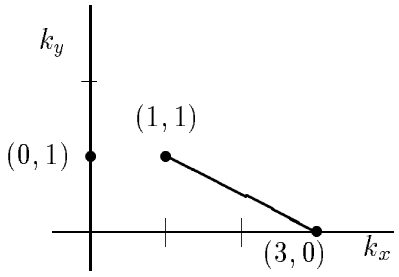

Figure 2

We have $\zeta_1^\bullet(t) = 1$ since each coordinate axis intersects only one Newton diagram. There is only one linear function (namely $a = k_x + 2\,k_y$) such that $\dim \Delta(a,\Gamma_1) = 1$. The one-dimensional volume $V_1(\Delta(a,\Gamma_1))$ of $\Delta(a,\Gamma_1)$ is equal to 1 and $V_1(\Delta(a,\Gamma_2)) = 0$. We have $m(a,\Gamma_1) = 3$ and $m(a,\Gamma_2) = 2$. Thus $\zeta_2^0(t) = (1-t)$, $\zeta_2^\infty(t) = 1$, $\zeta^0_{(\Gamma_1,\Gamma_2)}(t) = (1-t)^{-1}$ and $\zeta^\infty_{(\Gamma_1,\Gamma_2)}(t) = 1$, which coincides with the formulae for f written above.

EXAMPLE 1.24. Let $P = xyz + x^p + y^q + z^r$ be a $T_{p,q,r}$ singularity, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$, and let $Q = x^d + y^d + z^d$ be a homogeneous polynomial of degree d. Suppose that $p > q > r > d > 3$ and that p, q, and r are pairwise coprime. Let us compute the zeta-functions of $f = \frac{P}{Q}$ using the methods described above.

(a) It is clear that f is non-degenerate with respect to its Newton pair $\Lambda = (\Gamma_1, \Gamma_2)$. Thus

$$\zeta_f^\bullet(t) = \zeta_\Lambda^\bullet(t) = \zeta_1^\bullet(\zeta_2^\bullet)^{-1}\zeta_3^\bullet \quad (\bullet = 0 \text{ or } \infty).$$

One has $\zeta_1^\infty = \zeta_2^\infty = 1$ and the only covector which is necessary for computing ζ_3^∞ is $a = (1,1,1)$. In this case $m(a,\Gamma_1) = 3$, $m(a,\Gamma_2) = d$, $\Delta(a,\Gamma_1) = \{(1,1,1)\}$ and $\Delta(a,\Gamma_2)$ is the simplex $\{k_x + k_y + k_z = d,\ k_x \geq 0,\ k_y \geq 0,\ k_z \geq 0\}$, its two-dimensional volume is equal to $\frac{d^2}{2}$. Thus $\zeta_f^\infty = (1 - t^{d-3})^{d^2}$.

We have

$$\zeta_1^0 = (1 - t^{p-d})(1 - t^{q-d})(1 - t^{r-d}),$$

$$\zeta_2^0 = (1 - t^{r(q-d)})(1 - t^{r(p-d)})(1 - t^{q(p-d)})(1 - t^{r-d})^{2d}(1 - t^{q-d})^d.$$

To compute ζ_3^0 one has to take into account both the covectors $(rq - q - r, r, q)$, $(r, pr - p - r, p)$, and $(q, p, qp - p - q)$ corresponding to two-dimensional faces of Γ_1, and the covectors $(1, r-2, 1)$, $(r-2, 1, 1)$,

and $(q-2,1,1)$ corresponding to pairs of the form (one-dimensional face of Γ_1, one-dimensional face of Γ_2). E.g., for $a=(1,r-2,1)$, $\Delta(a,\Gamma_1)$ (respectively $\Delta(a,\Gamma_2)$) is the segment between $(0,0,r)$ to $(1,1,1)$ (respectively between $(d,0,0)$ and $(0,0,d)$). Note the absence of symmetry: the latter three covectors are not obtained from each other by permuting the coordinates and the numbers p, q, and r. Then

$$\zeta_3^0=(1-t^{r(q-d)})(1-t^{r(p-d)})(1-t^{q(p-d)})(1-t^{r-d})^{2d}(1-t^{q-d})^d$$

and

$$\zeta_f^0=(1-t^{p-d})(1-t^{q-d})(1-t^{r-d}).$$

(b) For computing the zeta-functions of f with the help of partial resolutions, let $\pi:(\mathcal{X},\mathcal{D})\to(\mathbb{C}^3,0)$ be the blowing-up of the origin in $\mathbb{C}^3$ and let $\widetilde{f}$ be the lifting $f\circ\pi$ of f to the space $\mathcal{X}$. The exceptional divisor $\mathcal{D}$ is the complex projective plane $\mathbb{CP}^2$. Let H_1 and H_2 be the strict transforms of the hypersurfaces $\{P=0\}$ and $\{Q=0\}$, $D_i=\mathcal{D}\cap H_i$. The curve D_1 consists of three transversal lines ℓ_1,ℓ_2,ℓ_3 and has three singular points $S_1=\ell_2\cap\ell_3=(0,0,1)$, $S_2=\ell_1\cap\ell_3=(0,1,0)$, and $S_3=\ell_1\cap\ell_3=(1,0,0)$. The curve D_2 is a smooth curve of degree d, it intersects D_1 at $3d$ distinct points $\{P_1,\dots,P_{3d}\}$.

