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On the Power Structure over the Grothendieck Ring of Varieties and Its Applications

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To Vladimir Igorevich Arnold with admiration

Abstract—We discuss the notion of a power structure over a ring and the geometric description of the power structure over the Grothendieck ring of complex quasi-projective varieties and show some examples of applications to generating series of classes of configuration spaces (for example, nested Hilbert schemes of J. Cheah) and wreath product orbifolds.

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To a pre-λ ring there corresponds a so-called *power structure*. This means, in particular, that one can give sense to an expression of the form

$$(1 + a_1 t + a_2 t^2 + \ldots)^m$$

for a_i and m from a ring R. (Generally speaking, there are many pre-λ structures on a ring that correspond to the same power structure.) A natural pre-λ structure on the Grothendieck ring $K_0(\mathcal{V}_\mathbb{C})$ of complex quasi-projective varieties is defined by the Kapranov zeta-function

$$\zeta_X(t) = 1 + [X]t + [S^2X]t^2 + [S^3X]t^3 + \ldots,$$

where $S^kX = X^k/S_k$ is the kth symmetric power of the variety X. In [8], we gave a geometric description of the corresponding power structure over the Grothendieck ring $K_0(\mathcal{V}_\mathbb{C})$. In some cases this permits one to give new (short and more transparent) proofs as well as certain refinements of formulae for generating series of classes of moduli spaces in the ring $K_0(\mathcal{V}_\mathbb{C})$ and/or of their invariants: the Euler characteristic and the Hodge–Deligne polynomial. An application of this sort (for the generating series of classes of Hilbert schemes of 0-dimensional subschemes of a smooth quasi-projective variety) was described in [9].

The aim of this paper is to describe the concept of a power structure (in a somewhat more general context introduced in [10]) and to show its applications to proofs and to some improvements of the results by J. Cheah in [4] on nested Hilbert schemes and by W.-P. Li and Zh. Qin in [12] on moduli spaces of some 1-dimensional subschemes. Finally, we rewrite some results of W. Wang and J. Zhou from [15, 16] on generating series of generalized orbifold Euler characteristics of wreath product orbifolds in terms of the power structure. The improvements of these results consist in the following: in the original papers, these results were formulated for some invariants (the Euler characteristic, the Hodge–Deligne polynomial, orbifold Euler characteristics, etc.), whereas here we formulate and prove them for the classes of the corresponding spaces in the Grothendieck ring $K_0(\mathcal{V}_\mathbb{C})$.

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1. POWER STRUCTURES

Definition. A pre-λ structure on a ring R is given by a series $\lambda_a(t) \in 1 + t \cdot R[[t]]$ defined for each $a \in R$ so that

(i) $\lambda_a(t) = 1 + at \mod t^2$,

(ii) $\lambda_{a+b}(t) = \lambda_a(t)\lambda_b(t)$ for $a, b \in R$.

Examples. One has the following important examples of pre-λ structures.

1. R is the ring $\mathbb{Z}$ of integers, and $\lambda_k(t) = (1-t)^{-k}$.

2. $R = \mathbb{Z}$ and $\lambda_k(t) = (1+t)^k$.

3. $R = \mathbb{Z}[u_1, \dots, u_r]$ (the ring of polynomials in r variables $u_1, \dots, u_r$); for a polynomial $P = P(\underline{u}) = \sum p_{\underline{k}}\, \underline{u}^{\underline{k}}$, $\underline{k} \in \mathbb{Z}^r_{\geq 0}$, $p_{\underline{k}} \in \mathbb{Z}$, we have

$$\lambda_P(t) = \prod_{\underline{k} \in \mathbb{Z}^r_{\geq 0}} (1 - \underline{u}^{\underline{k}}\, t)^{-p_{\underline{k}}},$$

where $\underline{u} = (u_1, \dots, u_r)$, $\underline{k} = (k_1, \dots, k_r)$, and $\underline{u}^{\underline{k}} = u_1^{k_1} \cdot \ldots \cdot u_r^{k_r}$ (see [9]).

4 (a more geometric example). Let R be the K-functor $K(X)$ of a space X, i.e., the Grothendieck ring of (say, real or complex) vector bundles over X. For a vector bundle E over X, let $\Lambda^k E$ be the kth exterior power of the bundle E. The series

$$\lambda_E(t) = 1 + [E]t + [\Lambda^2 E]t^2 + [\Lambda^3 E]t^3 + \ldots$$

defines a pre-λ structure on the ring $K(X)$.

With a pre-λ structure on a ring R, one can associate a *power structure* over R; this notion was introduced in [8].

