ON RATIONAL CUSPIDAL PROJECTIVE PLANE CURVES

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1. *Introduction.*

Let C be an irreducible projective plane curve in the complex projective space $\mathbb{P}^2$ with singular points $\{p_i\}_{i=1}^{\nu}$. From C one can extract the following information: its degree d, and the local embedded topological types T_i of the local singular germs $(C, p_i) \subset (\mathbb{P}^2, p_i)$. It is a very interesting, and still open problem, to characterize those collections of local embedded topological types $\{T_i\}_{i=1}^{\nu}$ (without fixing the positions of the points p_i) which can be realized by such a projective curve C of degree d. This remarkable problem is not only important for its own sake, but it is also connected with crucial properties, problems and conjectures in the theory of open surfaces.

For instance, the open surface $\mathbb{P}^2 \setminus C$ is $\mathbb{Q}$-acyclic if and only if C is a rational cuspidal curve. On the other hand, the *rigidity conjecture* proposed by Flenner and Zaidenberg in [**13**] (and supported by many examples, see [**9**], [**13**], [**14**], [**15**]) says that every $\mathbb{Q}$-acyclic affine surfaces Y with logarithmic Kodaira dimension $\bar{\kappa}(Y) = 2$ must be rigid. (E.g., if C has at least three cusps then $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 2$, cf. [**46**].) This conjecture for $Y = \mathbb{P}^2 \setminus C$ would imply the projective rigidity of the curve C in the sense that every equisingular deformation of C in $\mathbb{P}^2$ would be projectively equivalent to C.

Among other interesting related open problems we mention: every rational cuspidal curve can be transformed by a Cremona transformation into a line (this is the Coolidge-Nagata problem, see [**6**],[**26**]); or, the determination of the maximal number of cusps among all the rational cuspidal plane curves (proposed by F. Sakai in [**16**]) – this number is expected to be small (the maximal known by the authors is four). In a recent paper by K. Tono [**40**] it is proved that the maximal number is strictly less than nine. Finally, we also list the conjectured numerical inequality (2.3) [**34**] of S.Yu. Orevkov.

The first-mentioned 'characterization problem' (on the realization of prescribed topological types of singularities) has a long and rich history providing many interesting compatibility properties connecting local invariants of the germs $\{(C, p_i)\}_i$ with some global invariants of C (like its degree, or the log-Kodaira dimension of $\mathbb{P}^2 \setminus C$, etc.). Most of the compatibility properties are mere identities or inequalities (see e.g. the Matsuoka-Sakai inequality, or Orevkov's sharp inequality, both being consequences of the logarithmic version of the Bogomolov-Miyaoka-Yau inequality).

The more complex Varchenko's compatibility property is provided by the semicontinuity of the spectral numbers of isolated hypersurface singularities, and it consists of a finite set of inequalities.

Our goal is to propose a new compatibility property – valid for rational cuspidal curves C – which seems to be rather powerful (see section 3 for some comparison with some classical criterions). Its formulation is surprisingly very elementary. Consider a collection $(C, p_i)_{i=1}^{\nu}$ of locally irreducible plane curve singularities (i.e. cusps), let $\Delta_i(t)$ be the characteristic polynomials of the monodromy action associated with (C, p_i), and $\Delta(t) := \prod_i \Delta_i(t)$. Its degree is 2δ, where δ is the sum of the delta-invariants of the singular points. Then $\Delta(t)$ can be written as $1 + (t-1)\delta + (t-1)^2 Q(t)$ for some polynomial $Q(t)$. Let c_l be the coefficient of $t^{(d-3-l)d}$ in $Q(t)$ for any $l = 0, \ldots, d-3$.

CONJECTURE 1.   *Let $(C, p_i)_{i=1}^{\nu}$ be a collection of local plane curve singularities, all of them locally irreducible, such that $2\delta = (d-1)(d-2)$ for some integer d. Then if $(C, p_i)_{i=1}^{\nu}$ can be realized as the local singularities of a degree d (automatically rational and cuspidal) projective plane curve of degree d then*

$$c_l \leq (l+1)(l+2)/2 \ \text{ for all } \ l = 0, \ldots, d-3. \qquad (*_l)$$

In fact, the integers $N_l := c_l - (l+1)(l+2)/2$ are symmetric: $N_l = N_{d-3-l}$; and $N_0 = N_{d-3} = 0$ automatically. Moreover, there is a surprising phenomenon in the above conjecture:

If $\nu = 1$, then the conjecture is true if and only if in all the inequalities $(*_l)$, in fact, one has equality (cf. Theorem 3).

The conjecture can be reformulated in the language of the semigroups of the germs (C, p_i) (and the degree d) as well. E.g., if $\nu = 1$, then collection of vanishings of all the coefficients N_l can be described by a very precise and mysterious distribution of the elements of the semigroup of the unique singular point with respect to the intervals $I_l := ((l-1)d, ld]$ (cf. section 3):

If $\nu = 1$, the conjecture predicts that there are exactly $\min\{l+1, d\}$ elements of the semigroup in I_l for any $l > 0$.

The idea (and the main motivation) of the above conjecture came from the Seiberg-Witten invariant conjecture formulated for normal surface singularities by Nicolaescu and the forth author [29], respectively from the set of counterexamples for this conjecture provided by superisolated singularities [22] found by the last three authors. For the completeness of the presentation, this is described in short in section 2, but we emphasize that the present article is independent of the techniques of [29], in particular involves no Seiberg-Witten theory. (On the other hand, we are witnesses of a mysterious connection whose deeper understanding would be a wonderful mathematical goal.)

As supporting evidence, in the body of the paper we prove the following result:

THEOREM 1.   *If the logarithmic Kodaira dimension $\bar{\kappa} := \bar{\kappa}(\mathbb{P}^2 \setminus C)$ is ≤ 1, then the above conjecture is true. In fact, in all these cases $N_l = 0$ for any $l = 0, \ldots, d-3$ (regardless of ν).*

The proof of Theorem 1 consists of many steps. Its structure is the following.

(a) If $\bar{\kappa} = -\infty$ then $\nu = 1$ by [**46**]. Moreover, all these curves are classified by H. Kashiwara [**19**]. The family contains as an important subfamily the Abhyankar-Moh-Suzuki (AMS) curves.

In this case, our proof runs as follows: first we verify the vanishing of the coefficients N_l for the AMS curves (section 4) – this corresponds to the $\mathcal{F}_I$ family in Kashiwara's classification. The case of the other subfamily (the $\mathcal{F}_{II}$ curves) is treated in section 6. The proof is based on an induction where the starting point is given by some unicuspidal curves which have only one Puiseux pair. This case is completely resolved in section 5.

(b) The case $\bar{\kappa} = 0$ cannot occur by a result of Sh. Tsunoda [**41**], see also the paper by Orevkov [**34**].

(c) If $\bar{\kappa} = 1$ then by a result of Wakabayashi [**46**] one has $\nu \leq 2$. In the case $\nu = 1$, K. Tono writes the possible equations of the curves [**39**]. (Notice that Tsunoda' classification in [**42**] is incomplete.) We verify the conjecture for them in section 7. On the other hand, by another result of Tono [**38**], the case $\nu = 2$ corresponds exactly to the Lin-Zaidenberg bicuspidal rational plane curves. For them we verify the conjecture in section 8.

We wish to emphasize that in the process of the verification of the conjecture, in fact, *we list all the possible local topological types* (e.g. Eisenbud-Neumann splice diagrams) *of local plane curve singularities, together with the degrees d, which can be geometrically realized with $\bar{\kappa} \leq 1$*. But, from the point of view of the proof of Theorem 1, this information is far to be enough to verify the conjecture. Even if one knows the local topological types (e.g. the local resolution graphs, or even better, the generators of the corresponding semigroups), one needs sometimes additional rather involved arithmetical arguments to complete the proof.

It is important to notice that in the $\bar{\kappa} = 2$ case there are (infinitely many, and for arbitrarily large d) examples when *not all* the coefficients N_l vanish. The complete picture for curves with $d \leq 6$ is provided in section 2, together with some additional introductory examples with larger d. The following list provides those cases which are verified in the present article.

THEOREM 2. *If $\bar{\kappa} = 2$, then in the following cases the conjecture is true:*

(a) $d \leq 6$;
(b) *C is unicuspidal with one Puiseux pair (see section 5);*
(c) *Orevkov's curves C_{4k} and C_{4k}^* [**34**] (see section 9).*
(d) *d is even, $\nu = 2$, and the multiplicity sequence is $[d-2], [2_{d-2}]$ (see 2.4).*

See also Remark 1, valid for $\nu = 1$.

2. *The Main Conjecture. Motivation and first comments.*

For the convenience of the reader, we recall some notations and classical properties of plane curve singularities, which will be intensively used in the sequel.

2.1. *Invariants of germs of irreducible plane curve singularities.*

To encode the topology of a germ of an irreducible plane curve singularity $(C,0) \subset (\mathbb{C}^2, 0)$ several sets of invariants can be used: Puiseux pairs, Newton pairs, (minimal) embedded resolution graph, Eisenbud-Neumann splice diagram, semigroup, etc. We mainly use the Eisenbud-Neumann splice diagram (cf. [**7**], page 49). If the germ $(C,0)$ has g Newton pairs $\{(p_k, q_k)\}_{k=1}^g$ with $\gcd(p_k, q_k) = 1$ and $p_k \geq 2$ and $q_k \geq 2$ (and by convention, $q_1 > p_1$), define the integers $\{a_k\}_{k=1}^g$ by $a_1 := q_1$ and $a_{k+1} := q_{k+1} + p_{k+1} p_k a_k$ for $k \geq 1$. Then its Eisenbud-Neumann splice diagram decorated by the numerical data $\{(p_k, a_k)\}_{k=1}^g$ has the following shape:
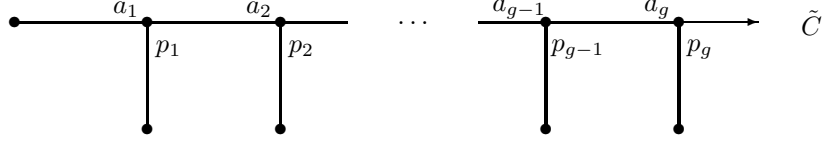
The characteristic polynomial $\Delta_{(C,0)}(t)$ of the monodromy acting on the first homology of the Milnor fiber of the singularity can be computed by A'Campo's formula [**2**] from the splice diagram. If we define

$$\beta_k := a_k p_k p_{k+1} \ldots p_g \quad \text{for } 1 \leq k \leq g;$$
$$\bar{\beta}_0 := p_1 p_2 \ldots p_g;$$
$$\bar{\beta}_k := a_k p_{k+1} \ldots p_g \qquad \text{for } 1 \leq k \leq g,$$

then $\Delta_{(C,0)}$ is given by:

$$\Delta_{(C,0)}(t) = \frac{(t-1) \prod_{1 \leq k \leq g}(t^{\beta_k} - 1)}{\prod_{0 \leq k \leq g} (t^{\bar{\beta}_k} - 1)}.$$

The polynomial $\Delta_{(C,0)}$ is a complete (embedded) topological invariant of the germ $(C,0) \subset (\mathbb{C}^2, 0)$, similarly as the semigroup $\Gamma_{(C,0)} \subset \mathbb{N}$ generated by all the possible intersection multiplicities $i(\{h = 0\}, C)$ at 0 for all $h \in \mathcal{O}_{(\mathbb{C}^2, 0)}$. The degree $2\delta_{(C,0)}$ of $\Delta_{(C,0)}$ is the conductor of the singularity, where the delta-invariant $\delta_{(C,0)}$ is the cardinality of the finite set $\mathbb{N} \setminus \Gamma_{(C,0)}$.

By [**17**], $\Delta_{(C,0)}(t) = (1 - t) \cdot L(t)$, where $L(t) := \sum_{k \in \Gamma_{(C,0)}} t^k$ is the Poincaré series of $\Gamma_{(C,0)}$. In fact, the minimal set of generators of $\Gamma_{(C,0)}$ consists of the $g+1$ elements $\bar{\beta}_i$ $(0 \leq i \leq g)$. It is also known that each element $\gamma \in \Gamma_{(C,0)}$ can be represented in a unique way in the form $\gamma = k_0 \bar{\beta}_0 + \sum_{1 \leq j \leq g} k_j \bar{\beta}_j$ with $k_0 \geq 0$ and $0 \leq k_j \leq p_j - 1$ for $1 \leq j \leq g$, see [**36**].

2.2. *Motivation. The Seiberg-Witten invariant conjecture (SWC).*

In [**29**] L. Nicolaescu and the forth author formulated the following conjecture (as a generalization of the "Casson invariant conjecture" of Neumann and Wahl [**32**]):

If the link of a normal surface singularity $(X,0)$ is a rational homology sphere then the geometric genus p_g of $(X,0)$ has an "optimal" topological upper bound. Namely,

$$p_g \leq \mathbf{sw}(M) - (K^2 + s)/8. \qquad (SWC)$$

Moreover, if $(X,0)$ is a $\mathbb{Q}$-Gorenstein (e.g. hypersurface) singularity then in (SWC) the equality holds.

Here, $\mathbf{sw}(M)$ is the (topological, or 'modified') Seiberg-Witten invariant of the link M of $(X,0)$ associated with its canonical $spin^c$ structure, K is the canonical cycle associated with a fixed resolution graph G of $(X,0)$, and s is the number of vertices of G (see [**29**] for more details).

The (SWC)-conjecture was verified successfully for many different families, see e.g. [**28**, **29**, **30**, **31**]. But the last three authors in [**22**] found some counterexamples based on superisolated singularities. This class "contains" in a canonical way the theory of complex projective plane curves, a fact which is crucial in the next discussion.

Hypersurface superisolated singularities were introduced in [**21**] by the second author and achieved the reputation of being a distinguished class of singularities and source of interesting examples and counterexamples. A hypersurface singularity $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$, $f = f_d + f_{d+1} + \ldots$ (where f_j is homogeneous of degree j) is superisolated if the projective plane curve $C := \{f_d = 0\} \subset \mathbb{P}^2$ is reduced with isolated singularities $\{p_i\}_{i=1}^{\nu}$, and these points are not situated on the projective curve $\{f_{d+1} = 0\}$. In this case the embedded topological type (and the equisingular type) of f depends only on the curve C. Notice also that the link of f is a rational homology sphere if the curve C is rational and cuspidal (i.e. if all the germs (C, p_i) are locally irreducible). *In the sequel we also will assume these two facts.*

In [**22**] the authors have shown that some hypersurface superisolated singularities with $\nu = \#Sing(C) \geq 2$ do not satisfy the above Seiberg-Witten invariant conjecture. Moreover, in all the counterexamples $p_g > \mathbf{sw}(M) - (K^2 + s)/8$ (contrary to the inequality predicted by the general conjecture 2.2 !). On the other hand, even after an intense search of the existing cases, the authors were not able to find any counterexample with $\nu = 1$. (In fact, one of the goal of the present article is to explain, at least partially, these phenomenons, cf. Theorem 3.)

In the next paragraphs, we plan to reformulate (SWC) relating with an even deeper conjectural property. We denote by Δ_i the characteristic polynomial of $(C, p_i) \subset (\mathbb{P}^2, p_i)$, set $\Delta(t) := \prod_i \Delta_i(t)$ and $2\delta := \deg \Delta(t)$. By the rationality of C one has $(d-1)(d-2) = 2\delta$. Clearly, δ is the sum of the delta-invariants of the germs $(C, p_i)_{i=1}^{\nu}$. Notice also that (see e.g. [**30**] (4.3)):

$$\Delta(1) = 1 \quad \text{and} \quad \Delta'(1) = \delta. \tag{1}$$

In the next discussion it is convenient to introduce the polynomials P and Q by

$$\Delta(t) = 1 + (t-1)P(t) = 1 + (t-1)\delta + (t-1)^2 Q(t).$$

The corresponding coefficients are denoted by

$$P(t) = \sum_{l=0}^{2\delta-1} a_k t^k, \quad \text{and} \quad Q(t) = \sum_{l \nmid d} b_l t^l + \sum_{l=0}^{d-3} c_l t^{(d-3-l)d}. \tag{2}$$

The following definition helps to describe the properties of these coefficients.

DEFINITION/LEMMA 1.   We say that a polynomial $D(t) = \sum_l \alpha_l t^l$ has the *negative distribution property* if $\sum_{0 \leq l \leq k} \alpha_l \leq 0$ for any $k \geq 0$. If $D(t) = N(t)(1-t)$ for some other polynomial $N(t)$, then $D(t)$ has the negative distribution property if and only if all the coefficients of $N(t)$ are nonpositive.

For the superisolated singularity $(X, 0) := \{f = 0\}$ those invariants which appear in the formula (SWC) (cf. 2.2) are computed in [**22**] as follows (here $\mathbf{sw}(M)$ is computed by the Reidemeister-Turaev torsion normalized by the Casson-Walker invariant) :

$$
\begin{cases}
K^2 + s = -(d-1)(d^2 - 3d + 1); \quad p_g = d(d-1)(d-2)/6; \quad \text{and} \\
\\
\mathbf{sw}(M) = \frac{1}{d} \sum_{\xi^d = 1 \neq \xi} \frac{\Delta(\xi)}{(\xi - 1)^2} + \frac{1}{2d} \Delta(t)''(1) - \frac{\delta(6\delta - 5)}{12d}.
\end{cases}
\tag{3}
$$

Consider now the (a priori) rational function

$$
R(t) := \frac{1}{d} \sum_{\xi^d = 1} \frac{\Delta(\xi t)}{(1 - \xi t)^2} - \frac{1 - t^{d^2}}{(1 - t^d)^3}.
\tag{4}
$$

PROPOSITION 1.   *With the above notations one has:*
(a) *$R(t)$ can be written in the form $D(t^d)/(1 - t^d)$, where*

$$
D(t) = (d-2)t + (d-3)t^2 + \ldots + t^{d-2} - \sum_{k=0}^{2\delta - 1} a_k t^{\lceil k/d \rceil}
$$

$$
= \sum_{k \geq 0} (1 - a_k) t^{\lceil k/d \rceil} - \frac{1 - t^d}{(1 - t)^2}.
$$

(b) *In (a), some combinations of the coefficients $\{a_k\}_k$ give no contribution in $R(t)$. Indeed, if one writes $P(t) = \delta + (t-1)Q(t)$ as above (cf. (2)), then the $\{b_l\}_l$ coefficients have no effect in R. More precisely, $R(t) = N(t^d)$, where*

$$
N(t) = \sum_{l=0}^{d-3} \left(c_l - \frac{(l+1)(l+2)}{2} \right) t^{d-3-l}.
$$

(c) *$N(t)$ is a symmetric polynomial (i.e. $N(t) = t^{d-3} \cdot N(1/t)$) with integral coefficients and with $N(0) = 0$.*
(d)

$$
R(1) = \mathbf{sw}(M) - \frac{K^2 + s}{8} - p_g.
$$

In particular, the following facts also hold:
(e) *$R(t) \equiv 0$ if and only if $D(t) \equiv 0$ if and only if $N(t) \equiv 0$. In this case evidently (SWC) is true (with equality).*
(f) *$D(t)$ has the negative distribution property if and only if all the coefficients of $R(t)$ (or, equivalently, of $N(t)$) are nonpositive. In this case $p_g \geq \mathbf{sw}(M) - (K^2 + s)/8$.*

Proof.   Write $\Delta(t)/(1 - t)^2 = 1/(1 - t)^2 - P(t)/(1 - t)$. By an elementary computation

$$
\frac{1}{d} \sum_{\xi^d = 1} \frac{1}{(1 - \xi t)^2} = \frac{1 + (d-1)t^d}{(1 - t^d)^2}.
$$

For the second contribution consider the monomials $a_k t^k$ of $P(t)$. Express any fixed k in the form $k = dn + r$ with $r = 1, 2, \ldots, d$. Then

$$\frac{1}{d} \sum_{\xi^d=1} \frac{\xi^k t^k}{1 - \xi t} = \frac{t^k}{d} \sum_{\xi^d=1} \frac{\xi^r}{1 - \xi t} = \frac{t^k}{d(1 - t^d)} \sum_{\xi^d=1} (1 + \xi t + \ldots + (\xi t)^{d-1}) \xi^r = \frac{t^{d(n+1)}}{1 - t^d}.$$

Notice also that

$$\frac{1 - t^d}{(1 - t)^2} = 1 + 2t + \ldots + (d-1)t^{d-2} + d(t^{d-1} + t^d + \ldots),$$

and

$$\frac{1 + (d-1)t}{(1 - t)} = 1 + d(t + t^2 + \ldots) = \sum_{k \geq 0} t^{\lceil k/d \rceil},$$

which ends the proof of (a). The proof of (b) is similar. (c) follows from (b) and from (4) using the fact that Δ is monic and symmetric. For (d) write

$$R(t) = \frac{1}{d} \sum_{\xi^d=1 \neq \xi} \frac{\Delta(\xi t)}{(1 - \xi t)^2} + \frac{(1 + t + \ldots + t^{d-1})^3 \Delta(t) - d(1 + t + \ldots + t^{d^2-1})}{d(1 - t)^2(1 + t + \ldots + t^{d-1})^3}.$$

Then compute $\lim_{t \to 1} R(t)$ by the L'Hospital rule, then use (1) and (3). □

The starting point of the present article is the following conjecture which is indeed a reformulation of Conjecture 1.