One has the following natural stratification of the exceptional divisor $\mathcal{D}$:

(i) 0-dimensional strata Λ_i^0 $(i=1,2,3)$, each consisting of one point S_i;

(ii) 0-dimensional strata Ξ_i^0 consisting of one point P_i each $(i=1,\dots,3d)$;

(iii) 1-dimensional strata $\Xi_i^1=\ell_i\setminus\{D_2\cup\ell_j\cup\ell_k\}$ $(i=1,2,3)$ and $\Xi_4^1=D_2\setminus D_1$;

(iv) the 2-dimensional stratum $\Xi^2=\mathcal{D}\setminus(D_1\cup D_2)$.

It is not difficult to see that $\zeta^0_{\Xi_2}(t)=1$, $\zeta^\infty_{\Xi_2}(t)=1-t^{d-3}$, and for each stratum Ξ from Ξ_i^0 $(1\le i\le 3d)$, Ξ_i^1 $(1\le i\le 4)$ one has $\zeta^\bullet_\Xi(t)=1$ $(\bullet=0$ or $\infty)$.

In what follows, the exceptional divisor $\mathcal{D}$ has the local equation $u=0$. At the point S_1 the lifting $\widetilde{f}$ of the function f is of the form $\frac{u^3x_1y_1+u^r+x_1^pu^p+y_1^qu^q}{u^dx_1^d+u^dy_1^d+u^d}$. This germ has the same Newton pair as the germ $\frac{u^3x_1y_1+u^r}{u^d}$. Using the Newton diagram formula one has $\zeta^\infty_{\Lambda_1^0}=1$, $\zeta^\infty_{\Lambda_1^0}=1-t^{r-d}$. At the point S_2 the function $\widetilde{f}$ has the form $\frac{u^3x_1z_1+z_1^ru^r+x_1^pu^p+u^q}{u^dx_1^d+u^d+z_1^du^d}$, and the same Newton pair as $\frac{u^3x_1z_1+z_1^ru^r+u^q}{u^d}$. Again from the Newton

diagrams one has $\zeta^{\infty}_{\Lambda^0_2}(t) = 1$, $\zeta^{0}_{\Lambda^0_2}(t) = 1 - t^{q-d}$. In the same way, $\zeta^{\infty}_{\Lambda^0_3}(t) = 1$, $\zeta^{0}_{\Lambda^0_3}(t) = 1 - t^{p-d}$. Combining these computations, one has the same results as we obtained above without using a partial resolution.

2. Global theory

We want to consider fibrations defined by meromorphic functions on manifolds. In order to have more general statements we prefer to use a notion of a meromorphic function slightly different from the standard one. Let M be an n-dimensional compact complex analytic manifold.

DEFINITION 2.1. *A meromorphic function f on the manifold M is a ratio $\frac{P}{Q}$ of two non-zero sections of a line bundle $\mathcal{L}$ over M. Two meromorphic functions $f = \frac{P}{Q}$ and $f' = \frac{P'}{Q'}$ (where P' and Q' are sections of a line bundle $\mathcal{L}'$) are equal if $P = U \cdot P'$ and $Q = U \cdot Q'$ where U is a section of the bundle* $\hom(\mathcal{L}', \mathcal{L}) = \mathcal{L} \otimes \mathcal{L}'^{*}$ *without zeroes (in particular, this implies that the bundles $\mathcal{L}$ and $\mathcal{L}'$ are isomorphic).*

A particular important case of meromorphic functions is given by rational functions $\frac{P(x_1,\dots,x_n)}{Q(x_1,\dots,x_n)}$ on the projective space $\mathbb{CP}^n$, where P and Q are homogeneous polynomials of the same degree.

A meromorphic function $f = \frac{P}{Q}$ defines a map f from the complement $M \setminus \{P = Q = 0\}$ of the set of common zeros of P and Q to the complex line $\mathbb{CP}^1$. The indeterminacy set $I(f) := \{P = Q = 0\}$ may have components of codimension one. For $c \in \mathbb{CP}^1$, let $F_c = f^{-1}(c)$.

Standard arguments (using a resolution of singularities; see, e.g., [V1]) give the following statement.

THEOREM 2.2. *The map $f : M \setminus \{P = Q = 0\} \to \mathbb{CP}^1$ is a C^{∞} locally trivial fibration outside a finite subset of the projective line $\mathbb{CP}^1$.*

Any fibre $F_{gen} = f^{-1}(c_{gen})$ of this fibration is called *a generic fibre* of the meromorphic function f. The smallest subset $B(f) \subset \mathbb{CP}^1$ for which f is a C^{∞} locally trivial fibration over $\mathbb{CP}^1 \setminus B(f)$ is called *the bifurcation set* of the meromorphic function f. Its elements are called *atypical values* of the meromorphic function f.