Definition. A *power structure* over a (semi)ring R with a unit is a map $(1 + t \cdot R[[t]]) \times R \to 1 + t \cdot R[[t]]\colon (A(t), m) \mapsto (A(t))^m$ that possesses the following properties:

(i) $(A(t))^0 = 1$,

(ii) $(A(t))^1 = A(t)$,

(iii) $(A(t) \cdot B(t))^m = (A(t))^m \cdot (B(t))^m$,

(iv) $(A(t))^{m+n} = (A(t))^m \cdot (A(t))^n$,

(v) $(A(t))^{mn} = ((A(t))^n)^m$,

(vi) $(1+t)^m = 1 + mt +$ terms of higher degree,

(vii) $(A(t^k))^m = (A(t))^m\big|_{t \mapsto t^k}$.

Remark. For a ring, property (i) follows from the other ones. It is necessary to keep this property only for a semiring.

Definition. A power structure is *finitely determined* if for each $M > 0$ there exists an $N > 0$ such that for any series $A(t)$ the M-jet of the series $(A(t))^m$ (i.e., $(A(t))^m \mod t^{M+1}$) is determined by the N-jet of the series $A(t)$.

Proposition 1. *To define a finitely determined power structure over a ring R, it is sufficient to define the series $(A_0(t))^m$ for any fixed series $A_0(t)$ of the form $1 + t +$ terms of higher degree and for each $m \in R$ so that*

(i) $(A_0(t))^m = 1 + mt +$ *terms of higher degree*,

(ii) $(A_0(t))^{m+n} = (A_0(t))^m (A_0(t))^n$.

Proof. By properties (vi) and (vii), each series $A(t) \in 1 + t \cdot R[[t]]$ can be uniquely represented as a product of the form $\prod_{i=1}^{\infty}(A_0(t^i))^{b_i}$ with $b_i \in R$. Then, by properties (iii) and (vii) (and the finite determinacy of the power structure), one has

$$(A(t))^m = \prod_{i=1}^{\infty}(A_0(t^i))^{b_i m}. \tag{1}$$

Conversely, one can easily see that the power structure defined by equation (1) possesses properties (i)–(vii). □

Proposition 1 means that a pre-λ structure on the ring R defines a finitely determined power structure over R. On the other hand, there are many pre-λ structures on the ring R that give the same power structure: these are the structures defined by the series $(A_0(t))^m$ for any fixed series $A_0(t)$ of the form $1 + t +$ terms of higher degree. In what follows we will prefer to use the series $A_0(t) = (1-t)^{-1} = 1 + t + t^2 + \ldots \in R[[t]]$.

Let $R[[\underline{t}]] = R[[t_1, \ldots, t_r]]$ be the ring of series in r variables $t_1, \ldots, t_r$ with coefficients in the ring R, and let $\mathfrak{m}$ be the ideal $\langle t_1, \ldots, t_r \rangle$. A power structure over the ring R allows one to give sense to expressions of the form $(A(\underline{t}))^m$ in a natural way, where $A(\underline{t}) \in 1 + \mathfrak{m}R[[\underline{t}]]$. Namely, the series $A(\underline{t})$ can be uniquely represented in the form

$$A(\underline{t}) = \prod_{\underline{k} \in \mathbb{Z}^r_{\geq 0} \setminus \{0\}} (1 - \underline{t}^{\underline{k}})^{-b_{\underline{k}}}, \qquad \underline{t}^{\underline{k}} = t_1^{k_1} \cdot \ldots \cdot t_r^{k_r}.$$

Then

$$(A(\underline{t}))^m = \prod_{\underline{k} \in \mathbb{Z}^r_{\geq 0} \setminus \{0\}} (1 - \underline{t}^{\underline{k}})^{-b_{\underline{k}} m}.$$

Let R_1 and R_2 be rings with power structures over them. A ring homomorphism $\varphi\colon R_1 \to R_2$ induces a natural homomorphism $R_1[[\underline{t}]] \to R_2[[\underline{t}]]$ (also denoted by φ) by the formula $\varphi\big(\sum a_{\underline{i}}\, \underline{t}^{\underline{i}}\big) = \sum \varphi(a_{\underline{i}})\, \underline{t}^{\underline{i}}$.

Proposition 2. *If a ring homomorphism $\varphi\colon R_1 \to R_2$ is such that $(1-t)^{-\varphi(m)} = \varphi((1-t)^{-m})$ for any $m \in R_1$, then $\varphi((A(\underline{t}))^m) = (\varphi(A(\underline{t})))^{\varphi(m)}$ for $A(\underline{t}) \in 1 + \mathfrak{m}R_1[[\underline{t}]]$, $m \in R_1$.*

A quasi-projective variety is the difference of two (complex) projective (algebraic) varieties.