CONJECTURE 2. *Assume that C is a rational cuspidal projective plane curve as above. Then the property (f) of Proposition 1 is always true. Namely, $N(t)$ (or equivalently, $R(t)$) has only nonpositive coefficients. In other words, $D(t)$ has the negative distribution property.*

In fact, one can ask about the stronger property (e) in Proposition 1, i.e. about the vanishing of $N(t)$ (or equivalently, of $R(t)$). Obviously, this does not hold in general: all the examples of [**22**] with $p_g > \mathbf{sw}(M) - (K^2 + s)/8$ have $R(t) \not\equiv 0$ (cf. also with 2.3). On the other hand, using Proposition 1(d) one gets the following: *if one can verify independently the inequality $p_g \leq \mathbf{sw}(M) - (K^2 + s)/8$ then the above Conjecture 2 is true if and only if, in fact, $R(t) \equiv 0$. This is the case, e.g., if $\nu = 1$:*

THEOREM 3. *Assume that C is an unicuspidal rational curve. Then Conjecture 2 is true if and only if $R(t)$ is identical zero.*

We will give two proofs of this theorem. The first one, given below, is based on a geometrical result which reveals the connection of Conjecture 2 with links of surface singularities and topological invariants of 3-manifolds. The second (see section 3.4) is rather elementary and surprising.

Proof. Recall that a good resolution graph of a normal surface singularity is called *AR* (*almost rational*) if the graph has at least one vertex with the following property: if one replaces the decoration (self-intersection) of the corresponding vertex by a smaller (more negative) integer then one gets a rational graph (in the sense

of Artin); for more details see [**28**]. The point is that if a normal surface singularity with rational homology sphere link admits an AR good resolution graph then it satisfies the inequality $p_g \leq \mathbf{sw}(M) - (K^2 + s)/8$, cf. [**28**] (9.6). [Here we have to mention that [**28**] verifies the inequality for the Ozsváth-Szabó 'Euler-characteristic' $\mathbf{sw}^{OSZ}(M)$, but, by [**35**], this agrees with the Reidemeister-Turaev sign refined torsion normalized by the Casson-Walker invariant, used here and in [**29**].] Therefore, it is enough to verify that $\{f = 0\}$ admits an AR good resolution graph. Recall that the minimal good resolution graph G of $\{f = 0\}$ can be obtained from the minimal good embedded resolution graph Γ of the unique singular point (C, p) of C when gluing an extra vertex w by a unique edge to the unique (-1) vertex v of Γ. The decoration of w (as any decoration) is negative, say $(-r)$, its precise value is not essential here – for details see e.g. [**22**]. Now, let G_s be the graph obtained from G by changing the decoration (-1) of v by (-2). Then G_s is the resolution graph of a sandwiched singularity. Indeed, consider again Γ. Blow up a generic point on the curve corresponding to v, and then blow up $r - 1$ generic points on the new curve. In this way one gets a non-minimal resolution graph of (C, p) which contains the graph G_s as a subgraph. $\qquad\square$

See also Example 3 for some additional comments about Theorem 3.

2.3. *Examples (d small; $\nu \geq 1$).*

Below, any singular point (C, p_i) will be identified by its multiplicity sequence. Since the number of occurrences of the multiplicity 1 in the multiplicity sequence equals the last multiplicity greater than 1, we omit the multiplicity 1: we denote such a sequence by $[m_1, \ldots, m_l]$ where $m_1 \geq m_2 \geq \ldots \geq m_l > m_{l+1} = 1$ for a suitable $l \geq 1$. In fact, we will write $[\hat{m}_{1_{r_1}}, \ldots, \hat{m}_{k_{r_k}}]$ for a multiplicity sequence which means that the multiplicity $\hat{m}_i$ occurs r_i times for $i = 1, \ldots, k$. For example, $[4_2, 2_3]$ means $[4, 4, 2, 2, 2, 1, 1]$.

For the classification of the cuspidal rational curves with degree $d \leq 5$, see e.g. the book of Namba [**27**]; for the classification of multiplicity sequences of rational cuspidal plane curves of degree 6 see e.g. Fenske's paper [**8**].

If $d = 3$, then C has a unique singularity of type $[2]$. If $d = 4$, then there are four possibilities; the corresponding multiplicity sequences of the singular points are $[3]$; $[2_3]$; $[2_2], [2]$ and $[2], [2], [2]$. By a verification (or by Proposition 1(c)), in all these cases $N(t) \equiv 0$, hence the conjecture is true.

d	ν	type of cusp	$-N(t)$
5	1	$[4]$	0
5	1	$[2_6]$	0
5	2	$[3, 2]$, $[2_2]$	0
5	2	$[3]$, $[2_3]$	$2t$
5	2	$[2_2]$, $[2_4]$	$2t$
5	3	$[3]$, $[2_2]$, $[2]$	$2t$
5	3	$[2_2]$, $[2_2]$, $[2_2]$	$6t$
5	4	$[2_3]$, $[2]$, $[2]$, $[2]$	$8t$

d	ν	type of cusp	$-N(t)$
6	1	$[5]$	0
6	1	$[4, 2_4]$	0
6	1	$[3_3, 2]$	0
6	2	$[3_3]$, $[2]$	0
6	2	$[3_2, 2]$, $[3]$	0
6	2	$[3_2]$, $[3, 2]$	0
6	2	$[4, 2_3]$, $[2]$	0
6	2	$[4, 2_2]$, $[2_2]$	0
6	2	$[4]$, $[2_4]$	$t + t^2$
6	3	$[4]$, $[2_3]$, $[2]$	0
6	3	$[4]$, $[2_2]$, $[2_2]$	0

TABLE 1. *Rational cuspidal curves of degrees 5 and 6*

TABLE 2. *Rational cuspidal curves of degree 7*

reference	d	ν	type of cusp	$-N(t)$
$x^6 z + y^7$	7	1	[6]	0
$x^5 z^2 + y^7$	7	2	$[5, 2_2]$, $[2_3]$	0
$x^4 z^3 + y^7$	7	2	$[4, 3]$, $[3_2]$	0
[8]	7	2	$[5]$, $[2_5]$	$2t + 2t^2 + 2t^3$
[8]	7	2	$[4, 2_3]$, $[3_2]$	$t + 2t^2 + t^3$
[8]	7	2	$[4, 2_2]$, $[3_2, 2]$	$2t + 2t^2 + 2t^3$
[8]	7	2	$[4]$, $[3_3]$	$2t + 2t^2 + 2t^3$
[15]	7	3	$[4, 2_2]$, $[3_2]$, $[2]$	$3t + 4t^2 + 3t^3$
[14]	7	3	$[5]$, $[2_4]$, $[2]$	$3t + 3t^3$
[14]	7	3	$[5]$, $[2_3]$, $[2_2]$	$3t + 3t^3$

For $d = 5$ and $d = 6$ Table 1 shows all the possible multiplicity sequences together with the polynomials $N(t)$. For all these cases the conjecture is also true.

For $d = 7$ there is no complete classification (known by the authors). Table 2 shows some examples (the first column provides either a possible equation, or the reference where the corresponding curve has been constructed).

Here the first example is of Abhyankar-Moh-Suzuki type, while the next two of Lin-Zaidenberg type, cf. with the next sections.

2.4. *Example (d large).*

Examples with arbitrarily large d and with non-vanishing $R(t)$ (but still satisfying the conjecture) exist. E.g., if C has two cusps of types $[d-2], [2_{d-2}]$ (see e.g. [8]), and d is even, then

$$-N(t) = \sum_{k=1}^{\frac{d}{2}-2} k(t^k + t^{d-3-k}).$$

It is instructive (and sometimes rather mysterious) to verify our conjecture for the other families listed e.g. in Fenske's article [8].

Theorem 1 has the following immediate consequence:

COROLLARY 1. *Let $f = f_d + f_{d+1} + \dots : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be a hypersurface superisolated singularity with $\bar{\kappa}(\mathbb{P}^2 \setminus \{f_d = 0\}) < 2$. Then the Seiberg-Witten invariant conjecture (cf. [29], see 2.2 here) is true for $(X, 0) = (\{f = 0\}, 0)$.*

It is a big question for the authors: what is the (conceptual) connection between the log Kodaira dimension (of what?) with the general conjecture 2.2 ?

3. $\nu = 1$ *revisited. Comparison with other criterions.*

Assume now that $\nu = 1$, and write $Sing(C) = \{p\}$. Recall that the characteristic polynomial Δ of $(C, p) \subset (\mathbb{P}^2, p)$ is a complete (embedded) topological invariant of this germ, similarly as the semigroup $\Gamma_{(C,p)} \subset \mathbb{N}$, see 2.1. In the next discussion we will replace Δ by $\Gamma_{(C,p)}$.

Recall, that by [17], $\Delta(t) = (1-t) \cdot L(t)$, where $L(t) = \sum_{k \in \Gamma_{(C,p)}} t^k$ is the Poincaré

series of $\Gamma_{(C,p)}$. Since $P(t)$ was defined by the identity $\Delta(t) = 1 - P(t)(1 - t)$, one gets $L(t) + P(t) = 1/(1 - t) = \sum_{k \geq 0} t^k$. In particular, $P(t) = \sum_{k \in \mathbb{N} \backslash \Gamma_{(C,p)}} t^k$.

We let the reader to verify that Proposition 1 implies the following facts (cf. also with Theorem 3).

3.1. Fact I.

If $\nu = 1$ then

$$\mathbf{sw}(M) - \frac{K^2 + s}{8} = \sum_{k \notin \Gamma(C,p)} \lceil k/d \rceil.$$

In particular, $p_g = \mathbf{sw}(M) - (K^2 + s)/8$ if and only if $D'(1) = 0$, or equivalently, if:

$$\sum_{k \notin \Gamma_{(C,0)}} \lceil k/d \rceil = d(d - 1)(d - 2)/6.$$

3.2. Fact II.

$$Q(t) = \frac{\left(\sum_{k \notin \Gamma_{(C,p)}} t^k\right) - \delta}{t - 1}.$$

Then Conjecture 2 (in the light of Theorem 3) predicts that the coefficient c_l of the monomial $t^{d(d-3-l)}$ of $Q(t)$ is exactly $(l + 1)(l + 2)/2$ (for any $0 \leq l \leq d - 3$).

3.3. Fact III.

$$D(t) = \sum_{k \in \Gamma_{(C,p)}} t^{\lceil k/d \rceil} - \frac{1 - t^d}{(1 - t)^2}.$$

In particular, the second reformulation of the Conjecture 2 (and Theorem 3) is the identity

$$\sum_{k \in \Gamma_{(C,p)}} t^{\lceil k/d \rceil} = 1 + 2t + \ldots + (d - 1)t^{d-2} + d(t^{d-1} + t^d + t^{d+1} + \ldots). \qquad (CP)$$

Property (CP) (similarly as the property of Fact II) connects the local invariant $\Gamma_{(C,p)}$ with the degree d of C. It predicts a very precise distribution law for the elements of the semigroup $\Gamma_{(C,p)}$ with respect to the intervals $I_l := ((l - 1)d, ld]$ ($l \in \mathbb{N}$). It says that for any $l \geq 0$ one has

$$\#\Gamma_{(C,p)} \cap I_l = \min\{l + 1, d\}. \qquad (CP_l)$$

By the symmetry of the semigroup (namely, $k \in \Gamma_{(C,p)}$ if and only if $2\delta - 1 - k \notin \Gamma_{(C,p)}$), one has that (CP_l) is true if and only if (CP_{d-2-l}) is true. In fact, (CP_l) is automatically true for $l = 0$ and any $l \geq d - 2$. But for $1 \leq l \leq d - 3$ it combines a lot of restrictions.

3.4. *A property of semigroups of plane curves.*

Now we will provide an alternative and very elementary proof of Theorem 3. It is based on a rather surprising and general property about the semigroups of

a (not necessarily rational) plane curve at any collection of points with the sole assumption that the curve is locally irreducible at them.

Let C be a (not necessarily rational) irreducible plane curve of degree d. Let $p_1, ..., p_r$ be a set of points of C such that the germ (C, p_i) is irreducible for any i. Denote by Γ_i the semigroup of (C, p_i).

PROPOSITION 2. *Suppose $l < d$. Let $(n_1, ..., n_r)$ be an r-uple of positive numbers such that $n_1 + ... + n_r \geq ld$. Then*

$$\sum_{i=1}^{r} \#(\Gamma_i \cap [0, n_i]) \geq (l+1)(l+2)/2.$$

Proof. Since the ideas of the proof appear when $r = 1$, and this is the case what we will use, for simplicity of notation we give the proof only for that case. Consider $n_1 \geq ld$. Observe that $(l+1)(l+2)/2$ is the dimension of the space of homogeneous polynomials P in three variables of degree l. Let $\gamma(t)$ be a local parametrisation of C at p_1. The composition $P(\gamma(t))$ can be written as $P(\gamma(t)) = \sum_{h=1}^{\infty} L_h(P)t^h$. Therefore the conditions, imposed to P, that the local intersection multiplicity $i_{p_1}(C, \{P = 0\})$ is $> n_1$, is given by the equations $L_h(P) = 0$ for $0 \leq h \leq n_1$. On the other hand, as the semigroup Γ_1 is the collection of intersection multiplicities of C with other curves at p_1, the number of independent conditions is at most $\#(\Gamma_1 \cap [0, n_1])$, i.e. $L_h(P) = 0$ for $h \leq n_1$ and $h \in \Gamma_1$ implies the vanishing $L_h(P)$ for all $h \leq n_1$. Therefore, if $\#(\Gamma_1 \cap [0, n_1]) < (l+1)(l+2)/2$, then there exists a non-zero polynomial P of degree l with $i_{p_1}(C, \{P = 0\}) > n_1$. But, by Bezout's theorem any such polynomial must have a component in common with C, but, as C is irreducible, this is impossible. $\square$

Proof of Theorem 3. Our Conjecture 2 in the unicuspidal case implies that the polynomial

$$D(t) = \sum_{l=0}^{d-1} (\#\Gamma_1 \cap ((l-1)d, ld] - (l+1))t^l$$

has the negative distribution property. We have proved in the previous proposition that $D(t)$ has the positive distribution property. Hence conjecture implies that $D(t)$ is identically 0. $\square$

3.5. *Comparing with consequences of log Bogomolov-Miyaoka-Yau type inequalities.*

We are rather surprised that the restrictions (CP_l), imposed already by the very first intervals I_l, are closely related with famous properties (conjectures) of rational plane curves. In order to exemplify this, we need some more notations.

Similarly as above, let $[m_1, m_2, ...]$ be the multiplicity sequence of (C, p), the unique singular point of C. Following Nagata [**26**], we define t to be the maximal positive integer such that $m_1 \geq m_2 + ... + m_t$.

We rewrite the elements of $\Gamma_{(C,p)}$ as $\{0 = \gamma_0 < \gamma_1 < \gamma_2 < ...\}$. Recall that the minimal set of generators of $\Gamma_{(C,p)}$ verifies $\bar{\beta}_0 < ... < \bar{\beta}_g$. In terms of the decorations of the Eisenbud-Neumann splice diagram $\bar{\beta}_0 = p_1 ... p_g = m_1$, $\bar{\beta}_k = a_k p_{k+1} ... p_g$ for $1 \leq k < g$, and $\bar{\beta}_g = a_g$, see 2.1. In fact, since $a_2 > a_1 p_1 p_2$ and $p_1 \geq 2$, one

has $\bar{\beta}_2 > 2\bar{\beta}_1$. In particular, the first (at least) three elements of $\Gamma_{(C,p)}$ depend only on $\bar{\beta}_0$ and $\bar{\beta}_1$ (hence, all their properties can be verified essentially at the level of germs with one Puiseux pair). Using these facts, one can verify that

$$\begin{cases} \gamma_1 = m_1; \quad \gamma_2 = m_1 + m_2; \quad \text{and} \\ m_1 + 2m_{t+1} \leq \gamma_3 \leq 3m_1. \end{cases} \tag{5}$$

3.5.1.   *Example.* (CP_1). By the above notations, (CP_1) says that $\gamma_2 \leq d < \gamma_3$. The first part $\gamma_2 \leq d$ can be verified easily. Indeed, if L_p denotes the tangent cone (line) of C at p, then $i_p(C, L_p) \in \Gamma_{(C,p)}$ and it is strict larger than m_1, hence $\gamma_2 \leq i_p(C, L_p) \leq d$ (the second inequality by Bézout's theorem).

For the second inequality, notice that if C satisfies the Nagata-Noether inequality $d < m_1 + 2m_{t+1}$, then by (5) it satisfies $d < \gamma_3$ too. Recall that any curve C which can be transformed by a Cremona transformation into a line satisfies the Nagata-Noether inequality [18, 23, 26]. On the other hand, it is conjectured that any cuspidal rational curve can be transformed into a line by a Cremona transformation (see [23]).

Notice also that via (5), (CP_1) (or $d < \gamma_3$) implies that $d < 3m_1$, an inequality valid for any cuspidal rational curve C, which was proved in [23] by the log Miyaoka inequality [20, 25]. In fact, $d < \gamma_3$ also implies that $i(C, L_p) = \gamma_2 = m_1 + m_2$.

3.5.2.   Finally, let us reconsider again the inequality $m_1 + m_2 = \gamma_2 \leq d$. Here, by a result of Yoshihara [47] (whose proof is based on the Abhyankar-Moh theorem), one has equality $m_1 + m_2 = d$ if and only if C is an Abhyankar-Moh-Suzuki curve (cf.Definition 1). The Abhyankar-Moh theorem [1] says that such a curve can be transformed into a line, hence by the above discussion it satisfies (CP_1). In the next section we will show that, in fact, it satisfies all the restrictions (CP_l), $l \geq 1$.