A loop in the complement $\mathbb{CP}^1 \setminus B(f)$ to the bifurcation set $B(f)$ gives rise to a monodromy transformation of the fibre bundle. The monodromy transformation is defined only up to homotopy (or rather,

up to isotopy), but the monodromy operator (the action of the monodromy transformation on a homology group of the generic fibre of the meromorphic function f) is well defined. Therefore the fundamental group $\pi_1(\mathbb{CP}^1 \setminus B(f))$ of the complement to the bifurcation set acts on the homology groups $H_*(F_{gen};\mathbb{C})$ of the generic fibre of the meromorphic function f. The image of the group $\pi_1(\mathbb{CP}^1 \setminus B(f))$ in the group of automorphisms of $H_*(F_{gen};\mathbb{C})$ is called *the monodromy group* of the meromorphic function f. It is generated by local monodromy operators corresponding to simple loops around the atypical values of f (see [AGV]).

Let $\zeta_f^c(t)$ be the zeta-function of the local monodromy corresponding to the value $c \in \mathbb{CP}^1$ (i.e., defined by a simple loop around the value c).

Note that the local monodromy and the corresponding zeta-function are defined for any value $c \in \mathbb{CP}^1$, not only for atypical ones. For a generic value of the meromorphic function f, the local monodromy is the identity and its zeta-function is equal to $(1-t)^{\chi(F_{gen})}$.

Let $f = \frac{P}{Q}$ be a meromorphic function on the complex manifold M.

The following statement is a direct consequence of the definitions.

STATEMENT 2.3. Let $\pi : \widetilde{M} \to M$ be a proper analytic map of an n-dimensional compact complex manifold $\widetilde{M}$ which is an isomorphism outside the union of the indeterminacy set $I(f)$ of the meromorphic function f and a finite number of level sets $f^{-1}(c_i)$. Let $\widetilde{f} = \frac{P\circ\pi}{Q\circ\pi}$ be the lifting of the meromorphic function $f = \frac{P}{Q}$ to $\widetilde{M}$. Then the generic fibre of $\widetilde{f}$ coincides with that of f, and for each $c \in \mathbb{CP}^1$ one has

$$\zeta_{\widetilde{f}}^c(t) = \zeta_f^c(t).$$

Note that, in order to ensure that the generic fibres of $\widetilde{f}$ and f coincide, one really needs to use the equivalence relation for meromorphic functions formulated above.

Let c be a point of the projective line $\mathbb{CP}^1$. For a point $x \in M$, let $\zeta_{f,x}^c(t)$ be the corresponding zeta-function of the germ of the meromorphic function f at the point x, and let $\chi_{f,x}^c$ be its degree $\deg \zeta_{f,x}^c(t)$.

THEOREM 2.4.

$$\zeta_f^c(t) = \int_{I(f)\cup F_c} \zeta_{f,x}^c(t)\, d\chi, \tag{1}$$

$$\chi(F_{gen}) - \chi(F_c) = \int_{F_c} (\chi_{f,x}^c - 1)\, d\chi + \int_{I(f)} \chi_{f,x}^c\, d\chi. \tag{2}$$

Proof. The proof follows the lines of the proof of 1.171.2 in [GLM1]. Without any loss of generality one can suppose that $c = 0$. There exists a modification $\pi : \mathcal{X} \to \mathcal{M}$ of the manifold M which is an isomorphism outside the set

$$\{P = Q = 0\} \cup \{f = 0\} \cup \{f = \infty\} = \{P = 0\} \cup \{Q = 0\}$$

such that $\mathcal{D} = \pi^{-1}(\{P = 0\} \cup \{Q = 0\})$ is a normal crossing divisor in the manifold $\mathcal{X}$. Then at each point of the exceptional divisor $\mathcal{D}$ in a local system of coordinates one has $P \circ \pi = u \cdot y_1^{k_1} \cdot \ldots \cdot y_n^{k_n}$, $Q \circ \pi = v \cdot y_1^{\ell_1} \cdot \ldots \cdot y_n^{\ell_n}$ with $u(0) \neq 0$, $v(0) \neq 0$, $k_i \geq 0$ and $\ell_i \geq 0$. There exist Whitney stratifications $\mathcal{S}$ and $\mathcal{S}^*$ of M and $\mathcal{X}$ respectively such that:

1. the map π is a stratified morphism with respect to these stratifications;

2. the set $\{P = 0\} \cup \{Q = 0\}$ is a stratified subspace of the stratified space $(M, \mathcal{S})$;

3. for each stratum $\Xi^* \in \mathcal{S}^*$, the germs of the liftings $P \circ \pi$ and $Q \circ \pi$ of the sections P and Q at points of Ξ^* have normal forms $u \cdot y_1^{k_1} \cdot \ldots \cdot y_n^{k_n}$ and $v \cdot y_1^{\ell_1} \cdot \ldots \cdot y_n^{\ell_n}$ where $(k_1, \ldots, k_n)$ and $(\ell_1, \ldots, \ell_n)$ do not depend on the point of Ξ^*;

4. for each stratum $\Xi \in \mathcal{S}$, the zeta-function $\zeta^c_{f,x}(t)$ does not depend on the point x for $x \in \Xi$.