Definition. The *Grothendieck ring $K_0(\mathcal{V}_\mathbb{C})$ of complex quasi-projective varieties* is the abelian group generated by the classes $[X]$ of all quasi-projective varieties X modulo the following relations:

(i) if varieties X and Y are isomorphic, then $[X] = [Y]$;

(ii) if Y is a Zariski closed subvariety of X, then $[X] = [Y] + [X \setminus Y]$.

The multiplication in $K_0(\mathcal{V}_\mathbb{C})$ is defined by the Cartesian product of varieties.

Remark. One can also consider the concept of the Grothendieck semiring $S_0(\mathcal{V}_\mathbb{C})$ of complex quasi-projective varieties substituting the word "semigroup" for the word "group" in the above definition. The elements of the semiring $S_0(\mathcal{V}_\mathbb{C})$ have somewhat more geometric sense: they are represented by "genuine" quasi-projective varieties (not by virtual ones).

The class $[\mathbb{A}^1_\mathbb{C}] \in K_0(\mathcal{V}_\mathbb{C})$ of the complex affine line is denoted by $\mathbb{L}$. In a number of cases it is reasonable (or rather necessary) to consider the localization $K_0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{-1}]$ of the Grothendieck ring $K_0(\mathcal{V}_\mathbb{C})$ by the class $\mathbb{L}$.

For a complex quasi-projective variety X, let $S^k X = X^k / S_k$ be the kth symmetric power of the space X (here S_k is the group of permutations of k elements; $S^k X$ is a quasi-projective variety as well).

Definition. The *Kapranov zeta function* of a quasi-projective variety X is the series

$$\zeta_X(t) = 1 + [X] \cdot t + [S^2 X] \cdot t^2 + [S^3 X] \cdot t^3 + \ldots \in K_0(\mathcal{V}_{\mathbb{C}})[[t]]$$

(see [11]).

One can see that

$$\zeta_{X+Y}(t) = \zeta_X(t) \cdot \zeta_Y(t). \tag{2}$$

This follows from the relation $S^k(X \amalg Y) = \coprod_{i=0}^k S^i X \times S^{k-i} Y$. In addition, we have

$$\zeta_{\mathbb{L}^n}(t) = \frac{1}{1 - \mathbb{L}^n t}.$$

For example, this implies that

$$\zeta_{\mathbb{C}\mathrm{P}^n}(t) = \prod_{i=0}^n \frac{1}{1 - \mathbb{L}^i t}.$$

Equation (2) means that the series $\zeta_X(t)$ defines a pre-λ structure on the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$. The geometric description of the corresponding power structure over the ring $K_0(\mathcal{V}_{\mathbb{C}})$ was given in [8]. We will formulate it here in the form adapted to series in r variables [10].

Let $A_{\underline{n}}$, $\underline{n} = (n_1, \ldots, n_r) \in \mathbb{Z}^r_{\geq 0} \setminus \{0\}$, and M be quasi-projective varieties and $A(\underline{t}) = 1 + \sum_{\underline{n} \in \mathbb{Z}^r_{\geq 0} \setminus \{0\}} [A_{\underline{n}}] \underline{t}^{\underline{n}} \in K_0(\mathcal{V}_{\mathbb{C}})[[\underline{t}]]$. Let $\mathfrak{A}$ be the disjoint union $\coprod_{\underline{k} \in \mathbb{Z}^r_{\geq 0} \setminus \{0\}} A_{\underline{k}}$, and let $\underline{k} \colon \mathfrak{A} \to \mathbb{Z}^r_{\geq 0}$ be the tautological map on it: it sends the points of $A_{\underline{k}}$ to $\underline{k} \in \mathbb{Z}^r_{\geq 0}$.