3.5.3.   The interested reader is invited to analyze some other particular restrictions (CP_l), e.g. (CP_l) for $l = 1, 2, 3$, combined together; cf. also with the last statement of Remarks 1(1).

REMARK.   Notice also that the above distribution law (CP) is very different from Arnold's prediction of the distribution of *generic* sub-semigroups of $\mathbb{N}$ [3]. This shows that the geometric realization of (C, p) as the unique singular point of a degree d rational curve implies that the semigroup of (C, p) "is far to be generic" (a fact already suggested by the local Abhyankar-Azevedo theorem as well).

3.6. *Comparison with Varchenko's criterion.*

The above negative distribution property has some analogies with the criterions provided by the semicontinuity of the spectrum [44, 45]. Namely, if $\{(C, p_i)\}_i$ are the local singular points of the degree d curve C, then the multisingularity $\sum_i (C, p_i)$ is in the deformation of the universal plane germ $(U, 0) := (x^d + y^d, 0)$. In particular, the collection of all spectral numbers Sp of the local plane curve singularities $(C, p_i)_i$ satisfies the semicontinuity property compared with the spectral numbers of $(U, 0)$ for any interval $(\alpha, \alpha + 1)$.

In order to exemplify the similarities, let us assume for simplicity that $\nu = 1$. Since the spectral numbers of $(U, 0)$ are of type l/d, the semicontinuity property

for intervals $(-1 + l/d, l/d)$ ($l = 2, 3, \ldots, d - 1$) reads as follows:

$$\#\{\alpha \in Sp \ : \ \alpha < l/d\} \leq (l - 2)(l - 1)/2. \tag{6}$$

On the other hand, the negative distribution property of $D(t)$ is equivalent to

$$\#\{k \in \Gamma_{(C,p)} \ : \ k \leq ld\} \leq (l + 1)(l + 2)/2 \tag{7}$$

for any $l = 1, 2, \ldots, d - 3$. Although the similarities are striking, it is not easy at all to compare the two set of inequalities: the transition from $Sp(C, p)$ to $\Gamma_{(C,p)}$ arithmetically is not very simple. (If we add to the discussion the semicontinuity intervals $(-1 + l/d, l/d)$ with $l > d$, then the comparison is even more difficult.) But even when $Sp(C, p)$ and $\Gamma_{(C,p)}$ can be easily compared (see e.g. below), still, the comparison of the inequalities is not obvious. Let us exemplify this in the case when (C, p) has only one Puiseux pair, say (a, b), $a < b$. In this case the semicontinuity transforms into

$$\#\{i \geq 0, j \geq 0 \ : \ ia + jb < abl/d\} \leq (l - 2)(l - 1)/2 + [al/d] + [bl/d] + 1.$$

(Recall that $(a - 1)(b - 1) = (d - 1)(d - 2)$.) On the other hand, our criterion (7) is:

$$\#\{i \geq 0, j \geq 0 \ : \ ia + jb \leq ld\} \leq (l + 2)(l + 1)/2.$$

In the next examples show that the two restrictions are (at least arithmetically) independent.

EXAMPLE 1. *The semicontinuity of the spectrum does not imply the negative distribution property.* Indeed, $(d, a, b) = (11, 4, 31)$ satisfies the semicontinuity but $N(t)$ has some positive coefficients (Γ in I_4 has six elements). A similar example is $(19, 7, 52)$; here the first l when CP_l fails is $l = 7$, I_l has 9 semigroup elements. (The reader is invited to verify that these triples (d, a, b) cannot be realized geometrically, a fact compatible with our conjecture.)

In fact, one can even enter in the game another restriction, namely the sharp inequality of Orevkov [34]:

$$d < \frac{3 + \sqrt{5}}{2}(a + 1) + \frac{1}{\sqrt{5}}, \tag{8}$$

and ask if the semicontinuity together with (8) would imply arithmetically the negative distribution property. The answer again is no, as (again) the above two examples show.

Nevertheless, *for d large, $\nu = 1$ and (C, p) with only one Puiseux pair*, computer experiment shows that Orevkov inequality and the semicontinuity of the spectrum imply the vanishing of $R(t)$.

EXAMPLE 2. Notice that for $\nu = 1$, Theorem 3 replaces the negative distribution property with the vanishing of $N(t)$. This is a non-trivial additional restriction. E.g., $(d, a, b) = (5, 3, 7)$ satisfies the semicontinuity property of the spectrum and also $N(t) = -t$. (Its geometric realization is excluded by Theorem 3.)

EXAMPLE 3. *The negative distribution property does not imply the semicontinuity of the spectrum.* Consider the case $(d, a, b) = (7, 3, 16)$. Then it satisfies the negative distribution property (with $N(t) = -t^2$) but it does not satisfy the semicontinuity of the spectrum.

Let us reconsider the triplet $(d, a, b) = (7, 3, 16)$ from the point of view of Theorem 3: this is a candidate for a unicuspidal curve but with $R(t) \neq 0$. According to Theorem 3 a geometric realization cannot exist, but the very existence of this triplet shows that the proof of Theorem 3 cannot be replaced by a pure arithmetical argument (having starting point the non-positivity of the coefficients of R, and the fact that (C, p) has only one Puiseux pairs, and proving the vanishing of R).

REMARK 1.   For $\nu = 1$, we do not know any example when R is zero but the situation geometrically is not realizable. (In particular, we do not know any example when R or N are vanishing, but the semicontinuity of the spectrum fails.) This motivates the following conjecture:

CONJECTURE 3.   Conjecture 3.2 (or, equivalently, 3.3) covers an 'if and only if' property: the local topological type $(C, p) \subset (\mathbb{P}^2, p)$ can be realized by a degree d unicuspidal rational curve if and only if the property (CP) is valid.

4.   *The validity of* (CP) *for Abhyankar-Moh-Suzuki curves.*

The *AMS type* curves appeared naturally in the study of rational plane curves C meeting with a line L at only one point $\{p\} = C \cap L$. By the Lin-Zaidenberg theorem [**48**] the curve C has at most another cusp as singularities. C is called a *Abhyankar-Moh-Suzuki* curve if $C \setminus L$ is smooth, respectively *Lin-Zaidenberg* curve if $C \setminus L$ is singular. K. Tono has given classifications, up to projective equivalence, of *AMS* curves in [**37**] and of *LZ* curves in [**38**]. Here we will not use these classifications, but we use the relation of these curves with automorphisms of the affine plane $\mathbb{C}^2 = \mathbb{P}^2 \setminus L$ instead.

DEFINITION 1.   An irreducible plane curve C is said to be of *Abhyankar-Moh-Suzuki type* (*AMS type* for short) if there exists a line $L \subset \mathbb{P}^2$ such that $C \setminus L$ is isomorphic to $\mathbb{C}$. In our situation this means that $\nu = 1$ and $C \cap L = Sing(C) = \{p\}$.

Not any curve with $\nu = 1$ is of *AMS* type, e.g. the examples (c)-(f) in Examples 1 are not (cf. also with 3.5.2). The simplest *AMS* curve is $\{zx^{d-1} + y^d = 0\}$. In this case $\Gamma_{(C,p)}$ is generated by two elements, $d - 1$ and d, and (CP) can be easily verified. The goal of the present section is to prove the general case.

THEOREM 4. (CP) *is satisfied by any AMS curve; in other words* $R(t) \equiv 0$.

Identify $\mathbb{C}^2$ with $\mathbb{P}^2 \setminus L$. In the next discussion we consider algebraic automorphisms $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ with components (f, g). Recall, that by [**1**], a curve C is of *AMS* type if and only if it is the compactification in $\mathbb{P}^2$ of the zero locus of a component of a certain automorphism ϕ. The embedding of $\mathbb{C}^2$ into $\mathbb{P}^2$ allows us to view any automorphism of $\mathbb{C}^2$ as a birational transformation of $\mathbb{P}^2$. The point is that the combinatorics of the minimal embedded resolution of (C, p) is closely related to the combinatorics of the minimal resolution of the indeterminacy of ϕ as a birational transformation of $\mathbb{P}^2$, and this last one can be described precisely. In the sequel we make this statement more precise. For details see [**10**, **11**].

Let $\pi' : X' \to \mathbb{P}^2$ be the minimal resolution of the indeterminacy of ϕ. Then
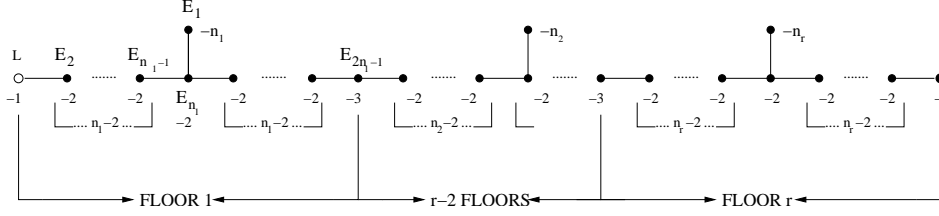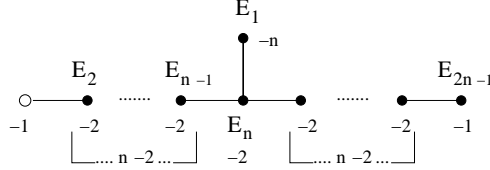
FIGURE 1.

FIGURE 2. *An elementary graph of degree n*

$(\pi')^*L$ is a divisor with normal crossings and its dual graph $\mathcal{A}$ (weighted by the corresponding self intersections) has the form of Figure 1.

Here the vertices of $\mathcal{A}$ are the "black" vertices. The strict transform of L is denoted by a "white" vertex with decoration L. This graph can be obtained from elementary pieces. An *elementary graph of degree n* is defined in Figure 2.

The graph $\mathcal{A}$ is obtained by putting r elementary graphs of degrees $n_1, \ldots, n_r$ one after the other, identifying the last vertex of each graph with the first vertex of the next, and weighting the identified vertices with -3. We say that the graph $\mathcal{A}$ has r floors. The morphism $\pi' : X' \to \mathbb{P}^2$ is a composition of blowing ups, which are totally ordered by appearance. This gives a total order of the vertices of the graph (which can be recovered combinatorially by successively contracting (-1)-vertices).

Now, we concentrate on the unique singular point p of C. The minimal embedded resolution $\pi : X \to \mathbb{P}^2$ of $(C, p) \subset (\mathbb{P}^2, p)$ is again a composition of blowing ups, and all these blowing ups appear in the minimal resolution of the indeterminacy of ϕ. In fact, there are two possibilities:

(a) π is the composition of all the blowing ups of π' except the last $n_r - 1$ of them;

(b) π is the composition of the blowing ups of the first $r - 1$ floors of π' except the last $n_{r-1} - 1$ of them.

If the second possibility holds then one can find a different automorphism $\psi = (f, g)$ such that C is the closure of $\{f = 0\}$ and the graph associated with ψ has $r - 1$ floors satisfying possibility (a). Therefore, one can always assume the validity of (a). This means that the embedded resolution graph of (C, p) can be obtained from the above graph $\mathcal{A}$ by deleting the last $n_r - 1$ (black) vertices, and changing the decoration of E_{n_r} into -1. The strict transform $\bar{C}$ intersects E_{n_r}.

In particular, the singularity p of the curve C has r Newton pairs, and the decorations $\{(p_k, a_k)\}_{k=1}^r$ of its Eisenbud-Neumann splice diagram (cf. 2.1) can be computed by standard graph-determinant computations (cf. [**7**], section 21). One

gets the following:

$$
\begin{cases}
d = i_p(z, \bar{C}) = n_1 \dots n_r; \\
p_1 = n_1 - 1; \text{ and } p_k = n_k \text{ for } k = 2, \dots, r; \\
a_1 = n_1, \text{ and } a_k = n_1^2 \dots n_{k-1}^2 n_k - 1 \text{ for } k = 2, \dots, r.
\end{cases}
\tag{9}
$$

In particular, the generators $\bar{\beta}_i$ of $\Gamma_{(C,p)}$ are $\bar{\beta}_0 = (n_1 - 1)n_2 \dots n_r$, $\bar{\beta}_1 = n_1 \dots n_r$, and for $1 < k \le r$ one has $\bar{\beta}_k = (n_1^2 \dots n_{k-1}^2 n_k - 1)n_{k+1} \dots n_r$.

We will verify (CP) by induction over r. Assume that $\mathcal{A} = \mathcal{A}_r$ has r floors of degrees $n_1, \dots, n_r$. Let Γ_r be the corresponding semigroup. If $r = 1$ then Γ_1 is generated by $n_1 - 1$ and n_1, $d_1 = n_1$ and (CP) can be easily verified. If $r > 1$, by (9) and 2.1, the degree d_r of of C is $d_r = n_1 \dots n_r = d_{r-1}n_r$, and Γ_r is generated by $n_r\Gamma_{r-1}$ and $\bar{\beta}_r := n_1^2 \dots n_{r-1}^2 n_r - 1$. In fact, any element $x \in \Gamma_r$ can be written in a unique way as $x = n_r y + b\bar{\beta}_r$ with $y \in \Gamma_{r-1}$ and $0 \le b < n_r$ (see e.g. [36]).

Now, the inductive step runs as follows:

$$
\sum_{x \in \Gamma_r} t^{\lceil x/d_r \rceil} = \sum_{b=0}^{n_r - 1} \sum_{y \in \Gamma_{r-1}} t^{\lceil (yn_r + b\bar{\beta}_r)/d_r \rceil} = \sum_{b=0}^{n_r - 1} \sum_{y \in \Gamma_{r-1}} t^{bd_{r-1} + \lceil y/d_{r-1} \rceil}
$$

$$
= \frac{1 - t^{d_r}}{1 - t^{d_{r-1}}} \cdot \frac{1 - t^{d_{r-1}}}{(1 - t)^2}.
$$

5. The case $\nu = 1$ and (C, p) with one Puiseux pair.

Assume that the unique cusp of C has exactly one Puiseux pair (a, b) (where $1 < a < b$). Then clearly $(a-1)(b-1) = (d-1)(d-2)$, where $d = \deg(C)$ as above.

In the sequel we denote by $\{\varphi_j\}_{j \ge 0}$ the Fibonacci numbers $\varphi_0 = 0$, $\varphi_1 = 1$, $\varphi_{j+2} = \varphi_{j+1} + \varphi_j$. The Fibonacci numbers have a remarkable amount of interesting properties, see e.g. Vajda's book [43]. We also will use some of them which will be crucial in the next arguments and also in the proof of the conjecture for Kashiwara's families (cf. section 6). E.g.:

(1) $\varphi_j^2 - \varphi_{j-1}\varphi_{j+1} = (-1)^{j+1}$;
(2) $\varphi_j^2 - \varphi_{j-2}\varphi_{j+2} = (-1)^j$.
(3) $\gcd(\varphi_j, \varphi_i) = \varphi_{\gcd(j,i)}$.

EXAMPLES 1.   We will consider the following pairs (a, b):
(a) $(a, b) = (d - 1, d)$;
(b) $(a, b) = (d/2, 2d - 1)$, where $d = \deg(C)$ is even;
(c) $(a, b) = (\varphi_{j-2}^2, \varphi_j^2)$ and $d = \varphi_{j-1}^2 + 1 = \varphi_{j-2}\varphi_j$, where j is odd and ≥ 5;
(d) $(a, b) = (\varphi_{j-2}, \varphi_{j+2})$ and $d = \varphi_j$, where j is odd and ≥ 5;
(e) $(a, b) = (\varphi_4, \varphi_8 + 1) = (3, 22)$ and $d = \varphi_6 = 8$;
(f) $(a, b) = (2\varphi_4, 2\varphi_8 + 1) = (6, 43)$ and $d = 2\varphi_6 = 16$.

All these cases are realizable: (a) e.g. by $\{zy^{d-1} = x^d\}$, (b) by $\{(zy - x^2)^{d/2} = xy^{d-1}\}$, or by the parametrization $[z(t) : x(t) : y(t)] = [1 + t^{d-1} : t^{d/2} : t^d]$. Both cases (a) and (b) satisfy Yoshihara's criterion $m_1 + m_2 = d$ [47] (cf. here with 3.5.2), hence any curve with these data is AMS curve.

The existence of (c) and (d) is guaranteed by Kashiwara classification [19], Corollary 11.4. These two cases can be realized by a rational pencil of type $(0, 1)$: the generic member of the pencil is (c), while the special member of the pencil is of

type (d) (cf. also with the next section). Orevkov in [**34**] denoted the curves (d) by C_j, where a different construction is also given for them.

The cases (e) and (f) correspond to the sporadic cases C_4 and C_4^* of Orevkov [**34**].

REMARKS 1. (1) The above list is not accidental. In [**12**] we prove that if C is a unicuspidal rational plane curve of degree d and if the singular point p of C has only one characteristic pair (a, b), then the triple (d, a, b) is one of the above cases. This classification is coordinated by the following integer. Let $\pi : X \to \mathbb{P}^2$ be the minimal embedded resolution of $C \subset \mathbb{P}^2$, and let $\bar{C}$ be the strict transform of C. Then $\bar{C}^2 = d^2 - ab = 3d - a - b - 1$, and in the above cases is as follows: it is positive for (a) and (b), it is zero for (c), equals -1 for (d), and $= -2$ for (e) and (f). [Notice that $\bar{C}^2 < -1$ if and only if $a + b > 3d$, i.e. if and only if the semigroup element $a + b$ is not sitting in the first three intervals I_l. Notice also that $\bar{C}^2 < -1$ happens exactly for the cases (a)-(d), i.e. when $\bar{\kappa}(\mathbb{P}^2 \setminus C) = -\infty$; cf. (3) below.]

(2) Since $\Gamma_{(C,p)}$ is the semigroup generated by a and b, the verification of (CP) for the above triples (d, a, b) is purely combinatorial depending on these integers.

On the other hand, not any triple (d, a, b) (with $(a - 1)(b - 1) = (d - 1)(d - 2)$) satisfies (CP). E.g., $(5, 3, 7)$ or $(17, 6, 49)$ do not. (But curves with these data do not exist, cf. [**12**].)

(3) The log Kodaira dimensions $\bar{\kappa}(\mathbb{P}^2 \setminus C)$ are the following (cf. [**34**]): $-\infty$ for (a)-(d), and 2 for the last two sporadic cases.

(4) Let $\alpha = (3 + \sqrt{5})/2$ be the root of $\alpha + \frac{1}{\alpha} = 3$. Notice that in family (d), d/a and b/d asymptotically equals α. In fact, for j odd, $\{\varphi_j/\varphi_{j-2}\}_j$ are the increasing convergents of the continued fraction of α. Using this, another remarkable property of the family (d) can be described as follows (cf. [**34**], page 658). The convex hull of all the pairs $(m, d) \in \mathbb{Z}^2$ satisfying $m + 1 \leq d < \alpha m$ (cf. with the sharp Orevkov inequality [**34**], or Example 1) coincides with the convex hull of all pairs (m, d) realizable by rational unicuspidal curves C (where $d = \deg(C)$ and $m = mult(C, p)$) with $\bar{\kappa}(\mathbb{P}^2 \setminus C) = -\infty$; moreover, this convex hull is generated by curves with numerical data (a) and (d). (For curves with $d > \alpha m$, see 9.)