Actually, point (4) follows from the first three. However it is convenient to include it in the list of conditions.

One applies the following version of the formula of A'Campo ([A'C]) and also its local variant for meromorphic germs (1.171.19). Let $S_{k,\ell}$ be the set of points of the manifold $\mathcal{X}$ in a neighbourhood of which the liftings $P \circ \pi$ and $Q \circ \pi$ of P and Q in some local coordinates have the forms $u \cdot y_1^k$ and $v \cdot y_1^\ell$ respectively $(u(0) \neq 0, v(0) \neq 0)$.

STATEMENT 2.5.

$$\zeta^0_f(t) = \prod_{k > \ell \geq 0} (1 - t^{k-\ell})^{\chi(S_{k,\ell})}.$$

Property (1) of the stratifications $\mathcal{S}$ and $\mathcal{S}^*$ implies that the morphism π is locally trivial over each stratum of $\mathcal{S}$: if the stratum Ξ of $\mathcal{S}$ is the image of the stratum Ξ^* of $\mathcal{S}^*$, $\Xi = \pi(\Xi^*)$, then $\pi : \Xi^* \to \Xi$ is a smooth locally trivial fibre bundle. In particular,

$$\chi(\Xi^*) = \chi(\Xi) \cdot \chi(\pi^{-1}(x) \cap \Xi^*), \qquad (x \in \Xi).$$

Let $\Xi_{k,\ell}$ be the set of strata from $\mathcal{S}^*$ such that the germs of the liftings $P\circ\pi$ and $Q\circ\pi$ of P and Q at their points are equivalent to y_1^k and y_1^ℓ respectively; $S_{k,\ell}=\bigcup\limits_{\Xi^*\in\Xi_{k,\ell}}\Xi^*$. We have

$$\zeta^0_f(t)=\prod_{k>\ell\geq 0}(1-t^{k-\ell})^{\sum\limits_{\Xi^*\in\Xi_{k,\ell}}\chi(\Xi^*)}=\prod_{k>\ell\geq 0}\prod_{\Xi^*\in\Xi_{k,\ell}}(1-t^{k-\ell})^{\chi(\Xi^*)}=$$

$$=\prod_{k>\ell\geq 0}\prod_{\Xi\in S}\prod_{\Xi^*\in\Xi_{k,\ell}\cap\pi^{-1}(\Xi)}(1-t^{k-\ell})^{\chi(\Xi^*)}=$$

$$=\prod_{\Xi\in\mathcal{S}}\prod_{k>\ell\geq 0}\prod_{\Xi^*\in\Xi_{k,\ell}\cap\pi^{-1}(\Xi)}(1-t^{k-\ell})^{\chi(\Xi)\cdot\chi(\pi^{-1}(x)\cap\Xi^*)}=$$

$$=\prod_{\Xi\in\mathcal{S}}\left(\prod_{k>\ell\geq 0}\prod_{\Xi^*\in\Xi_{k,\ell}\cap\pi^{-1}(\Xi)}(1-t^{k-\ell})^{\chi(\pi^{-1}(x)\cap\Xi^*)}\right)^{\chi(\Xi)}=$$

$$=\prod_{\Xi\in\mathcal{S}}[\zeta^0_{f,x}(t)]^{\chi(\Xi)}=\int_{\{P=Q=0\}\cup F_0}\zeta^0_{f,x}(t)\,d\chi.$$

As usual, the formula for the Euler characteristic of the generic fibre follows from the formula for the zeta-function, since the Euler characteristic is the degree of the zeta-function.

The difference between $(\chi^c_{f,x}-1)$ and $\chi^c_{f,x}$ in the two integrals in (2) reflects the fact that the Euler characteristic of the local level set $F_c\cap B_\varepsilon(x)$ (where $B_\varepsilon(x)$ is the ball of small radius ε centred at the point x) of the germ of the function f is equal to 1 for a point x of the level set F_c and is equal to 0 for a point x of the indeterminacy set $I(f)$. In the first case this local level set is contractible, and in the second it is the difference between two contractible sets. □

Let us denote $(-1)^{n-1}$ times the first and the second integrals in (2) by $\mu_f(c)$ and $\lambda_f(c)$ respectively. Let $\mu_f=\sum\limits_{c\in\mathbb{CP}^1}\mu_f(c)$, $\lambda_f=\sum\limits_{c\in\mathbb{CP}^1}\lambda_f(c)$ (in each sum only a finite number of summands are different from zero).