THEOREM 5. *The identity (CP) (i.e. $R(t) \equiv 0$) is satisfied in the above cases (a)-(f).*

Proof. The cases (a) and (b) are covered by Theorem 4. The cases (e) and (f) can be verified by hand: doing this the reader definitely will feel the mystery of this distribution pattern. In the sequel we will verify (c) and (d).

We start with (d). We fix j. The point is that $a + b = 3d$, hence a nice induction can be considered if we group the intervals I_l in blocks of three. Let us analyze the first block. I_1 contains a and $2a$; I_2 contains $3a$, $4a$, and $5a$; finally I_3 has $6a$, $7a$, b and $a + b$. All this can be verified by the definition of the Fibonacci numbers. E.g. $3a > d$ iff $3\varphi_{j-2} > \varphi_{j-1} + \varphi_{j-2}$ iff $2\varphi_{j-2} > \varphi_{j-1}$ iff $2\varphi_{j-2} > \varphi_{j-2} + \varphi_{j-3}$ which is true.

It is clear that it is enough to analyze the intervals I_l for $l < d$. So fix such an l. The point is that an inequality of type $kb > ld$ is true if and only if $k/l > 1/\alpha$. One direction is easy, since $d/b > 1/\alpha$. Assume, that $k/l > 1/\alpha$, then $d/b \geq k/l > 1/\alpha$ is not possible since d/b – being a convergent of α – is the best approximation of

α among fractions with denominator $\leq b$ (see e.g. [**33**] (7.13)). Hence $k/l > d/b$. In particular, $kb \in I_l$ if and only if $\lceil k\alpha \rceil = l$.

There is a similar statement for ia, but its proof is slightly different. If $ia \leq ld$ then using $d/a < \alpha$ one gets $i/l < \alpha$. Assume now that $i/l < \alpha$ then we wish to prove that $d/a < i/l < \alpha$ is not possible. For this notice that $d/a < b/d < \alpha$ (in fact d/a and b/d are two consecutive convergents of α). Then i/l cannot be between b/d and α since b/d is one of the convergents and $l < d$; and also i/l cannot be between d/a and b/d since $ab - d^2 = 1$ and $l < d$ (cf. [**33**], p. 165, 5(a)). Hence $ia \leq ld$ if and only if $i/l < \alpha$. This implies that $ia \in I_l$ if and only if $\lceil i/\alpha \rceil = l$.

Now, using the fact that $a + b = 3d$, we can move the first block to the second block (of three intervals) by adding $a + b = 3d$. In the relevant intervals ($l < d$) the only terms not in the image of this translation have form ia or kb, and they are all distinct. An easy counting shows that the only fact we have to show is that in any interval there are exactly 3 terms of these types. Namely, we have to verify the following: Consider the numbers S_1 of the form $\lceil i/\alpha \rceil$ and the numbers S_2 of the form $\lceil k\alpha \rceil$. Then the claim is that in $S_1 \cup S_2$ each positive integer appears exactly three times.

This can be proved as follows. Consider in the (x, y)-plane the semi-line ℓ_1 : $\{y = \alpha x\}$ (in the first quadrant) and the semi-line ℓ_2: $\{y = -x/\alpha\}$ (in the forth quadrant). For any positive integer l consider the vertical segment connecting the two intersection points of the line $\{x = l\}$ with ℓ_1 and ℓ_2. Notice that the length of this segment is exactly $3l$. Any horizontal line $\{y = i\}$ (resp. $\{y = -k\}$) intersects the segment iff $i/\alpha \leq l$ (resp. $k\alpha \leq l$). Since the segment can be intersected exactly by $3l$ horizontal lines of type $y =$ integer (and all the numbers of type $k\alpha$ and i/α are distinct) the claim follows.

Next, we prove (c). We start as in (d). Fix j and notice that $d = \varphi_{j-2}\varphi_j$.

Step 1. Assume that $l < \varphi_{j-2} + \varphi_j$. If $kb > ld$ then $k\varphi_j > l\varphi_{j-2}$, hence $k/l > \varphi_{j-2}/\varphi_j > 1/\alpha$. Conversely, assume that $k/l > 1/\alpha$. Consider the convergents $\varphi_{j-2}/\varphi_j > \varphi_j/\varphi_{j+2} > 1/\alpha$. Since $\varphi_{j+2} > l$ we conclude that k/l cannot be neither between φ_j/φ_{j+2} and $1/\alpha$ nor between φ_{j-2}/φ_j and φ_j/φ_{j+2} by similar argument as in (d). Hence either $k/l > \varphi_{j-2}/\varphi_j$ or $l = \varphi_j$. This shows that for $l \notin \{\varphi_j, \varphi_j + 1\}$, $kb \in I_l$ if and only if $\lceil k\alpha \rceil = l$.

Step 2. Assume again that $l < \varphi_{j-2} + \varphi_j$. If $ia \leq ld$ then $i/l \leq \varphi_j/\varphi_{j-2} < \alpha$. Conversely, assume that $i/l < \alpha$ and consider the three intervals $\varphi_j/\varphi_{j-2} < \varphi_{j+2}/\varphi_j < \varphi_{j+4}/\varphi_{j+2} < \alpha$. Then i/l cannot be in the second and third interval by similar arguments as above. Moreover, it cannot be in the first one either, since the two end-points are two elements of the φ_j-Farey sequence, and the denominator of any rational number between them is at least $\varphi_{j-2} + \varphi_j$ (cf. (6.4) [**33**]). Therefore, either $l = \varphi_j$ or $i/l \leq \varphi_j/\varphi_{j-2}$. Hence, if $l \notin \{\varphi_j, \varphi_j + 1\}$ then $ia \in I_l$ if and only if $\lceil i/\alpha \rceil = l$.

Step 3. The intervals I_l for $l = \varphi_j$ and $\varphi_j + 1$ can be analyzed independently: $kb \in I_{\varphi_j}$ if and only if $k = \varphi_{j-2}$ (and then $kb = \varphi_j d$), and $ia \in I_{\varphi_j}$ if and only if $i = \varphi_{j+2} - 1$ or $i = \varphi_{j+2} - 2$.

Similarly, $I_{\varphi_j + 1}$ contains no number of type kb, but contains three numbers of type ia for $i = \varphi_{j+2}, \varphi_{j+2} + 1, \varphi_{j+2} + 2$.

Combined the argument of (d) applied for Step 1 and Step 2 and the above two facts, we conclude that the distribution patern is true for any $l < \varphi_j + \varphi_{j-2}$.

Step 4. Notice that in the relevant intervals (i.e. for $l < d$), the semigroup element $ia + kb$ determines uniquely the integers i and k.

Step 5. Using the identity $a\varphi_j = \varphi_{j-2}d$, we construct a well-defined injective map $s_l : I_{l-\varphi_{j-2}} \cap \Gamma \to I_l \cap \Gamma$ by $x \mapsto x + a\varphi_j$. Denote by P_l the subset of semigroup elements of $I_l \cap \Gamma$ which have the form $ia + kb$ with $k \geq 0$ and $a > i \geq 0$. Then (via step 4) $I_l \cap \Gamma$ is the disjoint union of the image of s_l and P_l. Therefore, it is enough to show that

$$\#P_l = \varphi_{j-2} \qquad (10)$$

for any $\varphi_{j-2} \leq l < d$.

Step 6. Since the distribution property is true for any $l < \varphi_{j-2} + \varphi_j$ (step 3), we conclude (by the arguments of step 5) that (10) is true for any $\varphi_{j-2} - 1 \leq l \leq \varphi_{j-2} + \varphi_j - 1$.

Step 7. We verify that there is a bijection $P_l \to P_{l+\varphi_j}$ given by $(i, k) \mapsto (i, k + \varphi_{j-2})$ for any $l \geq \varphi_{j-2}$. The facts that the map is well-defined and injective are clear, for the surjectivity one has to verify an (easy) inequality satisfied by the Fibonacci numbers. Hence (10) follows by step 6 and induction. $\qquad \square$

6. $R(t) \equiv 0$ for $\bar{\kappa}(\mathbb{P}^2 \setminus C) = -\infty$ (i.e. for Kashiwara's curves).

We start with the following lemma which helps us to compare distinct semigroups.

LEMMA 1.   *Fix integers $k \geq 0$ and $m, d \geq 1$. Then*
(a) *in the set $B := \{\lceil mk/d \rceil + j\}_{j=0}^{m-1}$ the unique multiple of m is $\lceil k/d \rceil m$.*
(b) *Fix a subset $\Gamma \subset \mathbb{N}$ and define the series*

$$\eta(t) := \sum_{k \in \Gamma} t^{\lceil k/d \rceil} \quad and \quad \chi(t) := \sum_{k \in \Gamma} t^{\lceil mk/d \rceil}.$$

Then the series $\psi(t) := \chi(t)(1 - t^m)/(1 - t)$ satisfies the identity

$$\eta(t^m) = \frac{1}{m} \sum_{\xi^m = 1} \psi(\xi t).$$

Proof.   It is clear that there is only one multiple of m in B, say a. If $k = qd$ the proof is also clear. Otherwise, write $k = qd + r$ with $r \in \{1, \ldots, d-1\}$. Thus $mq/d = mq + rm/d$. Since $0 < r/d < 1$ then $0 < rm/d < m$ and $0 < \lceil mr/d \rceil \leq m$. Finally $mq < a \leq mq + m + m - 1 < m(q+2)$, and $a = (q+1)$. $\qquad \square$

6.1.   *The proof of the conjecture for Kashiwara's curves.*

As we have mentioned most of the previous curves appear as irreducible components of fibres of rational functions on $\mathbb{P}^2$ of type $(0, 1)$, that is, rational functions all whose fibres (once the indeterminacy point has been removed) are isomorphic to $\mathbb{C}$. Remark that any of these rational function ϕ has at most one indeterminacy point which will be the only possible singular point of any of its fibres. The classification of such rational functions is given in H. Kashiwara's paper [**19**]. Moreover, a rational cuspidal curve C verifies $\bar{\kappa}(\mathbb{P}^2 \setminus C) = -\infty$ if and only if C is an irreducible component of a fibre of a rational function of type $(0, 1)$.

The number of multiple fibres that a rational function ϕ on $\mathbb{P}^2$ of type $(0, 1)$ can have is at most two. Kashiwara's classification gives three strata inside the set of

rational functions of type $(0,1)$: $\mathcal{F}_0, \mathcal{F}_I, \mathcal{F}_{II}$, according to the number of multiple fibres will be $0, 1$ or 2 respectively.

(1) The stratum $\mathcal{F}_0$ consists of all linear rational functions on $\mathbb{P}^2$, therefore there are no cuspidal rational curves as fibres.

(2) If $\phi \in \mathcal{F}_I$ then the multiple fibre of ϕ turns out to be a line L, and ϕ is a component of an automorphism of $\mathbb{C}^2 = \mathbb{P}^2 \setminus L$, see Corollaire 8.1 in [**19**]. Moreover every fibre of ϕ, but the multiple one, defines a rational cuspidal curve of AMS-type for which we have already checked (CP) in Theorem 4.

(3) A rational function $\phi \in \mathcal{F}_{II}$ has two atypical values, say 0 and ∞. Its divisor will be denoted by $(\phi) = mS_0 - nS_\infty$ (for some integers that we may assume $\gcd(m,n) = 1$ taking ϕ primitive). From the topological point of view the rational function ϕ has only three different fibres S_0, S_∞ and the *generic fibre* ϕ_{ge}.

The minimal resolution graph of the indeterminacy of a rational function ϕ of type $(0,1)$ almost coincides with the minimal resolution graph of the generic fibre of ϕ. But this is not the case for special fibres. In her Théorème 6.1 the minimal resolution graphs of the indeterminacy point of the rational functions of type $(0,1)$ are given. The remaining case in which we are interested in is $\phi \in \mathcal{F}_{II}$ where five different graphs appear:

Case 1. $II(\ell)^*$ (with $\ell \geq 0$),

Case 2. $II^+(\ell, N; \lambda_1, \ldots, \lambda_N)^*$ (with N even ≥ 2),

Case 3. $II^+(\ell, N; \lambda_1, \ldots, \lambda_N)^*$ (with N odd ≥ 1),

Case 4. $II^-(\ell, N; \lambda_1, \ldots, \lambda_N)^*$ (with N even ≥ 2),

Case 5. $II^-(\ell, N; \lambda_1, \ldots, \lambda_N)^*$ (with N odd ≥ 2).

Here $\lambda_1, \ldots, \lambda_N$ are integers such that $\lambda_1, \ldots, \lambda_N \geq 0$ if $\ell \geq 1$ and $\lambda_1, \ldots, \lambda_N \geq 1$ if $\ell = 0$.

In Case 1, for a rational function ϕ^ℓ whose graph belongs to $II(\ell)^*$, there are three distinct cuspidal rational curves appearing as fibres of ϕ^ℓ: S_0^ℓ, S_∞^ℓ and the generic fibre ϕ_{ge}^ℓ. The rational function ϕ^ℓ is constructed by induction based on a rational function $\phi^{\ell-1}$ which implies that S_∞^ℓ is nothing but $S_0^{\ell-1}$, see e.g. Corollaire 11.4. Thus for each graph in $II(\ell)^*$ there are only two new unicuspidal rational plane curves: the generic member of the pencil, we will denote it by $II(\ell)_{ge}$, and the new special member of the pencil, we will denote it by $II(\ell)_{sp}$. We have already studied those curves (c) and (d) in Examples 1:

(c) $II(\ell)_{ge}$ is a unicuspidal rational curve of degree $d = \varphi_{2\ell+3}\varphi_{2\ell+5}$ and only one characteristic pair $(a,b) = (\varphi_{2\ell+3}^2, \varphi_{2\ell+5}^2)$;

(d) $II(\ell)_{sp}$ is a unicuspidal rational curve of degree $d = \varphi_{2\ell+3}$ and only one characteristic pair $(a,b) = (\varphi_{2\ell+1}, \varphi_{2\ell+5})$.

Therefore in Theorem 5 we have proved the identity (CP) for $II(\ell)_{ge}$ and $II(\ell)_{sp}$, $\ell \geq 0$.

The same fact happens for all rational functions whose graphs belongs to any of the remaining four cases, see e.g. Corollaire 11.6. The irreducible component of the ∞ fibre of the ℓ-rational function is nothing but the 0-fibre of the previous $(\ell-1)$-rational function. Therefore in each group we have just two new unicuspidal rational curves: the generic member of the pencil and one special member of the pencil.

The only difference between the generic rational curve in $II^+(\ell, N; \lambda_1, \ldots, \lambda_N)^*$ (with N even ≥ 2), (Case 2), and the generic rational curve in $II^+(\ell, N; \lambda_1, \ldots, \lambda_N)^*$ (with N odd ≥ 1), (Case 3), is the number of Puiseux pairs.

In fact, it is possible to codify the invariants of the generic members of Cases 2 and 3 in one and the same sequence of Eisenbud-Neumann splice diagrams, which we denote by $II^+(\ell, N; \lambda_1, \ldots, \lambda_N)_{ge}$. The same can be done with the special members of the pencils, which we will denote by $II^+(\ell, N; \lambda_1, \ldots, \lambda_N)_{sp}$.

Finally the same can be done also with Cases 4 and 5, that is for rational functions with graphs in $II^-(\ell, N; \lambda_1, \ldots, \lambda_N)^*$ (with N even ≥ 2) or $II^-(\ell, N; \lambda_1, \ldots, \lambda_N)^*$ (with N odd ≥ 2). The generic curve will be denoted by $II^-(\ell, N; \lambda_1, \ldots, \lambda_N)_{ge}$ and the special curve by $II^-(\ell, N; \lambda_1, \ldots, \lambda_N)_{sp}$.

6.2. *Degrees of curves and generators of semigroups.*

Fix $\ell \geq 0$ and $N \geq 1$ and non-negative integers $\lambda_1, \ldots, \lambda_N$.

6.2.1. $II^+(\ell, N; \lambda_1, \ldots, \lambda_N)_{ge}$. For $1 \leq i \leq N$ define

$$\begin{cases} n_i := \lambda_i \varphi_{2\ell+3}^2 + \varphi_{2\ell+3}\varphi_{2\ell-1} - 1 & \text{for } i \text{ odd;} \\ n_i := \lambda_i \varphi_{2\ell+3}^2 + \varphi_{2\ell+3}(\varphi_{2\ell+3} - \varphi_{2\ell-1}) - 1 & \text{for } i \text{ even.} \end{cases} \tag{11}$$

The degree of the curve $II^+(\ell, N; \lambda_1, \ldots, \lambda_N)_{ge}$ is $d = \varphi_{2\ell+3}\varphi_{2\ell+5}n_1 \ldots n_N$, (see Proposition 7.2 in [**19**]). We remark that the integer m_ℓ in [**19**] is nothing but the Fibonacci number $\varphi_{2\ell+3}$. This curve has only one singularity whose splice diagram can be easily deduced from the resolution graph of the corresponding rational function. This singularity has $N+1$ Newton pairs and the decorations $\{(p_k, a_k)\}_{k=1}^{N+1}$ of the corresponding Eisenbud-Neumann splice diagram can be done computing the graph determinants (as it is explained in [**7**]). Thus

$$\begin{cases} p_k = n_k \text{ for } 1 \leq k \leq N, \text{and } p_{N+1} = \varphi_{2\ell+3}^2; \\ a_k = (\varphi_{2\ell+5}^2 n_1^2 \ldots n_{k-1}^2 n_k - 1)/\varphi_{2\ell+3}^2, \text{ for } 1 \leq k \leq N \text{ and} \\ a_{N+1} = \varphi_{2\ell+5}^2 n_1^2 \ldots n_N^2. \end{cases} \tag{12}$$

To check that the a_k's are integers, let b_k denote the numerator $\varphi_{2\ell+5}^2 n_1^2 \ldots n_{k-1}^2 n_k - 1$ of a_k, then $b_k = b_{k-1}n_{k-1}n_k + n_{k-1}n_k - 1$. By induction, to show that $b_k = 0 \bmod \varphi_{2\ell+3}^2$ it is enough to prove that $n_{k-1}n_k - 1 = 0 \bmod \varphi_{2\ell+3}^2$. From the definition of n_k, see (11), $n_{k-1}n_k - 1 = (\varphi_{2\ell+3}\varphi_{2\ell-1} - 1)(-\varphi_{2\ell+3}\varphi_{2\ell-1} - 1) - 1 = -(\varphi_{2\ell+3}^2\varphi_{2\ell-1}^2 - 1) - 1 = 0 \bmod \varphi_{2\ell+3}^2$, after property (ii) of the Fibonacci numbers.

Thus the generators $\{\bar{\beta}_k\}$ of its semigroup $\Gamma_{II^+(\ell,N;\lambda)_{ge}}$ are $\bar{\beta}_0 = \varphi_{2\ell+3}^2 n_1 \ldots n_N$, $\bar{\beta}_k = (\varphi_{2\ell+5}^2 n_1^2 \ldots n_{k-1}^2 n_k - 1)n_{k+1} \ldots n_N$ for $1 \leq k \leq N$, and $\bar{\beta}_{N+1} = \varphi_{2\ell+5}^2 n_1^2 \ldots n_N^2$.

6.2.2. $II^+(\ell, N; \lambda_1, \ldots, \lambda_N)_{sp}$. The curve $II^+(\ell, N; \lambda_1, \ldots, \lambda_N)_{sp}$ is rational and unicuspidal and has degree $d = \varphi_{2\ell+5}n_1 \ldots n_N$, (see Proposition 7.2 in [**19**]). This curve has also only one singularity which is a cusp with $N+1$ Newton pairs. To compute the decorations $\{(p_k, a_k)\}_{k=1}^{N+1}$ of the corresponding splice diagram, observe that all the determinants, but the last two, are the same as in the previous case. Thus the invariants in this case are:

$$\begin{cases} p_k = n_k \text{ for } 1 \leq k \leq N, \text{ and } p_{N+1} = \varphi_{2\ell+3}; \\ a_k = (\varphi_{2\ell+5}^2 n_1^2 \ldots n_{k-1}^2 n_k - 1)/\varphi_{2\ell+3}^2, \text{for } 1 \leq k \leq N \text{ and} \\ a_{N+1} = (\varphi_{2\ell+5}^2 n_1^2 \ldots n_N^2 + 1)/\varphi_{2\ell+3}. \end{cases} \tag{13}$$

Moreover, a_{N+1} is also integer because $n_i = -1 \bmod \varphi_{2\ell+3}$ and then $\varphi_{2\ell+5}^2 n_1^2 \ldots n_N^2 + 1 = \varphi_{2\ell+5}^2 - 1 = (\varphi_{2\ell+3}\varphi_{2\ell+7} - 1) + 1 = 0 \bmod \varphi_{2\ell+3}$.

The generators $\{\bar{\beta}_k\}$ of its semigroup $\Gamma_{II^+(\ell,N;\lambda)_{sp}}$ are $\bar{\beta}_0 = \varphi_{2\ell+3}n_1\ldots n_N$, $\bar{\beta}_k = (\varphi_{2\ell+5}^2 n_1^2 \ldots n_{k-1}^2 n_k - 1)n_{k+1}\ldots n_N/\varphi_{2\ell+3}$ for $1 \leq k \leq N$, and $\bar{\beta}_{N+1} = (\varphi_{2\ell+5}^2 n_1^2 \ldots n_N^2 + 1)/\varphi_{2\ell+3}$.

6.2.3. $II^-(\ell, N; \lambda_1, \ldots, \lambda_N)_{ge}$.   For $1 \leq i \leq N$, define

$$\begin{cases} \tilde{n}_i := \lambda_i \varphi_{2\ell+3}^2 + \varphi_{2\ell+3}\varphi_{2\ell-1} - 1 & \text{for } i \text{ even;} \\ \tilde{n}_i := \lambda_i \varphi_{2\ell+3}^2 + \varphi_{2\ell+3}(\varphi_{2\ell+3} - \varphi_{2\ell-1}) - 1 & \text{for } i \text{ odd.} \end{cases} \tag{14}$$

Essentially the data of the curve $II^-(\ell, N; \lambda_1, \ldots, \lambda_N)_{ge}$ can be obtained from the data of the curve $II^+(\ell, N; \lambda_1, \ldots, \lambda_N)_{ge}$ replacing $\varphi_{2\ell+5}$ by $\varphi_{2\ell+1}$. The combinatorial reason for that is the identity $\varphi_{2\ell+3}^2 = \varphi_{2\ell+1}\varphi_{2\ell+5} - 1$. The degree of the rational curve $II^-(\ell, N; \lambda_1, \ldots, \lambda_N)_{ge}$ is $d = \varphi_{2\ell+3}\varphi_{2\ell+1}\tilde{n}_1 \ldots \tilde{n}_N$, (see Proposition 7.2 in [**19**]). This curve has only one singular point which is a cusp with $N+1$ Newton pairs. The decorations $\{(p_k, a_k)\}_{k=1}^{N+1}$ of the corresponding Eisenbud-Neumann splice diagram obtained computing the graph determinants are

$$\begin{cases} p_k = \tilde{n}_k \text{ for } 1 \leq k \leq N, \text{ and } p_{N+1} = \varphi_{2\ell+3}^2; \\ a_k = (\varphi_{2\ell+1}^2 \tilde{n}_1^2 \ldots \tilde{n}_{k-1}^2 \tilde{n}_k - 1)/\varphi_{2\ell+3}^2, \text{ for } 1 \leq k \leq N \text{ and} \\ a_{N+1} = \varphi_{2\ell+1}^2 \tilde{n}_1^2 \ldots \tilde{n}_N^2. \end{cases} \tag{15}$$

In the same way as before one checks that the a_k's are integers. The minimal set of generators of its semigroup $\Gamma_{II^-(\ell,N;\lambda)_{ge}}$ are $\bar{\beta}_0 = \varphi_{2\ell+3}^2 \tilde{n}_1 \ldots \tilde{n}_N$, $\bar{\beta}_k = (\varphi_{2\ell+1}^2 n_1^2 \ldots \tilde{n}_{k-1}^2 \tilde{n}_k - 1)\tilde{n}_{k+1} \ldots \tilde{n}_N$ for $1 \leq k \leq N$, and $\bar{\beta}_{N+1} = \varphi_{2\ell+1}^2 \tilde{n}_1^2 \ldots \tilde{n}_N^2$.

6.2.4. $II^-(\ell, N; \lambda_1, \ldots, \lambda_N)_{sp}$.   The degree of the unicuspidal rational curve $II^-(\ell, N; \lambda_1, \ldots, \lambda_N)_{sp}$ is $d = \varphi_{2\ell+1}n_1 \ldots n_N$, (see Proposition 7.2 in [**19**]). Its singularity has $N+1$ Newton pairs and the decorations $\{(p_k, a_k)\}_{k=1}^{N+1}$ of the corresponding Eisenbud-Neumann splice diagram are

$$\begin{cases} p_k = \tilde{n}_k \text{ for } 1 \leq k \leq N, \text{and } p_{N+1} = \varphi_{2\ell+3}; \\ a_k = (\varphi_{2\ell+1}^2 \tilde{n}_1^2 \ldots \tilde{n}_{k-1}^2 \tilde{n}_k - 1)/\varphi_{2\ell+3}^2, \text{ for } 1 \leq k \leq N \text{ and} \\ a_{N+1} = (\varphi_{2\ell+1}^2 \tilde{n}_1^2 \ldots \tilde{n}_N^2 + 1)/\varphi_{2\ell+3}. \end{cases} \tag{16}$$

Again a_{N+1} is integer and the generators of its semigroup $\Gamma_{II^-(\ell,N;\lambda)_{sp}}$ are $\bar{\beta}_0 = \varphi_{2\ell+3}\tilde{n}_1 \ldots \tilde{n}_N$, $\bar{\beta}_k = (\varphi_{2\ell+1}^2 \tilde{n}_1^2 \ldots \tilde{n}_{k-1}^2 \tilde{n}_k - 1)\tilde{n}_{k+1} \ldots \tilde{n}_N/\varphi_{2\ell+3}$ for $1 \leq k \leq N$, and $\bar{\beta}_{N+1} = (\varphi_{2\ell+1}^2 \tilde{n}_1^2 \ldots \tilde{n}_N^2 + 1)/\varphi_{2\ell+3}$.

6.3. (CP) for generic members of the pencils.

The generic members of the pencils are the curves $II^\varepsilon(\ell, N; \lambda_1, \ldots, \lambda_N)_{ge}$, $\varepsilon = \pm$. To deal with the elements of their semigroups $\Gamma_{II^\pm(\ell,N;\lambda)_{ge}}$ it is better to multiply them by $\varphi_{2\ell+3}$. Define

$$\chi_{ge}^\varepsilon(t) := \sum_{k \in \Gamma_{II^\varepsilon(\ell,N;\lambda)_{ge}}} t^{\lceil \varphi_{2\ell+3}k/d \rceil} \text{ and } \psi_{ge}^\varepsilon(t) := \frac{\chi_{ge}^\varepsilon(t)(1 - t^{\varphi_{2\ell+3}})}{1 - t}, \text{ with } \varepsilon = \pm,$$

where either $d = \varphi_{2\ell+3}\varphi_{2\ell+5}n_1 \ldots n_N$ if $\varepsilon = +$ or $d = \varphi_{2\ell+3}\varphi_{2\ell+1}\tilde{n}_1 \ldots \tilde{n}_N$ otherwise. If

$$CP_{ge}^\varepsilon(t) := \sum_{k \in \Gamma_{II^\varepsilon(\ell,N;\lambda)_{ge}}} t^{\lceil k/d \rceil} \text{ with } \varepsilon = \pm,$$

then Lemma 1 implies

$$CP_{ge}^{\varepsilon}(t^{\varphi_{2\ell+3}}) = \frac{1}{\varphi_{2\ell+3}} \sum_{\xi^{\varphi_{2\ell+3}}=1} \psi_{ge}^{\varepsilon}(\xi t) \text{ with } \varepsilon = \pm. \tag{17}$$

PROPOSITION 3.
  (a) *For the curve* $II^+(\ell, N; \lambda_1, \ldots, \lambda_N)_{ge}$ *the following identity holds:*

$$\chi_{ge}^+(t) = \frac{(1 - t^{\varphi_{2\ell+1}\varphi_{2\ell+5}})(1 - t^{\varphi_{2\ell+3}^2\varphi_{2\ell+5}n_1\ldots n_N})}{(1 - t^{\varphi_{2\ell+3}^2})(1 - t^{\varphi_{2\ell+1}})(1 - t^{\varphi_{2\ell+5}})}.$$

  (b) *For the curve* $II^-(\ell, N; \lambda_1, \ldots, \lambda_N)_{ge}$ *the following identity holds:*

$$\chi_{ge}^-(t) = \frac{(1 - t^{\varphi_{2\ell+1}\varphi_{2\ell+5}})(1 - t^{\varphi_{2\ell+3}^2\varphi_{2\ell+1}\tilde{n}_1\ldots\tilde{n}_N})}{(1 - t^{\varphi_{2\ell+3}^2})(1 - t^{\varphi_{2\ell+1}})(1 - t^{\varphi_{2\ell+5}})}.$$

Proof.   We start with the curve $II^+(\ell, N; \lambda_1, \ldots, \lambda_N)_{ge}$ which has degree $d = \varphi_{2\ell+3}\varphi_{2\ell+5}n_1\ldots n_N$ and whose generators of its semigroup $\Gamma_{II^+(\ell,N;\lambda)_{ge}}$ has been described in (12). By 2.1, each element x in the semigroup can be written in a unique way as

$$x = x_0\varphi_{2\ell+3}^2 \prod_{i=1}^{N} n_i + \sum_{k=1}^{N} y_k \, a_k \, \varphi_{2\ell+3}^2 n_{k+1}\ldots n_N + z_0 \, \varphi_{2\ell+5}^2 \, n_1^2 \ldots n_N^2,$$

with $x_0 \geq 0$, $0 \leq y_k \leq n_k - 1$, for $1 \leq k \leq N$, and $0 \leq z_0 \leq \varphi_{2\ell+3}^2 - 1$. Thus

$$\frac{\varphi_{2\ell+3}x}{d} = x_0\frac{\varphi_{2\ell+3}^2}{\varphi_{2\ell+5}} + \sum_{k=1}^{N} y_k\frac{(\varphi_{2\ell+5}^2 n_1^2 \ldots n_{k-1}^2 n_k - 1)}{\varphi_{2\ell+5}n_1\ldots n_k} + z_0\varphi_{2\ell+5}n_1\ldots n_N.$$

Write $x_0 = q_0\varphi_{2\ell+5} + r_0$ with $0 \leq r_0 \leq \varphi_{2\ell+5} - 1$. Using $\varphi_{2\ell+3}^2 = \varphi_{2\ell+5}\varphi_{2\ell+1} - 1$, we get

$$\lceil\varphi_{2\ell+3}x/d\rceil = q_0\varphi_{2\ell+3}^2 + r_0\varphi_{2\ell+1} + \sum_{k=1}^{N} y_k\varphi_{2\ell+5}n_1\ldots n_{k-1} + z_0\varphi_{2\ell+5}n_1\ldots n_N,$$

because $-1 < (-r_0 n_1 \ldots n_N - \sum_{k=1}^{N} y_k n_{k+1}\ldots n_N)/\varphi_{2\ell+5}n_1\ldots n_N \leq 0$, for every $0 \leq y_k \leq n_k - 1$, for $1 \leq k \leq N$, and $0 \leq r_0 \leq \varphi_{2\ell+5} - 1$. The result is proved because $\chi_{ge}^+(t)$ is equal to

$$\sum_{q_0\geq 0}^{} \sum_{r_0=0}^{\varphi_{2\ell+5}-1} \sum_{k=1}^{N}\sum_{y_k=0}^{n_k-1} \sum_{z_0=0}^{\varphi_{2\ell+3}^2-1} t^{q_0\varphi_{2\ell+3}^2+r_0\varphi_{2\ell+1}+\sum_{k=1}^{N} y_k\varphi_{2\ell+5}n_1\ldots n_{k-1}+z_0\varphi_{2\ell+5}n_1\ldots n_N}$$

$$= \frac{(1 - t^{\varphi_{2\ell+1}\varphi_{2\ell+5}})\prod_{k=1}^{N}(1 - t^{\varphi_{2\ell+5}n_1\ldots n_k})(1 - t^{\varphi_{2\ell+3}^2\varphi_{2\ell+5}n_1\ldots n_N})}{(1 - t^{\varphi_{2\ell+3}^2})(1 - t^{\varphi_{2\ell+1}})\prod_{k=1}^{N}(1 - t^{\varphi_{2\ell+5}n_1\ldots n_{k-1}})(1 - t^{\varphi_{2\ell+5}n_1\ldots n_N})}.$$

The proof for the curve $II^-(\ell, N; \lambda_1, \ldots, \lambda_N)_{ge}$ is essentially the same replacing $\varphi_{2\ell+5}$ by $\varphi_{2\ell+1}$.   □

THEOREM 6. (CP) *is true for the generic curves* $II^+(\ell, N; \lambda_1, \ldots, \lambda_N)_{ge}$ *and* $II^-(\ell, N; \lambda_1, \ldots, \lambda_N)_{ge}$.

Proof. We do the case $\varepsilon = +$ leaving the other one to the reader. Using Proposition 3 and (17) then

$$CP_{ge}^+(t^{\varphi_{2\ell+3}}) = \frac{(1 - t^{\varphi_{2\ell+3}})(1 - t^{\varphi_{2\ell+3}^2\varphi_{2\ell+5}n_1\ldots n_N})}{(1 - t^{\varphi_{2\ell+3}^2})} \cdot \frac{1}{\varphi_{2\ell+3}}$$

$$\cdot \sum_{\xi^{\varphi_{2\ell+3}}=1} \frac{(1 - (\xi t)^{\varphi_{2\ell+1}\varphi_{2\ell+5}})}{(1 - (\xi t)^{\varphi_{2\ell+1}})(1 - (\xi t)^{\varphi_{2\ell+5}})(1 - \xi t)}.$$

Let $\Delta(t)$ be the characteristic polynomial of the rational unicuspidal curve of Example (d) in Examples 1. That is, it has degree $\varphi_{2\ell+3}$ and one characteristic pair $(a, b) = (\varphi_{2\ell+1}, \varphi_{2\ell+5})$. For such a curve we have already proved $R(t) \equiv 0$ in Theorem 5. Thus

$$\frac{1}{\varphi_{2\ell+3}} \sum_{\xi^{\varphi_{2\ell+3}}=1} \frac{(1 - (\xi t)^{\varphi_{2\ell+1}\varphi_{2\ell+5}})}{(1 - (\xi t)^{\varphi_{2\ell+1}})(1 - (\xi t)^{\varphi_{2\ell+5}})(1 - \xi t)}$$

$$= \frac{1}{\varphi_{2\ell+3}} \sum_{\xi^{\varphi_{2\ell+3}}=1} \frac{\Delta(\xi t)}{(1 - \xi t)^2} = \frac{(1 - t^{\varphi_{2\ell+3}^2})}{(1 - t^{\varphi_{2\ell+3}^2})^3}.$$

Since the degree of $II^+(\ell, N; \lambda_1, \ldots, \lambda_N)_{ge}$ is $\varphi_{2\ell+3}\varphi_{2\ell+5}n_1\ldots n_N$ then the result follows because

$$CP_{ge}^+(t^{\varphi_{2\ell+3}}) = \frac{(1 - t^{\varphi_{2\ell+3}^2\varphi_{2\ell+5}n_1\ldots n_N})}{(1 - t^{\varphi_{2\ell+3}^2})^2}.$$

$\square$

6.4. (CP) *for special members of the pencils.*

The special members of the pencils are the rational curves $II^\varepsilon(\ell, N; \lambda_1, \ldots, \lambda_N)_{sp}$, with $\varepsilon = \pm$. As in the previous case, to deal with the elements of their semigroups $\Gamma_{II^\pm(\ell,N;\lambda)_{sp}}$ we multiply them by $\varphi_{2\ell+3}$. Define

$$\chi_{sp}^\varepsilon(t) := \sum_{k \in \Gamma_{II^\varepsilon(\ell,N;\lambda)_{sp}}} t^{\lceil \varphi_{2\ell+3}k/d \rceil} \text{ and } \psi_{sp}^\varepsilon(t) := \frac{\chi_{sp}^\varepsilon(t)(1 - t^{\varphi_{2\ell+3}})}{1 - t}, \text{ with } \varepsilon = \pm,$$

where either $d = \varphi_{2\ell+5}n_1\ldots n_N$ if $\varepsilon = +$ or $d = \varphi_{2\ell+1}\tilde{n}_1\ldots \tilde{n}_N$ otherwise. If

$$CP_{sp}^\varepsilon(t) := \sum_{k \in \Gamma_{II^\varepsilon(\ell,N;\lambda)_{sp}}} t^{\lceil k/d \rceil} \text{ with } \varepsilon = \pm,$$

then Lemma 1 implies

$$CP_{sp}^\varepsilon(t^{\varphi_{2\ell+3}}) = \frac{1}{\varphi_{2\ell+3}} \sum_{\xi^{\varphi_{2\ell+3}}=1} \psi_{sp}^\varepsilon(\xi t) \text{ with } \varepsilon = \pm. \quad (18)$$

THEOREM 7. *(CP) is true for the special curves $II^+(\ell, N; \lambda_1, \ldots, \lambda_N)_{sp}$ and $II^-(\ell, N; \lambda_1, \ldots, \lambda_N)_{sp}$.*

Proof. We do the proof for the curve $II^+(\ell, N; \lambda_1, \ldots, \lambda_N)_{sp}$ and the proof for the curve $II^-(\ell, N; \lambda_1, \ldots, \lambda_N)_{sp}$ is essentially the same replacing n_i by $\tilde{n}_i$ and $\varphi_{2\ell+5}$ by $\varphi_{2\ell+1}$.