THEOREM 2.6.

$$\mu_f+\lambda_f=(-1)^{n-1}\left(2\cdot\chi(F_{gen})-\chi(M)+\chi(I(f))\right).$$

Proof. One has

$$\int_{\mathbb{CP}^1}\chi(F_c)\,d\chi=\chi(M\setminus\{P=Q=0\})=\chi(M)-\chi(I(f)).$$

Therefore

$$\chi(M)-\chi(\{P=Q=0\}) = \int_{\mathbb{CP}^1} \chi(F_{gen})\, d\chi + \int_{\mathbb{CP}^1} (\chi(F_c)-\chi(F_{gen}))\, d\chi =$$

$$= 2\cdot\chi(F_{gen}) - (-1)^{n-1} \sum_{c\in\mathbb{CP}^1} (\mu_f(c)+\lambda_f(c)) = 2\cdot\chi(F_{gen}) + (-1)^n(\mu_f+\lambda_f). \qquad \square$$

Let $\widetilde{f}$ be the restriction of f to $M \setminus \{Q=0\}$,

$$\widetilde{f} : M \setminus \{Q=0\} \to \mathbb{C} = \mathbb{CP}^1 \setminus \{\infty\}.$$

Note that the fibres of f and $\widetilde{f}$ over values $c \in \mathbb{C}$ coincide.

COROLLARY 2.7.

$$\chi(F_{gen}) = \chi(M) - \chi(\{Q=0\}) + (-1)^{n-1}(\lambda_f - \lambda_f(\infty) + \mu_f - \mu_f(\infty)).$$

For the meromorphic function on the complex projective space $\mathbb{CP}^n$ defined by a polynomial P in n variables with isolated critical points, $\mu_f(c)$ is the sum of the Milnor numbers of the critical points of the polynomial P with critical value c, and $\lambda_f(c)$ is equal to the invariant $\lambda_P(c)$ studied in [ALM1]. Therefore $\mu_f(c)$ and $\lambda_f(c)$ can be considered as generalizations of those invariants (they have sense also in the case when critical points of the polynomial P are not isolated). One has $\mu_f = \mu_P + \mu_f(\infty)$, $\lambda_f = \lambda_P + \lambda_f(\infty)$, where $\mu_P = \sum_{c\in\mathbb{C}} \mu_P(c)$, $\lambda_P = \sum_{c\in\mathbb{C}} \lambda_P(c)$. Note that in this case Corollary 1 turns into the well known formula $\chi(F_{gen}) = 1 + (-1)^{n-1}(\lambda_P + \mu_P)$.

3. Applications

3.1. Polynomial functions

A polynomial $P : \mathbb{C}^n \to \mathbb{C}$ defines a meromorphic function $f = \frac{\widetilde{P}}{x_0^d}$ on the projective space $\mathbb{CP}^n$ ($d = \deg P$, $\widetilde{P}$ is the homogenization of P). For any $c \in \mathbb{CP}^1$, the local monodromy of the polynomial P and its zeta-function $\zeta_P^c(t)$ are defined (in fact they coincide with those of the meromorphic function f). The technique described can be applied to this case. For instance,

1. For $c \in \mathbb{C} \subset \mathbb{CP}^1$,

$$\zeta_P^c(t) = \left(\int_{\{\widetilde{P}=0\}\cap\mathbb{CP}^{n-1}_\infty} \zeta_{f,x}^c(t)\, d\chi \right) \cdot \left(\int_{\{P=c\}} \zeta_{P,x}^c(t)\, d\chi \right). \qquad (3)$$

2. For the infinite value,

$$\zeta_P^{\infty}(t) = \int_{\mathbb{CP}^{n-1}_{\infty}} \zeta^{\infty}_{f,x}(t)\, d\chi.$$

Note that the zeta-function of the monodromy at infinity of the polynomial P is nothing but $\zeta_P^{\infty}(t)$.

The bifurcation set consists of critical values of the polynomial P (in the affine part) and of atypical ("critical") values at infinity.

In order to study the level set $\{P = c\}$, one can consider the zero level set of the polynomial $(P - c)$. Thus let us consider the zero level set $V_0 = \{P = 0\} \subset \mathbb{C}^n$ of the polynomial P. Let us suppose that V_0 has only isolated singular points (in the affine part $\mathbb{C}^n$). For $\rho > 0$, let B_ρ be the open ball of radius ρ centred at the origin in $\mathbb{C}^n$ and $S_\rho = \partial B_\rho$ be the $(2n-1)$-dimensional sphere of radius ρ with the centre at the origin. There exists $R > 0$ such that, for all $\rho \geq R$, the sphere S_ρ is transversal to the level set $V_0 = \{P = 0\}$. The restriction $P|_{\mathbb{C}^n \setminus B_R} : \mathbb{C}^n \setminus B_R \to \mathbb{C}$ of the function P to the complement of the ball B_R defines a C^∞ locally trivial fibration over a punctured neighbourhood of the origin in $\mathbb{C}$. The loop $\varepsilon_0 \cdot \exp(2\pi i\tau)$ $(0 \leq \tau \leq 1$, $\|\varepsilon_0\|$ small enough) defines the monodromy transformation $h : V_{\varepsilon_0} \setminus B_R \to V_{\varepsilon_0} \setminus B_R$. Let us denote its zeta-function $\zeta_h(t)$ by $\widetilde{\zeta}_P^0(t)$. We use the following definition.