The degree of $II^+(\ell, N; \lambda_1, \ldots, \lambda_N)_g$ is $d = \varphi_{2\ell+5}n_1\ldots n_N$ and the generators of

the semigroup $\Gamma_{II+(\ell,N;\lambda_1,\ldots,\lambda_N)_{sp}}$ have been described in (13). Each element x in $\Gamma_{II+(\ell,N;\lambda)_{sp}}$ can be written in a unique way as

$$x = x_0\varphi_{2\ell+3}\prod_{i=1}^{N} n_i + \sum_{k=1}^{N} y_k \, a_k \, \varphi_{2\ell+3} \, n_{k+1}\ldots n_N + z_0\frac{\varphi_{2\ell+5}^2 n_1^2\ldots n_N^2 + 1}{\varphi_{2\ell+3}},$$

with $x_0 \geq 0$, $0 \leq y_k \leq n_k - 1$, for $1 \leq k \leq N$, and $0 \leq z_0 \leq \varphi_{2\ell+3} - 1$. Thus

$$\frac{\varphi_{2\ell+3}x}{d} = x_0\frac{\varphi_{2\ell+3}^2}{\varphi_{2\ell+5}} + \sum_{k=1}^{N} y_k\frac{(\varphi_{2\ell+5}^2 n_1^2\ldots n_{k-1}^2 n_k - 1)}{\varphi_{2\ell+5}n_1\ldots n_k} + z_0\frac{\varphi_{2\ell+5}^2 n_1^2\ldots n_N^2 + 1}{\varphi_{2\ell+5}n_1\ldots n_N}.$$

Write $x_0 = q_0\varphi_{2\ell+5} + r_0$ with $0 \leq r_0 \leq \varphi_{2\ell+5} - 1$. Since $\varphi_{2\ell+3}^2 = \varphi_{2\ell+5}\varphi_{2\ell+1} - 1$ then

$$\lceil\varphi_{2\ell+3}x/d\rceil = q_0\varphi_{2\ell+3}^2 + \alpha(r_0,\mathbf{y},z_0) + \lceil\beta(r_0,\mathbf{y},z_0)\rceil.$$

where $\beta(r_0,\mathbf{y},z_0) := (z_0 - r_0 n_1\ldots n_N - \sum_{k=1}^{N} y_k n_{k+1}\ldots n_N)/\varphi_{2\ell+5}n_1\ldots n_N$ and

$$\alpha(r_0,\mathbf{y},z_0) := r_0\varphi_{2\ell+1} + \sum_{k=1}^{N} y_k\varphi_{2\ell+5}n_1\ldots n_{k-1} + z_0\varphi_{2\ell+5}n_1\ldots n_N.$$

Set $A := \{(r_0,\mathbf{y},z_0) : 0 \leq r_0 \leq \varphi_{2\ell+5} - 1, 0 \leq z_0 \leq \varphi_{2\ell+3} - 1, 0 \leq y_k \leq n_k - 1$, for $k = 1,\ldots,N\}$. Since for $(r_0, y_1,\ldots,y_N,z_0) \in A$, one gets $-1 < \beta(r_0,\mathbf{y},z_0) < 1$ then $\lceil\beta(r_0,\mathbf{y},z_0)\rceil$ is either 0 or 1. Consider the subset P of A defined by $\{z_0 - r_0 n_1\ldots n_N - \sum_{k=1}^{N} y_k n_{k+1}\ldots n_N > 0\}$ and Q its complement in A. Thus $\lceil\beta(r_0,\mathbf{y},z_0)\rceil$ is 1 if and only if $(r_0,\mathbf{y},z_0) \in P$ and it is zero otherwise. This implies

$$\chi_{sp}^+(t) = \frac{\sum_{(r_0,\mathbf{y},z_0)\in P} t^{\alpha(r_0,\mathbf{y},z_0)+1} + \sum_{(r_0,\mathbf{y},z_0)\in Q} t^{\alpha(r_0,\mathbf{y},z_0)}}{(1 - t^{\varphi_{2\ell+3}^2})}.$$

REMARK 2. If $(r_0,\mathbf{y},z_0) \in P$ then $0 < z_0 - r_0 n_1\ldots n_N - \sum_{k=1}^{N} y_k n_{k+1}\ldots n_N < \varphi_{2\ell+3}$.

By (18), $CP_{sp}^+(t^{\varphi_{2\ell+3}})$ is equal to

$$\frac{(1 - t^{\varphi_{2\ell+3}})}{(1 - t^{\varphi_{2\ell+3}^2})}\cdot\frac{1}{\varphi_{2\ell+3}}\sum_{\xi^{\varphi_{2\ell+3}}=1}\frac{\sum_{(r_0,\mathbf{y},z_0)\in P}(\xi t)^{\alpha(r_0,\mathbf{y},z_0)+1} + \sum_{(r_0,\mathbf{y},z_0)\in Q}(\xi t)^{\alpha(r_0,\mathbf{y},z_0)}}{(1 - \xi t)}.$$

Claim: $CP_{sp}^+(t^{\varphi_{2\ell+3}})$ is also equal to:

$$\frac{(1 - t^{\varphi_{2\ell+3}})}{(1 - t^{\varphi_{2\ell+3}^2})}\cdot\frac{1}{\varphi_{2\ell+3}}\sum_{\xi^{\varphi_{2\ell+3}}=1}\frac{\sum_{(r_0,\mathbf{y},z_0)\in P}(\xi t)^{\alpha(r_0,\mathbf{y},z_0)} + \sum_{(r_0,\mathbf{y},z_0)\in Q}(\xi t)^{\alpha(r_0,\mathbf{y},z_0)}}{(1 - \xi t)}.$$

For each $(r_0,\mathbf{y},z_0) \in P$ consider the series $t^{\alpha(r_0,\mathbf{y},z_0)+1}/(1 - t) = t^{\alpha(r_0,\mathbf{y},z_0)+1}(1 + t + t^2 + ..)$. Then

$$\frac{1}{\varphi_{2\ell+3}}\sum_{\xi^{\varphi_{2\ell+3}}=1}\frac{(\xi t)^{\alpha(r_0,\mathbf{y},z_0)+1}}{(1 - \xi t)}$$

is nothing but the coefficients whose power is a multiple of $\varphi_{2\ell+3}$ in $t^{\alpha(r_0,\mathbf{y},z_0)+1}(1 + t + t^2 + ..)$. Therefore to prove the **claim** it is enough to check that for every

$(r_0, \mathbf{y}, z_0) \in P$ then $\alpha(r_0, \mathbf{y}, z_0) \neq 0 \mod \varphi_{2\ell+3}$. Write $\alpha(r_0, \mathbf{y}, z_0)$ as

$$r_0(\varphi_{2\ell+3} - \varphi_{2\ell+2}) + \left(\sum_{k=1}^{N} y_k n_1 \dots n_{k-1} + z_0 n_1 \dots n_N \right) (\varphi_{2\ell+2} + 2\varphi_{2\ell+3}).$$

Since $\gcd(\varphi_{2\ell+2}, \varphi_{2\ell+3}) = 1$ (see property (iii) of the Fibonacci numbers) then $\alpha(r_0, \mathbf{y}, z_0) = 0 \mod \varphi_{2\ell+3}$ if and only if

$$-r_0 + \sum_{k=1}^{N} y_k n_1 \dots n_{k-1} + z_0 n_1 \dots n_N = 0 \mod \varphi_{2\ell+3}.$$

Moreover, since for all k, $n_k = -1 \mod \varphi_{2\ell+3}$ then

$$-r_0 + \sum_{k=1}^{N} y_k (-1)^{k-1} + z_0 (-1)^N = 0 \mod \varphi_{2\ell+3}$$

$$\Longleftrightarrow \quad -(-1)^N r_0 - \sum_{k=1}^{N} y_k (-1)^{N-k} + z_0 = 0 \mod \varphi_{2\ell+3}.$$

But this last identity implies $z_0 - r_0 n_1 \dots n_N - \sum_{k=1}^{N} y_k n_{k+1} \dots n_N = s\varphi_{2\ell+3}$ for $s \in \mathbb{Z}$ which contradicts the definition of the set P, see also Remark 2.

Since

$$\frac{\sum_{(r_0, \mathbf{y}, z_0) \in P} t^{\alpha(r_0, \mathbf{y}, z_0)} + \sum_{(r_0, \mathbf{y}, z_0) \in Q} t^{\alpha(r_0, \mathbf{y}, z_0)}}{(1 - t^{\varphi_{2\ell+3}^2})}$$

$$= \frac{(1 - t^{\varphi_{2\ell+1}\varphi_{2\ell+5}})(1 - t^{\varphi_{2\ell+3}\varphi_{2\ell+5}n_1 \dots n_N})}{(1 - t^{\varphi_{2\ell+3}^2})(1 - t^{\varphi_{2\ell+1}})(1 - t^{\varphi_{2\ell+5}})},$$

the **claim** implies that $CP_{sp}^+(t^{\varphi_{2\ell+3}})$ can be written as:

$$\frac{(1 - t^{\varphi_{2\ell+3}})(1 - t^{\varphi_{2\ell+3}\varphi_{2\ell+5}n_1 \dots n_N})}{(1 - t^{\varphi_{2\ell+3}^2}) \cdot \varphi_{2\ell+3}} \cdot \sum_{\xi^{\varphi_{2\ell+3}}=1} \frac{\Delta(\xi t)}{(1 - \xi t)^2} = \frac{(1 - t^{\varphi_{2\ell+3}\varphi_{2\ell+5}n_1 \dots n_N})}{(1 - t^{\varphi_{2\ell+3}})^2}$$

where again $\Delta(t)$ is the characteristic polynomial of the rational unicuspidal curve of degree $\varphi_{2\ell+3}$ which has one characteristic pair $(a, b) = (\varphi_{2\ell+1}, \varphi_{2\ell+5})$, see (d) in Examples 1. Since the degree of $II^+(\ell, N; \lambda_1, \dots, \lambda_N)_{sp}$ is $\varphi_{2\ell+5}n_1 \dots n_N$ then Theorem 7 is proved. $\square$

7.   The case $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 1$ and $\nu = 1$.

7.1.   Tono's classification theorem.

We recall the following classification result of K. Tono [**39**]. Let $[x : y : z]$ be a system of homogeneous coordinates in $\mathbb{P}^2$. Assume that C is a unicuspidal rational plane curve with $\bar{\kappa} = 1$. Then C is projectively equivalent to one of the following curves C':

Type I. C' is given by

$$((f_1^s y + \sum_{i=2}^{s+1} a_i f_1^{s+1-i} x^{ia-a+1})^a - f_1^{as+1})/x^{a-1} = 0,$$

where $f_1 = x^{a-1}z + y^a$, $a \geq 3$, $s \geq 1$, $a_2, \ldots, a_{s+1} \in \mathbb{C}$ with $a_{s+1} \neq 0$.

In this case, $d = a^2 s + 1$, and the multiplicity sequence of (C', p) is $[(a^2 - a)s, (sa)_{2a-1}, a_{2s}]$.

Type IIa. C' is given by

$$((yf_2^n + x^{2n+1})^{4n+1} - f_3^{2n+1})/f_2^n = 0,$$

where $f_2 = xz - y^2$, $f_3 = f_2^{2n}z + 2x^{2n}yf_2^n + x^{4n+1}$ and $n \geq 2$.

In this case $d = 8n^2 + 4n + 1$ and the multiplicity sequence of (C', p) is $[(n(4n+1))_4, (4n+1)_{2n}, 3n+1, n_3]$.

Type IIb. C' is given by

$$((f_3^{2s-1}(f_2^n y + x^{2n+1}) + \sum_{i=1}^{s} a_i f_3^{2(s-i)} f_2^{i(4n+1)-n})^{4n+1} - f_3^{2((4n+1)s-n)})/f_2^n = 0,$$

where $n \geq 2$, $s \geq 1$, $a_1, \ldots, a_s \in \mathbb{C}$ with $a_s \neq 0$.

The degree of C' is $d = 2(4n+1)^2 s - 4n(2n+1)$. Set $a^* := 4n+1$ and $s^* := 4s-1$. The multiplicity sequence for $s = 1$ is $[(3na^*)_4, (3a^*)_{2n}, (a^*)_3, 3n+1, n_3]$, otherwise it is

$$[(s^*a^*n)_4, (s^*a^*)_{2n}, (sa^*)_3, (s-1)a^*, (a^*)_{2(s-1)}, 3n+1, n_3].$$

7.2. (C, p) in the case Type I.

After a computation, one has the following facts:

If $s = 1$, then (C', p) has two characteristic pairs. The decorations of the splice diagram (for notation see 3.5) are the following:

$$p_1 = a - 1, \; p_2 = a, \; a_1 = a, \; \text{and} \; a_2 = a(d + 1) + 1 = a^3 + 2a + 1.$$

In particular, the semigroup Γ is generated by the elements $a^2 - a$, a^2 and $a^3 + 2a + 1$.

If $s > 1$ the (C', p) has three pairs: $p_1 = a - 1$, $p_2 = s$, $p_3 = a$, $a_1 = a$, $a_2 = d$ and $a_3 = as(d + 1) + 1$. The semigroup is generated by

$$\bar{\beta}_0 = (a^2 - a)s, \; \bar{\beta}_1 = a^2 s, \; \bar{\beta}_2 = a(a^2 s + 1) = ad, \; \bar{\beta}_3 = 1 + as(d + 1) = a_3.$$

Notice that if in this second case we consider $s = 1$, we get a non-minimal splice diagram (with $p_2 = 1$) of the first case. In particular, any argument for general $s > 1$ can be adopted to the $s = 1$ situation by a simple substitution $s = 1$ and by elimination of the semigroup generator $\bar{\beta}_2 = ad$. Therefore, in the sequel we will consider the general $s \geq 1$ situation (which can be specialized to $s = 1$ as described above).

THEOREM 8. *Type I satisfies the distribution property (CP): $R(t) \equiv 0$.*

Proof. We will use the notations of 3.3. We will prove the inequality

$$\#\Gamma \cap I_{l+1} \leq 1 + \#\Gamma \cap I_l, \qquad\qquad (in_l)$$

for any l. Notice that this implies the negative distribution property, hence the vanishing of $R(t)$ would follow from Theorem 3.

Since $\Gamma \cap I_0 = \{0\}$ and $\Gamma \cap I_1 = \{(a^2 - a)s, a^2 s\}$, the inequality (in_0) follows. Since $d(d - 3) + 1$ is the largest element in $\mathbb{N} \setminus \Gamma$, (in_l) for $l \geq d - 2$ also follows. Next, we fix an l with $1 \leq l \leq d - 3$. Denote by $i_0 := (l - 1)d + 1$ and consider the

map

$$\phi_l : \Gamma \cap I_l \setminus \{i_0\} \to \Gamma \cap I_{l+1}$$

defined by $\phi_l(x) = x + \bar{\beta}_1 = x + a^2 s$. Clearly, ϕ_l is injective.

The following facts can be verified by elementary computations (using the short hints):

(a) Using 2.1 (especially the formula for Δ) one gets that any $\gamma \in \Gamma$ can be written in a unique way in the form

$$\gamma = k\bar{\beta}_3 + m\bar{\beta}_0 + t\bar{\beta}_2 + n\bar{\beta}_1, \tag{19}$$

where $0 \le k < a$, $0 \le m < a$, $0 \le t < s$ and $n \ge 0$. Here k is the remainder of γ modulo a.

(b) If in (19) $n > 0$ then $\gamma = \gamma' + a^2 s$ for some $\gamma' \in \Gamma$. In particular, $\Gamma \cap I_{l+1} \setminus im(\phi_l)$ has two types of elements:

Type A. $\gamma = k\bar{\beta}_3 + m\bar{\beta}_0 + t\bar{\beta}_2$ with $0 \le k < a$, $0 \le m < a$, $0 \le t < s$; or

Type B. $\gamma = (l+1)d$.

Here we do not exclude the situation when some semigroup element has both types. Let S_A (resp. S_B) be the set of elements of type A (resp. B).

(c) $\#S_A \le 2$.

Indeed, assume that $\gamma = k\bar{\beta}_3 + m\bar{\beta}_0 + t\bar{\beta}_2$ and $\gamma' = k'\bar{\beta}_3 + m'\bar{\beta}_0 + t'\bar{\beta}_2$ are both elements of S_A. If $k' < k$ then $(a-1)(a^2-a)s + (s-1)ad \ge m'\bar{\beta}_0 + t'\bar{\beta}_2 > ld - k'\bar{\beta}_3 \ge -d + (k-k')\bar{\beta}_3 \ge -d + \bar{\beta}_3$, a contradiction. Hence $k = k'$. Next, consider the difference $\gamma - \gamma' = (m - m')\bar{\beta}_0 + (t - t')da$. Since $|\gamma - \gamma'| < d$ and $|m - m'|\bar{\beta}_0 < (a-1)d$ one gets $|t - t'|ad < ad$, hence $t = t'$. Therefore, if $\gamma' > \gamma$ then $\gamma' = \gamma + (a^2 - a)s$ (since $2(a^2 - a)s \ge d$).

(d) One has the following identity for any γ of type A:

$$k\bar{\beta}_3 + m\bar{\beta}_0 + t\bar{\beta}_2 = (kas + m + at)d + (k - m)(1 + sa), \tag{20}$$

with $|k - m| \le a - 1$.

If $S_A = \emptyset$, or $S_A = \{\gamma\}$ and $(l+1)d \notin \Gamma$, or $S_A = \{(l+1)d\}$ then obviously (in_l) follows. Hence we only have to analyze the following two cases:

Case 1. $\#S_A = 2$.

Write $S_A = \{\gamma, \gamma'\}$ where $\gamma' = \gamma + (a^2 - a)s$ (see the proof of (c)). Set $\gamma = ld + r$ for some $r > 0$. Since $ld + r + (a^2 - a)s = \gamma' \le ld + d$, one gets $r \le as + 1$.

Case 1a. First we verify that the case $0 < r \le as$ cannot occur. Indeed, assume that $r \le as$ and write γ in the form (20). Here there are two possibilities: either $kas + m + at = l$ and $k - m > 0$, or $kas + m + at = l + 1$ and $k - m < 0$. The first possibility is eliminated by $r = (k-m)(as+1) > as \ge r$. Hence $kas + m + ta = l + 1$, and $d - r = (m - k)(1 + as)$. If $m - k \le a - 2$ then $a^2 s + 1 - as \le d - r \le (a-2)(1 + sa)$, a contradiction. Finally, if $m - k = a - 1$ then necessarily $k = 0$ and $m = a - 1$, hence $\gamma' = t\bar{\beta}_2 + (a-1)\bar{\beta}_1$ is not of type A, again a contradiction.

Case 1b. Therefore, $r = as + 1$. In other words, $\gamma = ld + as + 1$ and $\gamma' = (l+1)d$. In particular, γ' is of also of type B. This implies that $\#\Gamma \cap I_{l+1} = 2 + \#im(\phi_l)$. Next, write γ as in (20). Then an elementary computation shows that $kas + m + at = l + 1$

is not possible. Therefore $kas + m + at = l$ and $k - m = 1$. In particular,

$$i_0 = m\bar{\beta}_3 + (m + a - 1)\bar{\beta}_0 + t\bar{\beta}_2 + (a(s-1)+1)\bar{\beta}_1,$$

hence $i_0 \in \Gamma \cap I_l$. This implies $\#\Gamma \cap I_l = 1 + \#im(\phi_l)$, hence (in_l) follows with equality.

Case 2. $S_A = \{\gamma\}$, $(l+1)d \in \Gamma$ and $\gamma \neq (l+1)d$.

Our goal is to show that $i_0 \in \Gamma$. Since $(l+1)d$ is not of type A, it follows that $(l+1)d = \tilde{\gamma} + a^2 s$ for some $\tilde{\gamma} \in \Gamma$ – in fact, for $\tilde{\gamma} = ld + 1$. Hence we have to verify the following fact: If $\tilde{\gamma} = ld + 1 \in \Gamma$, and there exists $\gamma \neq (l+1)d$ of type A in $\Gamma \cap I_{l+1}$, then $i_0 \in \Gamma$.

Write $\gamma = k\bar{\beta}_3 + m\bar{\beta}_0 + t\bar{\beta}_2$ with $0 \leq k < a$, $0 \leq m < a$, $0 \leq t < s$. Similarly, set some representation $\tilde{\gamma} = k'\bar{\beta}_3 + m'\bar{\beta}_0 + t'\bar{\beta}_2 + n'\bar{\beta}_1$. Here we do not impose any inequality for the coefficients k', t', m', n'; hence k' is uniquely determined by $\tilde{\gamma}$, but m' and n' not (because of the relation $a(a^2 - a) = a^2(a - 1)$). Notice also that $i_0 = \tilde{\gamma} - d$. Therefore, the identity

$$d = \bar{\beta}_3 + 2\bar{\beta}_0 - (1 + sa)\bar{\beta}_1$$

shows that if there is a representation of $\tilde{\gamma}$ with $k' \geq 1$ and some $m' \geq 2$ then $i_0 \in \Gamma$.

First we verify that (in any representation of $\tilde{\gamma}$ as above) $k' \geq 1$. Indeed, assume that $k' = 0$. Then (taking $\tilde{\gamma} = ld + 1$ modulo a) one gets that $l = -1 + La$ for some $1 \leq L < as$. Since $ld + 1 \leq \gamma \leq (l+1)d - 1$, one has

$$La(a^2 s + 1) - a^2 s \leq k\bar{\beta}_3 + m\bar{\beta}_0 + t\bar{\beta}_2 \leq La(a^2 s + 1) - 1. \qquad (21)$$

This implies that

$$\left\lceil \frac{k\bar{\beta}_3 + m\bar{\beta}_0 + t\bar{\beta}_2}{a(a^2 s + 1)} \right\rceil = L.$$

But one also has

$$(t + ks)a(a^2 s + 1) < k\bar{\beta}_3 + m\bar{\beta}_0 + t\bar{\beta}_2 \leq (t + ks + 1)a(a^2 s + 1).$$

Hence $L = t + ks + 1$. But for this L the left inequality of (21) fails.

Next, we show that there exists a representation with $m' \geq 2$. E.g., if in some "bad" representation of $\tilde{\gamma}$ one has $m' = 0$, then taking $k'\bar{\beta}_3 + t'\bar{\beta}_2 + n'\bar{\beta}_1$ modulo d one gets $n' \equiv -1 + k'(as + 1)$ modulo d. Since $1 \leq k' \leq a - 1$, this implies that $n' \geq as$, hence $n'a^2$ can be rewritten as $(n' - a + 1)a^2 + a(a^2 - a)$. Similar argument works for $m' = 1$ as well. $\square$

7.3. *Addendum to the proof of Theorem 8.*

In the verification of the conjecture for Type II curves we will use the results valid for the Type I curves. We will need the following additional facts.

Assume that (C', p) is of Type I as above, and we keep the notations of 7.2 and 8. 8 shows that $(*) \#\Gamma \cap [0, ld] = (l+1)(l+2)/2$ for any $0 \leq l < d$. We first assume that $d \equiv 0$, $a \equiv 1$ and $s \equiv -1 \bmod 4$.

Recall that any $\gamma \in \Gamma$ can be written in a unique way as $\gamma = k\bar{\beta}_3 + m\bar{\beta}_0 + t\bar{\beta}_2 + n\bar{\beta}_1$. It is easy to see that $\gamma \equiv -n \bmod 4$. Set $\#_{l,i} := \#\{\gamma \in \Gamma, \ \gamma \leq ld, \ n \equiv i \bmod 4\}$ for $i = 0, 1, 2, 3$ and $0 \leq l < d$. Then, using the function $x \mapsto x + \bar{\beta}_1 = x + d - 1$,

one gets

$$\#_{l,1} = \#_{l-1,2}, \ \#_{l,2} = \#_{l-1,3}, \ \#_{l,3} = \#_{l-1,0}, \ \#_{l,0} = \#_{l-1,1} + A$$

for some $A \in \mathbb{N}$. Here one uses that $4|ld$. Taking the sum over i and using $(*)$, one gets that $A = l + 1$. Hence $\#_{l,i}$ can be computed inductively. In particular, for any $l < d$, $\#_{l,0} = (l + 1) + \#_{l-4,0} = (l + 1) + (l - 3) + \#_{l-8,0} = \ldots$. More precisely, if $l = 4k + r$ (with $0 \leq r \leq 3$), then

$$\#_{l,0} = (k + 1)(2k + r + 1) \text{ for any } l < d.$$

E.g., this can be rewritten into

$$\#_{2l,0} = (l + 1)(l + 2)/2, \text{ whenever } 2l < d. \qquad (\#_{2l})$$

The very last identity is true even if $s = 1$, $a \equiv 1$ and $d \equiv 2 \ \mathrm{mod}\, 4$. Indeed, in this case $\bar{\beta}_2$ is eliminated, $\bar{\beta}_0 \equiv \bar{\beta}_3 \equiv \bar{\beta}_1 - 1 \equiv 0 \ \mathrm{mod}\, 4$, hence $\gamma \equiv n \ \mathrm{mod}\, 4$. Therefore, $\#_{l,1} = \#_{l-1,0}$ and $\#_{l,3} = \#_{l-1,2}$ similarly as above, since ld is even. Although $\#_{l,2} = \#_{l-1,1}$ may not be true in general, but for l even it is true. Writing $\#_{l,0} = \#_{l-1,3} + A$, the formula for $(\#_{2l})$ works again by a similar argument as above.

7.4. (C', p) in the case Type II.

One can verify the following facts:

In case IIa one has two characteristic pairs, the splice decorations are: $p_1 = n$, $p_2 = 4n + 1$, $a_1 = 4n + 1$ and $a_2 = (2n + 1)d - n(4n + 1)$.

In case IIb one has three characteristic pairs, and the splice decorations can be uniformly described for any $s \geq 1$. They are: $p_1 = n$, $p_2 = s^*$, $p_3 = a^*$, $a_1 = a^*$, $a_2 = d/2$ and $a_3 = a^*s^*d/2 + (s - 1)a^* + 3n + 1$.

In fact, the IIa case also can be considered as a 'specialization': if in the formulas of IIb (including the formula of d as well) we substitute $s = 1/2$ then we get the non-minimal splice diagram (with $p_2 = s^* = 1$) of the IIa case. Hence in the sequel we will handle the case II uniformly (using the formulas of IIb, where s is any positive integer or $1/2$).

It is easy to verify that for Type II the degree d satisfies $2d = (a^*)^2 s^* + 1$, and the generators of Γ are

$$\bar{\beta}_0 = \frac{(a^*)^2 - a^*}{4} s^*; \quad \bar{\beta}_1 = 2d - 1 = (a^*)^2 s^*; \quad \bar{\beta}_2 = a^* d/2; \quad \bar{\beta}_3 = \frac{a^* s^*}{4}(2d + 1) + \frac{1}{4}.$$

If $s = 1/2$, or $s^* = 1$, then one should eliminate $\bar{\beta}_2$.

THEOREM 9.   *The Conjecture is true for Type II curves.*

Proof.   One can also think about the integers a^* and s^* as parameters of a Type I curve (a^*, s^* corresponding to a and s) with semigroup Γ^*. Their other Type I invariants are (cf. 7.2):

$$d^* = (a^*)^2 s^* + 1; \ \bar{\beta}_0^* = ((a^*)^2 - a^*)s^*; \ \bar{\beta}_1^* = d^* - 1 = (a^*)^2 s^*;$$

$$\bar{\beta}_2^* = a^* d^*; \ \bar{\beta}_3^* = a^* s^*(d^* + 1) + 1.$$

Notice that

$$d^* = 2d, \ \bar{\beta}_0^* = 4\bar{\beta}_0, \ \bar{\beta}_1^* = \bar{\beta}_1; \bar{\beta}_2^* = 4\bar{\beta}_2; \ \bar{\beta}_3^* = 4\bar{\beta}_3.$$

We fix an integer $l < d$ and we wish to determine

$$\#\{\gamma \in \Gamma : \gamma = k\bar{\beta}_3 + m\bar{\beta}_0 + t\bar{\beta}_2 + n\bar{\beta}_1 \leq ld\}.$$

Multiplying by 4, one gets that this equals

$$\#\{\gamma^* \in \Gamma^* : \gamma^* = k\bar{\beta}_3^* + m\bar{\beta}_0^* + t\bar{\beta}_2^* + 4n\bar{\beta}_1^* \leq 2ld^*\}.$$

If $s \geq 1$, then $d^* \equiv 0$; $a^* \equiv 1$; $s^* \equiv -1 \bmod 4$. If $s = 1/2$, then $d^* \equiv 2$ and $a^* \equiv 1$ $\bmod 4$, and $s^* = 1$. Hence this wanted number is $\#_{2l,0}$ computed in 7.3 (notice that $2l < 2d = d^*$), and equals $(l+1)(l+2)/2$. Hence

$$\#\{\gamma \in \Gamma : \gamma = k\bar{\beta}_3 + m\bar{\beta}_0 + t\bar{\beta}_2 + n\bar{\beta}_1 \leq ld\} = (l+1)(l+2)/2$$

for any s. In particular, conjecture follows for Type II as well. $\qquad\square$

8. Lin-Zaidenberg curves: $\bar{\kappa} = 1$ and $\nu = 2$.

In this section we use an equivalent definition of *Lin-Zaidenberg type* curves given in terms of automorphisms of the affine plane, (see section 4 for another definition). Consider a coordinate system x, y in $\mathbb{C}^2$. Take a pair (p, q) of relatively prime numbers with $p > q$. Let $C_{p,q}$ be the curve of $\mathbb{C}^2$ defined by the equation $y^p + x^q = 0$. Consider $\mathbb{P}^2$ together with a projective reference (X, Y, Z). We embed $\mathbb{C}^2$ into $\mathbb{P}^2$ declaring that the image of the embedding is the open subset U_Z defined by $Z \neq 0$ and that $(x, y) = (X/Z, Y/Z)$. We will denote by the same symbol the curve $C_{p,q}$ and its compactification to $\mathbb{P}^2$. Let $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ be an algebraic automorphism. The embedding of $\mathbb{C}^2$ into $\mathbb{P}^2$ allows us to view any automorphism of $\mathbb{C}^2$ as a birational transformation of $\mathbb{P}^2$.

DEFINITION 2. A curve C is said to be of *Lin-Zaidenberg type*, (*LZ type* for short) if it is equal to $\phi^{-1}(C_{p,q})$ for a certain automorphism ϕ viewed as birational transformation of $\mathbb{P}^2$. (Taking the inverse ϕ^{-1} instead of ϕ is just for notational convenience).

For a rational bicuspidal plane curve, K. Tono [**38**] has proved that C is of *LZ type* if and only if $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 1$.

THEOREM 10. *(CP) is satisfied by any LZ type curve; in other words $R(t) \equiv 0$.*

Let C be a *LZ type* curve. The curve C has two singularities, one at the origin of $\mathbb{C}^2$ and the other at infinity. Let $L_1(t)$ and $L_2(t)$ denote respectively the Poincaré series of their corresponding semigroups, see 2.1. From the very definition (4) of $R(t)$, the vanishing $R(t) \equiv 0$ is equivalent to

$$\frac{1}{d} \sum_{\xi^d = 1} L_1(\xi t) L_2(\xi t) = \frac{1 - t^{d^2}}{(1 - t^d)^3}. \tag{22}$$

The singularity at the origin is isomorphic to $y^p + x^q = 0$, therefore

$$L_1(t) = \frac{(1 - t^{pq})}{(1 - t^p)(1 - t^q)}. \tag{23}$$

The combinatorics of the minimal embedded resolution of the singularity at infinity of C are closely related to the combinatorics of the minimal resolution of the

indeterminacy of ϕ as a birational transformation of $\mathbb{P}^2$, and to the combinatorics of the minimal embedded resolution of the singularity that $C_{p,q}$ has at infinity.

We will need the facts on algebraic automorphisms of $\mathbb{C}^2$ from section 4; we refer to [**10**], [**11**] for details and more precise statements. Let $\pi : X \to \mathbb{P}^2$ the minimal resolution of the indeterminacy of ϕ. Let $L := \mathbb{P}^2 \setminus \mathbb{C}^2$ denote the line at infinity and consider $\mathcal{A}$ the dual graph of the normal crossing divisor π^*L. Recall that we have a total order of the vertices of the graph (this order is recovered combinatorially by successively contracting (-1) vertices and adjusting weights at each step). Denote by $\mathcal{A}^*$ the graph $\mathcal{A}$ minus its first vertex (the one corresponding to the line at infinity). The automorphism ϕ admits a unique decomposition as $\psi \circ H$ where H is an affine transformation of $\mathbb{C}^2$ and ψ is the automorphism of $\mathbb{C}^2$ which is composition of π^{-1} and the successive contraction of all the irreducible components of π^*L with self intersection (-1). Moreover all the components except the one corresponding to the last vertex of $\mathcal{A}$ are contracted. For our purposes it is sufficient to consider automorphisms for which H is the identity; we will assume this in the sequel. The automorphism ϕ admits a decomposition as $\phi = \phi_r \circ ... \circ \phi_1$, where ϕ_i is the result considering first the blowing ups whose exceptional divisors belong to the i-th floor and then contracting all possible components with self-intersection (-1). The graph associated to the resolution of indeterminacy of ϕ_i is the elementary graph of length n_i.

The singularity at infinity of $C_{p,q}$ is at the point $(1 : 0 : 0)$ of $\mathbb{P}^2$ and is defined by the equation $y^p + z^{p-q} = 0$. Let $\sigma : Y \to \mathbb{P}^2$ the composition of blowing ups giving its minimal embedded resolution of this singularity. By the irreducibility of $C_{p,q}$ at $(1 : 0 : 0)$ the blowing ups whose composition gives rise to σ are totally ordered. Let $E = \sigma^{-1}(1 : 0 : 0)$ be the exceptional divisor of σ; denote by $\mathcal{G}$ its dual graph. Each vertex of $\mathcal{G}$ corresponds to an irreducible component of E, and hence to one of the blowing-ups whose composition is σ, and therefore they are totally ordered. It is known that $\mathcal{G}$ is a linear graph whose first vertex is one of the extremes. Although the ordering given to the vertices is not the same that the order given by the linearity of the graph we will denote by v_1 and v_2 the two extremal vertices of the graph (the vertex v_1 is the first vertex in the given order, but v_2 need not be the second). We decorate the graph adding an arrow to the vertex corresponding with the irreducible component of E.

We must distinguish two cases:

Case 1: the first indetermination point of ϕ^{-1} is different to the point at infinity of $C_{p,q}$. Let $\psi : Z \to \mathbb{P}^2$ be the blowing up process giving the minimal embedded resolution of the singularity that C has at infinity. It is easy to check that ψ is the sequence of blowing ups needed to resolve the indeterminacy of ϕ, followed by the sequence of blowing ups providing the embedded resolution of the singularity of $C_{p,q}$ at infinity. The divisor ψ^*C can be decomposed as

$$\psi^*C = \tilde{C} + \sum_{i=1}^{k} m_k E_k,$$

where $\tilde{C}$ is the strict transform of C and $E_1,..., E_k$ the irreducible components of the exceptional divisor ordered by appearance.

In this case it is easy to see that the decorated dual graph $\mathcal{B}$ of the exceptional divisor is obtained by joining with an edge the last vertex of $\mathcal{A}^*$ with the first vertex v_1 of $\mathcal{G}$. We give a total order to the vertices of the resulting graph are totally

ordered by considering first the vertices of $\mathcal{A}^*$ and after the vertices of $\mathcal{G}$ with the previously given order. This order coincides with the natural order obtained defined identifying the vertices with the divisors $E_1,...,E_k$.

With this information it is easy to compute the multiplicity sequence of the singularity, and to deduce from it the degree of C and the coefficients $m_1,...,m_k$. The degree is $d := \deg(C) = pn_1\ldots n_r$. For computing the Poincaré series L_2 we need to know the coefficients m_i of the univalent and trivalent vertices of $\mathcal{B}$. We observe that the univalent vertices are the first vertex of each floor of $\mathcal{A}^*$ (which in the picture are the univalent central vertices of the floors), the second vertex of the first floor, and the vertex v_2 of $\mathcal{G}$. We have exactly one trivalent vertex for each floor in addition to the last vertex of $\mathcal{G}$, where the arrow decorating the graph is attached. The Poincaré series of the semigroup of the singularity at infinity obtained is:

$$L_2(t) = \frac{(1 - t^{n_1^2\ldots n_r^2 p^2 - pq})\prod_{i=1}^r(1 - t^{(n_1^2\ldots n_i^2 - 1)n_i\ldots n_r p})}{(1 - t^{n_1^2\ldots n_r^2 p - q})(1 - t^{n_1\ldots n_r p})\prod_{i=1}^r(1 - t^{(n_1^2\ldots n_i^2 - 1)n_{i+1}\ldots n_r p})}. \tag{24}$$

Case 2: the first indetermination point of ϕ^{-1} is equal to the point at infinity of $C_{p,q}$. We can further assume that the singularities at infinity of $\phi_r^{-1}(C_{p,q})$ and C do not have the same resolution graph: if this were the case we could work with $\phi_r^{-1}(C_{p,q})$ instead of $C_{p,q}$ and with $\phi_{r-1} \circ \ldots \circ \phi_1 \circ H$ instead of ϕ, and we would be in Case 1. Let k be the only integer such that $p - kq < q < p - (k-1)q$; then the condition that $\phi_r^{-1}(C_{p,q})$ and C do not have the same resolution graph is equivalent to $n_r \geq k + 1$.

Let $\pi' : X \to \mathbb{P}^2$ be the minimal blowing-up process resolving the indeterminacy of ϕ^{-1}. Viewing ϕ as a composition of a sequence of blowing ups followed by a sequence of contractions it is clear that the dual graph of $(\pi')^*L$ can be naturally identified with the graph $\mathcal{A}$, but with a different ordering of the vertices (in particular the first vertices are those of the r-th floor of $\mathcal{A}$). Noticing that $n_r \geq k + 1$, a comparison between the blowing up sequences π' and σ shows that their first $k + 1$ blowing up processes are the same in both sequences. Call σ' the composition of these blowing-ups and let W:=$\{w_1', ..., w_{k+1}'\}$ be the ordered set of vertices of $\mathcal{G}$ corresponding to the exceptional divisors of these blowing-ups. We can decompose σ as $\sigma = \sigma_2 \circ \sigma_1$, where σ_1 is the composition of blowing-ups whose associated vertex does not belong to W and σ_2 is the composition of the remaining blowing-ups. Suppose that w_i' is the first of the vertices of W such that its associated exceptional divisor meets the strict transform of $C_{p,q}$ by σ_2. Clearly w_i' can be regarded also as a vertex of $\mathcal{A}$. Denote by π_1 the composition of those blowing ups of π such that the vertex associated to their exceptional divisor is smaller or equal than w_i' with the ordering in $\mathcal{A}$ induced by π.