DEFINITION 3.1. *The value* 0 *is atypical at infinity for the polynomial* P *if the restriction* $P|_{\mathbb{C}^n \setminus B_R}$ *of the function* P *to the complement of the ball* B_R *is not a* C^∞ *locally trivial fibration over a neighbourhood of the origin in* $\mathbb{C}$.

This definition does not depend on choice of coordinates, i.e., it is invariant with respect to polynomial diffeomorphisms of the space $\mathbb{C}^n$. One can see that an atypical value at infinity is atypical, i.e. it belongs to $B(P)$. Moreover the bifurcation set $B(P)$ is the union of the set of critical values of the polynomial P (in $\mathbb{C}^n$) and of the set of values atypical at infinity in the sense described. If the singular locus of the level set $V_0 = \{P = 0\}$ is not finite, the value 0 can hardly be considered as typical at infinity.

THEOREM 3.2. *The zeta-function near infinity* $\widetilde{\zeta}_P^0(t)$ *of the local monodromy (corresponding to the value* 0*) of the polynomial* P *is the first factor in formula* (3)*, with* $c = 0$*. If this zeta-function is different from* 1*, then the value* 0 *is atypical at infinity.*

EXAMPLE 3.3. Let $P(x, y, z) = x^a y^b (x^c y^d - z^{c+d}) + z$, $(ad - bc) \neq 0$, and let $\mathbb{D} = \deg(P) = a + b + c + d$. The curve $\{P_{\mathbb{D}} = 0\} \subset \mathbb{CP}^2_{\infty}$

consists of three components: the line $C_1 = \{x = 0\}$ with multiplicity a, the line $C_2 = \{y = 0\}$ with multiplicity b, and the reduced curve $C_3 = \{x^c y^d - z^{c+d} = 0\}$. Let $Q_1 = C_2 \cap C_3 = (1 : 0 : 0)$, $Q_2 = C_1 \cap C_3 = (0 : 1 : 0)$, $Q_3 = C_1 \cap C_2 = (0 : 0 : 1)$. At each point x of the infinite hyperplane $\mathbb{CP}^2_\infty$ except Q_1 and Q_2, one has $\zeta^0_{P,x}(t) = 1$. At the point Q_1, the germ of the meromorphic function P has the form $\frac{y^b(y^d - z^{c+d}) + zu^{\mathbb{D}-1}}{u^{\mathbb{D}}}$. Its zero zeta-function can be obtained by the Varchenko type formula (1.171.21). If $(ad - bc) < 0$, then $\zeta^0_{P,Q_1}(t) = 1$. If $(ad - bc) > 0$, then

$$\zeta^0_{P,Q_1}(t) = (1 - t^{\frac{ad-bc}{h}})^h,$$

where $h = g.c.d(c, d) \cdot g.c.d.(\frac{ad-bc}{g.c.d(c,d)}, \mathbb{D} - 1)$. The situation at the point Q_2 is given by symmetry. Finally,

$$\widetilde{\zeta}^0_P(t) = (1 - t^{\frac{|ad-bc|}{h}})^h.$$

Thus the value 0 is atypical at infinity. In the same way $\widetilde{\zeta}^0_{P-c}(t) = 1$, for $c \neq 0$.

EXAMPLE 3.4. The polynomial function $P(x, , y, z) = x + x^2yz$ has 0 as an atypical value at infinity and $\widetilde{\zeta}^0_P(t) = 1$. Hence the converse of the above theorem does not hold.

3.2. Yomdin-at-infinity polynomials

For a polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$, let P_i be its homogeneous part of degree i. Let P be of the form $P = P_d + P_{d-k} + terms\ of\ lower\ degree$, $k \geq 1$. Consider the hypersurfaces in $\mathbb{CP}^{n-1}_\infty$ defined by $\{P_d = 0\}$ and $\{P_{d-k} = 0\}$. Let $Sing(P_d)$ be the singular locus of the hypersurface $\{P_d = 0\}$ (including all points where $\{P_d = 0\}$ is not reduced). One says that P is *a Yomdin-at-infinity polynomial* if $Sing(P_d) \cap \{P_{d-k} = 0\} = \emptyset$ (in particular this implies that $Sing(P_d)$ is finite).