Let $\psi : Z \to \mathbb{P}^2$ be the blowing up process giving the minimal embedded resolution of the singularity that C has at infinity. Then it is easy to check that $\psi = \pi_1 \circ \sigma_1$. Due to this decomposition the decorated dual graph $\mathcal{B}$ of the exceptional divisor of ψ can be obtained as follows:

Let $\mathcal{A}'$ be the graph obtained from $\mathcal{A}^*$ by deleting the last k vertices. If $k > 1$ let $\mathcal{G}'$ be the graph obtained from $\mathcal{G}$ by deleting the first $k - 1$ vertices; if $k = 1$ let $\mathcal{G}'$ be the graph obtained by deletion of the first vertex of $\mathcal{G}$. The graph $\mathcal{G}'$ is linear. If $k = 1$ then its first vertex is a extreme of it, and is denoted by w. If $k > 0$ both its first and second vertices are extremes of it; in this case we alter the ordering in the vertices of $\mathcal{G}'$ by interchanging the order of the first two vertices, and denote by w the first vertex of $\mathcal{G}'$ with the altered order. The graph $\mathcal{B}$ is is obtained by

identifying (not joining with an edge) the last vertex of $\mathcal{A}'$ with the vertex w of $\mathcal{G}'$. We give a total order to the vertices of the resulting graph by considering first the vertices of $\mathcal{A}'$ and after the vertices of $\mathcal{G}'$ with the altered order. This order coincides with the natural order obtained defined identifying the vertices with the irreducible components of the exceptional divisor of ψ. The univalent vertices of $\mathcal{B}$ are the first vertex of each floor of $\mathcal{A}$ (in the picture these are the univalent central vertices of the floors), the second vertex of the first floor, and the extreme of $\mathcal{G}'$ different from w. We have exactly one trivalent vertex for each floor in addition to the last vertex of $\mathcal{G}'$, where the arrow decorating the graph is attached. The degree of C is $d = n_1 \ldots n_r q$ and the Poincaré series of the singularity at infinity is:

$$L_2(t) = \frac{(1 - t^{n_1^2 \ldots n_r^2 q^2 - pq}) \prod_{i=1}^r (1 - t^{(n_1^2 \ldots n_i^2 - 1) n_i \ldots n_r q})}{(1 - t^{n_1^2 \ldots n_r^2 q - p})(1 - t^{n_1 \ldots n_r q}) \prod_{i=1}^r (1 - t^{(n_1^2 \ldots n_i^2 - 1) n_{i+1} \ldots n_r q})}. \tag{25}$$

REMARK 3.   The formulas for the degree and $L_2(t)$ obtained in Case 2 are the result of interchanging the role of p and q in the formulas obtained in Case 1.

Due to (22), (23), (24), (25) and 3, we need to prove the following fact. For any two coprime positive integers p and q (without imposing $p > q$), for $n_1, \ldots, n_r$ positive integers and $d := n_1 \ldots n_r p$ the following identity holds:

$$\frac{1}{d} \sum_{\xi^d = 1} \frac{(1 - (\xi t)^{pq})(1 - (\xi t)^{n_1^2 \ldots n_r^2 p^2 - pq})}{(1 - (\xi t)^p)(1 - t^q)(1 - (\xi t)^{n_1^2 \ldots n_r^2 p - q})(1 - (\xi t)^{n_1 \ldots n_r p})}$$

$$\cdot \prod_{i=1}^r \frac{(1 - (\xi t)^{(n_1^2 \ldots n_i^2 - 1) n_i \ldots n_r p})}{(1 - (\xi t)^{(n_1^2 \ldots n_i^2 - 1) n_{i+1} \ldots n_r p})} = \frac{1 - t^{d^2}}{(1 - t^d)^3}.$$

Multiplying in both sides by $d(1 - t^d)^3$ and extracting common factors of numerator and denominator we see that the last equality is equivalent to:

$$(1 - t^d) \sum_{\xi^d = 1} [(\sum_{j=0}^{p-1} (\xi t)^{jq})(\sum_{j=0}^{p-1} (\xi t)^{j(n_1^2 \ldots n_r^2 p - q)})(\sum_{j=0}^{n_1 \ldots n_r - 1} (\xi t)^{jp}) \tag{26}$$

$$\cdot \prod_{i=1}^r (\sum_{j=0}^{n_i - 1} (\xi t)^{j(n_1^2 \ldots n_{i-1}^2 n_i - 1) n_{i+1} \ldots n_r p})] = d(1 - t^{d^2}).$$

The left hand side of (26) is equal to $(1 - t^d)$ times

$$\sum_{\xi^d = 1} [\sum_{i,j=0}^{p-1} \sum_{l=0}^{n_1 \ldots n_r - 1} \sum_{\alpha=1}^r \sum_{k_\alpha = 0}^{n_\alpha - 1} (\xi t)^{iq + j(n_1^2 \ldots n_r^2 p - q) + lp + \sum_{\alpha=1}^r k_\alpha (n_1^2 \ldots n_{\alpha-1}^2 n_\alpha - 1) n_{\alpha+1} \ldots n_r p}].$$

$$\tag{27}$$

The exponent of (ξt) in the last expression can be written as $A(i, j, l, \mathbf{k}) + dB(i, j, l, \mathbf{k})$, for

$$A(i, j, l, \mathbf{k}) := (i - j)q + lp - \sum_{\alpha=1}^r k_\alpha n_{\alpha+1} \ldots n_r p,$$

$$B(i, j, l, \mathbf{k}) := jn_1 \ldots n_r + \sum_{\alpha+1}^r k_\alpha n_1 \ldots n_{\alpha-1}.$$

But $\sum_{\xi^m=1} \xi^m = 0$ for any integer m non-divisible by d. In the present case $m = A(i,j,l,\mathbf{k})$.

LEMMA 2.   *If $A(i,j,l,\mathbf{k})$ is divisible by d then $i = j$ and $l = \sum_{\alpha=1}^{r} n_{\alpha+1} \dots n_r$.*

As $d = n_1 \dots n_r p$ we deduce that $p | A(i,j,l,\mathbf{k})$. It immediately follows that $i = j$ and that

$$n_1 \dots n_r | l - \sum_{\alpha=1}^{r} k_\alpha n_{\alpha+1} \dots n_r.$$

We claim that then $0 = l - \sum_{\alpha=1}^{r} k_\alpha n_{\alpha+1} \dots n_r$. The claim is proved by induction on r. For $r = 1$ it is obvious since $|l - k_1| < n_1$. For the induction step we write

$$l - \sum_{\alpha=1}^{r} k_\alpha n_{\alpha+1} \dots n_r = l - k_r + n_r \sum_{\alpha=1}^{r-1} k_\alpha n_{\alpha+1} \dots n_{r-1};$$

as $n_1 \dots n_r$ divides the last quantity we have $n_r | l - k_r$. Therefore we can express l as $l = k_r + n_r l'$. Then

$$n_1 \dots n_{r-1} | l' - \sum_{\alpha=1}^{r-1} k_\alpha n_{\alpha+1} \dots n_{r-1},$$

and we conclude by induction.

Therefore, by Lemma 2 and a computation, the left hand side of (26) becomes

$$(1 - t^d) \cdot d \sum_{j=0}^{p-1} \sum_{\alpha=1}^{r} \sum_{k_\alpha=0}^{n_\alpha-1} t^{jn_1^2 \dots n_r^2 p + \sum_{\alpha=1}^{r} n_1^2 \dots n_{\alpha-1}^2 n_\alpha \dots n_r p}.$$

A computation shows that the result equals $d(1 - t^{d^2})$, as desired.

9.   *(CP) for Orevkov's curves.*

In this section we will consider the families $\{C_{4k}\}$ and $\{C_{4k}^*\}$ ($k \geq 2$) of Orevkov [**34**]. There is a special interest in these curves: Orevkov proved that they satisfy $d > \alpha m$ (where $\alpha = (3 + \sqrt{5})/2$ as above); moreover he conjectured that these are the only rational cuspidal curves (together with the curves C_4 and C_4^* from Examples 1(e)-(f)) satisfying this inequality. Also, for these curves one has $\bar{\kappa}(\mathbb{P}^2 \setminus C) = 2$, in contrast with the previous sections 3-7. In particular, Theorem 11 shows that (CP) is not a speciality of curves with $\bar{\kappa} < 2$ (and of some finitely many sporadic curves of general type).

In this case the number of characteristic pairs of (C,p) is again two, which makes the structure of the semigroups rather interesting. The numerical invariants of the curves are the following (cf. [**34**]). In both cases $j \equiv 0 \mod 4$, and $j \geq 8$.

9.1. C_j has degree $d = \varphi_{j+2}$.

The germ (C_j,p) has two characteristic pairs with numerical invariants:

$$p_1 = \varphi_j/3, \ p_2 = 3; \ a_1 = \varphi_{j+4}/3, \ a_2 = 1 + \varphi_j \varphi_{j+4}/3;$$

$$\bar{\beta}_0 = \varphi_j, \ \bar{\beta}_1 = \varphi_{j+4}, \ \bar{\beta}_2 = a_2.$$

9.2. C_j^* has degree $d^* = 2\varphi_{j+2}$.

The germ (C_j, p) has two characteristic pairs with numerical invariants:

$$p_1 = \varphi_j/3, \ p_2^* = 6; \ a_1 = \varphi_{j+4}/3, \ a_2^* = 1 + 2\varphi_j\varphi_{j+4}/3;$$

$$\bar{\beta}_0^* = 2\varphi_j, \ \bar{\beta}_1^* = 2\varphi_{j+4}, \ \bar{\beta}_2^* = a_2^*.$$

THEOREM 11. C_{4k} and C_{4k}^* satisfy (CP).

Proof.   We start with the case C_j.

Step 1. Assume that $l < d$. Notice that $\gcd(\bar{\beta}_1, d) = 1$, hence $k\bar{\beta}_1 = ld$ implies $l = k = 0$. In all the other cases, by a similar argument as in the proof of 5, $k\bar{\beta}_1 < ld$ is equivalent to $k/l < 1/\alpha$. Hence $k\bar{\beta}_1 \in I_l$ if and only if $\lceil k\alpha \rceil = l$.

Step 2. Assume $l < d$. If $i\bar{\beta}_0 > ld$ then $i/l > \varphi_{j+2}/\varphi_j > \alpha$. Conversely, if $i/l > \alpha$ then consider $\varphi_{j+2}/\varphi_j > \varphi_{j+4}/\varphi_{j+2} > \alpha$. Similarly as in the proof of 5, we get that $i/l \geq \varphi_{j+2}/\varphi_j$. Hence, if $l \notin \{\varphi_j, \varphi_j + 1\}$ then $i\bar{\beta}_0 \in I_l$ if and only if $\lceil i/\alpha \rceil = l$.

Step 3. Notice that $x := \varphi_j\varphi_{j+4}/3 = (\varphi_{j+2}^2 - 1)/3$ is not a multiple of d, hence x and $a_2 = x + 1$ are in the same interval I_{l_0}. Let Γ_0 be the semigroup generated by $\bar{\beta}_0$ and $\bar{\beta}_1$. Hence for $l < l_0$ one has $\Gamma_0 \cap I_l = \Gamma \cap I_l$.

Moreover, $l_0 - 1 < \varphi_j$. Indeed, $(l_0 - 1)\varphi_{j+2} < x = (\varphi_{j+2}^2 - 1)/3 < \varphi_{j+2}\varphi_j$ (because $\varphi_{j+2} < 3\varphi_j$).

Notice also that the very first semigroup element $i\varphi_j + k\varphi_{j+4}$ of Γ_0 which can be written in two different ways is $x = (\varphi_{j+4}/3)\varphi_j = (\varphi_j/3)\varphi_{j+4}$.

Hence, for any $l \leq l_0 - 1$, by similar argument as in the proof of 5(d), and using Step 1 and Step 2, we get that the distribution patern is true.

The proof now bifurcates in two cases.

First assume that $d = \varphi_{j+2} = 3t - 1$ for some t. Then $x = 3t^2 - 2t$ and $l_0 = t$.

Step 4. One can verify that the following numbers are in increasing order: $x - 2\varphi_j$, $(l_0 - 1)d$, $x - \varphi_j$, x, $l_0 d$, $x + \varphi_j$, $x + 3\varphi_j$, $(l_0 + 1)d$, $x + 4\varphi_j$, $x + 6\varphi_j$, $(l_0 + 2)d$, $x + \varphi_{j+4}$, $x + 7\varphi_j$. E.g., $x + \varphi_j > l_0 d$ reduces to $\varphi_j > t$, or to $3\varphi_j > \varphi_{j+2} + 1$ which is true. The other verifications are similar.

Step 5. Since $\varphi_{j+4} + \varphi_j = 3\varphi_{j+2}$, the map $s_l : I_{l-3} \cap \Gamma_0 \to I_l \cap \Gamma_0$, given by $y \mapsto y + \varphi_{j+4} + \varphi_j$ is well-defined. Clearly, it is injective. For $l = l_0$ the complement of its image has two elements, namely x and $x - \varphi_j$ (cf. Steps 3 and 4).

For $l > l_0$ the map s_l is surjective. This follows from Step 4 and the identities $i\varphi_j = (i - \varphi_{j+4}/3)\varphi_j + (\varphi_j/3)\varphi_{j+4}$ (for $i > \varphi_{j+4}/3$) and $k\varphi_{j+4} = (k - \varphi_j/3)\varphi_{j+4} + (\varphi_{j+4}/3)\varphi_j$ (for $k > \varphi_j/3$). Therefore

$$\sum_{k \in \Gamma_0} t^{\lceil k/d \rceil} = 1 + 2t + \ldots + l_0 t^{l_0 - 1} + \left(l_0 + (l_0 - 1)t + l_0 t^2\right) \cdot (t^{l_0} + t^{l_0 + 3} + t^{l_0 + 6} + \ldots).$$

Step 6. Consider now the intervals $l_0 \leq l \leq 2l_0 - 1$. Since (cf. Step 4) $2x + d > 2x + 2\varphi_j > 2l_0 d$, we get that $2a_2$ is not situating in these intervals. By a mod 3 argument, for any such l, $\Gamma \cap I_l$ is the disjoint union of $I_l \cap \Gamma_0$ and $I_l \cap (a_2 + \Gamma_0)$. Moreover, in all these intervals any element of $a_2 + \Gamma_0$ has a unique representation in the form $a_2 + i\varphi_j + k\varphi_{j+4}$.

In particular, we have to understand the distribution of $a_2 + \Gamma_0$. By a similar computation as in Step 4, we get that I_{l_0} contains only a_2, I_{l_0+1} contains $a_2 + i\varphi_j$

for $i = 1, 2, 3$; I_{l_0+2} contains $a_2 + i\varphi_j$ for $i = 4, 5, 6$. In particular, for intervals $l = l_0, l_0 + 1, l_0 + 2$ the distribution of Γ follows.

Step 7. Assume again $l_0 \leq l \leq 2l_0 - 1$. Consider the injective map $s'_l : I_{l-3} \cap (a_2 + \Gamma_0) \to I_l \cap (a_2 + \Gamma_0)$ given by $y \mapsto y + \varphi_j + \varphi_{j+4}$. We claim that for $l_0 - 3 \leq l \leq 2l_0 - 1$ the complement of the image of s'_l has exactly three elements. There are two types of elements in the complement. The first type is $a_2 + k\varphi_{j+4} = 1 + (k + \varphi_j/3)\varphi_{j+4} = 1 + k'\varphi_{j+4}$. Since $k'\varphi_{j+4}$ is never a multiple of d (in the relevant intervals) we get (via Step 1) that $1 + k'\varphi_{j+4} \in I_l$ if and only if $k'\varphi_{j+4} \in I_l$ if and only if $\lceil k'\alpha \rceil = l$.

The other type is $a_2 + i\varphi_j = 1 + (i + \varphi_{j+4}/3)\varphi_j = 1 + i'\varphi_j$. Notice that $i'\varphi_j$ can be a multiple of d in these intervals, namely for $i' = d$. This fact combined with Step 2 we get that $1 + i'\varphi_j \in I_l$ if and only if $\lceil i'/\alpha \rceil = l$. Now, by the end of the proof of 5(d) the above claim follows.

In particular, the distribution is true for any $l < 2l_0$.

Step 8. Since (CP_l) is true if and only if (CP_{d-2-l}) is true, one gets (CP_l) for all the remaining cases.

The proof in the other case $d = \varphi_{j+2} = 3t + 1$ is similar. The only differences are the following. First one has the following increasing numbers:

$x - \varphi_j$, $(l_0 - 1)d$, x, $x + \varphi_j$, $l_0 d$, $x + 2\varphi_j$, $x + 4\varphi_j$, $(l_0 + 1)d$, $x + 5\varphi_j$, $x + 6\varphi_j$, $(l_0 + 2)d$, $x + 7\varphi_j$. Also $x + \varphi_{j+4} < (l_0 + 2)d$. In particular,

$$\sum_{k \in \Gamma_0} t^{\lceil k/d \rceil} = 1 + 2t + \ldots + l_0 t^{l_0 - 1} + \Big((l_0 - 1) + (l_0 - 1)t + l_0 t^2 \Big) \cdot (t^{l_0} + t^{l_0+3} + t^{l_0+6} + \ldots).$$

Moreover, $a_2 + \Gamma_0$ has 2, 3, resp. 3 elements in I_l for $l = l_0, l_0 + 1$, resp. $l_0 + 2$, namely $a_2 + i\varphi_j$ for $i = 0, 1$ ($l = l_0$); $a_2 + i\varphi_j$ for $i = 2, 3, 4$ ($l = l_0 + 1$); and finally $a_2 + i\varphi_j$ for $i = 5, 6$ and $a_2 + \varphi_{j+4}$ in the last case. Otherwise all the arguments are similar.

In the next paragraph we show that the case C_j^* can be reduced to the case C_j.

Assume first that $l < \varphi_{j+2}$. In these intervals $I_l(C_j)$, for any element of the semigroup of C_j, the coefficient c_2 of a_2 is ≤ 2. Indeed, $3a_2 = 3 + \varphi_j\varphi_{j+4} = 2 + \varphi_{j+2}^2$. Moreover, by a computation one can verify that there no elements of type $2a_2 + i\varphi_j + k\varphi_{j+4}$ (i.e. with $c_2 = 2$) which equals some $l\varphi_{j+2} + 1$ (the first entry of some interval). Using this, one gets a bijection $\Gamma(C_j) \cap I_l(C_j) \to \Gamma(C_j^*) \cap I_l(C_j^*)$ given by $x \mapsto 2x - c_2$ (which sends $\bar{\beta}_i$ into $\bar{\beta}_i^*$). Therefore, (CP_l) is true for C_j^* for any $l \leq d^*/2 - 1$. By symmetry, (CP_l) is true for any $l \geq d^* - 2 - (d^*/2 - 1) = d^*/2 - 1$, hence for any l. □

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ON RATIONAL CUSPIDAL PROJECTIVE PLANE CURVES

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ABSTRACT FOR ON-LINE PUBLICATION

In 2002 L. Nicolaescu and the forth author formulated a very general conjecture which relates the geometric genus of a Gorenstein surface singularity with rational homology sphere link with the Seiberg-Witten invariant (or one of its candidates) of the link. Recently, the last three authors found some counterexamples using superisolated singularities. The theory of superisolated hypersurface singularities with rational homology sphere link is equivalent with the theory of rational cuspidal projective plane curves. In the case when the corresponding curve has only one singular point one knows no counterexample. In fact, in this case the above Seiberg-Witten conjecture led us to a very interesting and deep set of 'compatibility properties' of these curves (generalising the Seiberg-Witten invariant conjecture, but sitting deeply in algebraic geometry) which seems to generalise some other famous conjectures and properties as well (e.g. the Noether-Nagata or the log Bogomolov-Miyaoka-Yau inequalities). Namely, we provide a set of 'compatibility conditions' which conjecturally is satisfied by a local embedded topological type of a germ of plane curve singularity and an integer d if and only if the germ can be realized as the unique singular point of a rational unicuspidal projective plane curve of degree d. The conjectured compatibility properties have a weaker version too, valid for any rational cuspidal curve with more than one singular point. The goal of the present article is to formulate these conjectured properties, and to verify them in all the situations when the logarithmic Kodaira dimension of the complement of the corresponding plane curves is strictly less than two.