Y. Yomdin ([Y]) considered critical points of holomorphic functions which are local versions of such polynomials. He gave a formula for their Milnor numbers. In [Si1] the zeta-function of the classical monodromy transformation of such a germ was described; see also the contribution of D. Siersma in this volume, [Si2]. The generic fibre (level set) of a Yomdin-at-infinity polynomial is homotopy equivalent to the bouquet of n-dimensional spheres ([D1]). Its Euler characteristic χ_P (or rather the (global) Milnor number) was determined in [ALM2]. For $k = 1$ the zeta-function of such a polynomial was obtained in [GN].

Let $P(z_1,\dots,z_n)=P_d+P_{d-k}+\dots$ be a Yomdin-at-infinity polynomial. Let $Sing(P_d)$ consist of s points $Q_1,\dots,Q_s$. One has the following natural stratification of the infinite hyperplane $\mathbb{CP}^{n-1}_\infty$:

1. the $(n-1)$-dimensional stratum $\Xi^{n-1}=\mathbb{CP}^{n-1}_\infty\setminus\{P_d=0\}$;
2. the $(n-2)$-dimensional stratum $\Xi^{n-2}=\{P_d=0\}\setminus\{Q_1,\dots,Q_s\}$;
3. the 0-dimensional strata Ξ^0_i $(i=1,\dots,s)$, each consisting of one point Q_i.

The Euler characteristic of the stratum Ξ^{n-1} is equal to

$$\chi(\mathbb{CP}^{n-1}_\infty)-\chi(\{P_d=0\})=n-\chi(n-1,d)+(-1)^{n-2}\sum_{i=1}^{s}\mu_i,$$

where $\chi(n-1,d)=n+\frac{(1-d)^n-1}{d}$ is the Euler characteristic of a non-singular hypersurface of degree d in the complex projective space $\mathbb{CP}^{n-1}_\infty$, μ_i is the Milnor number of the germ of the hypersurface $\{P_d=0\}\subset\mathbb{CP}^{n-1}_\infty$ at the point Q_i. At each point of the stratum Ξ^n, the germ of the meromorphic function P has (in some local coordinates $u,y_1,\dots,y_n$ where $u=0$ defines $\mathbb{CP}^{n-1}$) the form $\frac{1}{u^d}$, and its infinite zeta-function $\zeta^\infty_{\Xi^n}(t)$ is equal to $(1-t^d)$.

At each point of the stratum Ξ^{n-2}, the germ of the polynomial P has (in some local coordinates $u,y_1,\dots,y_{n-1}$) the form $\frac{y_1}{u^d}$. Its infinite zeta-function $\zeta^\infty_{\Xi^{n-2}}(t)$ is equal to 1 and thus it does not contribute a factor to the zeta-function of the polynomial P.

At a point Q_i $(i=1,\dots,s)$, the germ of the meromorphic function P has the form $\varphi(u,y_1,\dots,y_{n-1})=\dfrac{g_i(y_1,\dots,y_{n-1})+u^k}{u^d}$, where g_i is a local equation of the hypersurface $\{P_d=0\}\subset\mathbb{CP}^{n-1}_\infty$ at the point Q_i. Thus μ_i is its Milnor number.

In order to compute the infinite zeta-function $\zeta^\infty_\varphi(t)$ of the meromorphic germ φ, let us consider a resolution $\pi:(\mathcal{X},\mathcal{D})\to(\mathbb{C}^{n-1},0)$ of the singularity g_i, i.e., at each point of the exceptional divisor $\mathcal{D}$, the lifting $g_i\circ\pi$ of the function g_i to the space $\mathcal{X}$ of the modification has (in some local coordinates) the form $y_1^{m_1}\cdot\ldots\cdot y_{n-1}^{m_{(n-1)}}$ $(m_i\geq 0)$. Let us consider the modification

$$\widetilde{\pi}=id\times\pi:(\mathbb{C}_u\times\mathcal{X},0\times\mathcal{D})\to(\mathbb{C}^n,0)=(\mathbb{C}_u\times\mathbb{C}^{n-1},0)$$

of the space $(\mathbb{C}^n,0)$. Let $\widetilde{\varphi}=\varphi\circ\widetilde{\pi}$ be the lifting of the meromorphic function φ to the space $\mathbb{C}_u\times\mathcal{X}$ of the modification $\widetilde{\pi}$. At points of $\{0\}\times\mathcal{D}$ the function $\widetilde{\phi}$ has the form $\dfrac{y_1^{m_1}\cdot\ldots\cdot y_{n-1}^{m_{n-1}})+u^k}{u^d}$. Let $\mathcal{M}^\infty_{\widetilde{\varphi}}=\widetilde{\pi}^{-1}(\mathcal{M}^\infty_\varphi)$ ($\mathcal{M}^\infty_\varphi$ is the infinite Milnor fibre of the germ φ) be

the local level set of the meromorphic function $\widetilde{\varphi}$ (close to the infinite one). In the natural way one has the (infinite) monodromy $h^{\infty}_{\widetilde{\varphi}}$ acting on $\mathcal{M}^{\infty}_{\widetilde{\varphi}}$ and its zeta-function $\zeta^{\infty}_{\widetilde{\varphi}}(t)$.

The problem is that the modification $\widetilde{\pi}$ is not an isomorphism outside the union of the zero sets of the numerator and denominator of φ. Therefore $\zeta^{\infty}_{\varphi}(t)$ does not coincide with $\zeta^{\infty}_{\widetilde{\varphi}}(t)$. However one can show that

$$\zeta^{\infty}_{\widetilde{\varphi}}(t) = (1-t^{d-k})^{\chi(\mathcal{D})-1}\,\zeta^{\infty}_{\varphi}(t).$$

The infinite zeta-function of the germ φ can be computed using the Varchenko type formula (1.171.21), by taking into account the local normal form of the germ of $\widetilde{\phi}$ described above and the fact that the zeta-function $\zeta_h(t)$ of a self-transformation $h : X \to X$ of a space X determines the zeta-function $\zeta_{h^k}(t)$ of the k-th power of h. In particular, if $\zeta_h(t) = \prod\limits_{m\geq 1}(1-t^m)^{a_m}$, then

$$\zeta^k_h(t) = \prod_{m\geq 1}\left(1-t^{\frac{m}{g.c.d.(k,m)}}\right)^{g.c.d.(k,m)\cdot a_m}.$$

THEOREM 3.5. ([GLM5]) *For a Yomdin-at-infinity polynomial $P = P_d + P_{d-k} + \ldots$, its zeta-function at infinity is equal to*

$$\zeta_P(t) = (1-t^d)^{\chi(\Xi^{n-1})}(1-t^{d-k})^s\left(\prod_{i=1}^{s}\zeta^k_{g_i}(t^{d-k})\right)^{-1},$$

where $\chi(\Xi^{n-1}) = \frac{1-(1-d)^n}{d} + (-1)^{n-2}\sum_{i=1}^{s}\mu(g_i)$.

3.3. The Euler characteristic of a singular hypersurface

Let X be a compact complex manifold and let $\mathcal{L}$ be a holomorphic line bundle on X. Let s be a section of the bundle $\mathcal{L}$ not identically equal to zero, $Z := \{s = 0\}$ its zero locus (a hypersurface in the manifold X). Let s' be another section of the bundle $\mathcal{L}$, whose zero locus Z' is nonsingular and transversal to a Whitney stratification of the hypersurface Z. A. Parusiński and P. Pragacz have proved (see [PP1], Proposition 7) a statement which can be written as follows:

$$\chi(Z') - \chi(Z) = \int_{Z\setminus Z'}(\chi_x(Z)-1)\,d\chi, \tag{4}$$

where $\chi_x(Z)$ is the Euler characteristic of the Milnor fibre of the germ of the section s at the point x. We shall indicate a more general formula which includes this one as a particular case.

THEOREM 3.6. *Let s be as above and let s' be a section of the bundle $\mathcal{L}$ whose zero locus Z' is non-singular. Let f be the meromorphic function s/s' on the manifold X. Then*

$$\chi(Z') - \chi(Z) = \int_{Z \setminus Z'} (\chi_x(Z) - 1)\, d\chi + \int_{Z \cap Z'} \chi^0_{f,x} d\chi, \qquad (5)$$

where $\chi^0_{f,x}$ is the Euler characteristic of the 0-Milnor fibre of the meromorphic germ f at the point x.

Proof. Let F_t be the level set $\{f = t\}$ of the (global) meromorphic function f on the manifold X (with indeterminacy set $\{s = s' = 0\}$), i.e., $F_t = \{s - ts' = 0\} \setminus \{s = s' = 0\}$. We know that, for a generic value t, one has

$$\chi(F_{gen}) - \chi(F_0) = \int_{F_0} (\chi^0_{f,x}(Z) - 1)\, d\chi + \int_{\{s=s'=0\}} \chi^0_{f,x} d\chi,$$

where $\chi^0_{f,x}$ is the Euler characteristic of the 0-Milnor fibre of the meromorphic germ f at the point x. One has $F_0 = Z \setminus (Z \cap Z')$, $F_\infty = Z' \setminus (Z \cap Z')$, and in this case F_∞ is a generic level set of the meromorphic function f (since its closure is non-singular). Therefore $\chi(F_0) = \chi(Z) - \chi(Z \cap Z')$, $\chi(F_{gen}) = \chi(Z') - \chi(Z \cap Z')$. Finally, for $x \in F_0$, the germ of the function f at the point x is holomorphic and thus $\chi^0_{f,x} = \chi_x(Z)$.

If the hypersurface Z' is transversal to all strata of a Whitney stratification of the hypersurface Z, then, for $x \in Z \cap Z'$, the Euler characteristic $\chi^0_{f,x} = 0$ (Proposition 5.1 from [PP2]) and therefore formula (5) reduces to (4).

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