Geometric description of the power structure over the ring $K_0(\mathcal{V}_{\mathbb{C}})$. The coefficient at $\underline{t}^{\underline{n}}$ in the series $A(\underline{t})^{[M]}$ is represented by the configuration space of pairs (K, φ), where K is a finite subset of the variety M and φ is a map from K to $\mathfrak{A}$ such that $\sum_{x \in K} \underline{k}(\varphi(x)) = \underline{n}$. To describe such a configuration space as a quasi-projective variety, one can write it as

$$\sum_{\mathbf{k} \colon \sum \underline{i}\, k_{\underline{i}} = \underline{n}} \left[\left(\left(\prod_{\underline{i}} M^{k_{\underline{i}}} \right) \setminus \Delta \right) \times \prod_{\underline{i}} A_{\underline{i}}^{k_{\underline{i}}} \Big/ \prod_{\underline{i}} S_{k_{\underline{i}}} \right], \tag{3}$$

where $\mathbf{k} = \{k_{\underline{i}} \colon \underline{i} \in \mathbb{Z}^r_{\geq 0} \setminus \{0\},\ k_{\underline{i}} \in \mathbb{Z}\}$ and Δ is the "large diagonal" in $M^{\Sigma k_{\underline{i}}}$, which consists of $\left(\sum k_{\underline{i}}\right)$-tuples of points of M at least two of which coincide; the permutation group $S_{k_{\underline{i}}}$ acts by permuting the corresponding $k_{\underline{i}}$ factors in $\prod_s M^{k_{\underline{i}}} \supset \left(\prod_{\underline{i}} M^{k_{\underline{i}}}\right) \setminus \Delta$ and the spaces $A_{\underline{i}}$ simultaneously (the connection between this formula and the description above is clear).

One can show (see [8]) that the operation described indeed gives a power structure over the ring $K_0(\mathcal{V}_{\mathbb{C}})$; i.e., it satisfies conditions (i)–(vii) of the definition. The fact that this structure corresponds to the Kapranov zeta function follows from the equation

$$(1 + t + t^2 + \ldots)^{[M]} = 1 + [M] \cdot t + [S^2 M] \cdot t^2 + [S^3 M] \cdot t^3 + \ldots. \tag{4}$$

Indeed, since there is only one map from M to a point (a coefficient in the series $1 + t + t^2 + \ldots$), the coefficient of t^n on the left-hand side of equation (4) is represented by the space a point of which is a finite set of points of the variety M with positive multiplicities such that the sum of these multiplicities is equal to n. This is just the definition of the nth symmetric power of the variety M.

It is also useful to describe the binomial $(1 + t)^{[M]}$. The coefficient of t^n in it is represented by the space a point of which is a finite subset of the variety M with n elements, i.e., the configuration space $(M^n \setminus \Delta)/S_n$ of unordered n-tuples of distinct points of M.

It seems that the power structure can be used to prove some combinatorial identities. For instance, applying formula (3) to a finite set M with m elements and to finite sets $A_{\underline{n}}$, one gets a formula for the power of a series:

$$\left(1+\sum_{\underline{n}\in\mathbb{Z}^r_{\geq 0}\setminus\{0\}} a_{\underline{n}}\,\underline{t}^{\underline{n}}\right)^m = 1+\sum_{\underline{n}\in\mathbb{Z}^r_{\geq 0}\setminus\{0\}}\left(\sum_{\mathbf{k}\colon \sum \underline{i}k_{\underline{i}}=\underline{n}} \frac{m!}{(m-\Sigma k_{\underline{i}})!\prod_{\underline{i}} k_{\underline{i}}!}\prod_{\underline{i}} a_{\underline{i}}^{k_{\underline{i}}}\right)\underline{t}^{\underline{n}}.$$

There are two natural homomorphisms from the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ to the ring $\mathbb{Z}$ of integers and to the ring $\mathbb{Z}[u,v]$ of polynomials in two variables: the Euler characteristic (with compact support) $\chi\colon K_0(\mathcal{V}_{\mathbb{C}})\to\mathbb{Z}$ and the Hodge–Deligne polynomial $e\colon K_0(\mathcal{V}_{\mathbb{C}})\to\mathbb{Z}[u,v]$: $e(X)(u,v)=\sum e^{p,q}(X)u^pv^q$.

Macdonald's formula [13]

$$\chi\big(1+[X]t+[S^2X]t^2+[S^3X]t^3+\ldots\big)=(1-t)^{-\chi(X)}$$

and the corresponding formula for the Hodge–Deligne polynomial (see [2; 3, Proposition 1.2])

$$e\big(1+[X]t+[S^2X]t^2+\ldots\big)(u,v)=\prod_{p,q}\left(\frac{1}{1-u^pv^qt}\right)^{e^{p,q}(X)}$$

imply that these homomorphisms respect the above-described power structures over these rings (see Example 3 and Proposition 2 or [9]). Therefore, if series in $K_0(\mathcal{V}_{\mathbb{C}})[[\underline{t}]]$ satisfy a relation written in terms of the power structure, then the corresponding relations hold for the Euler characteristics and the Hodge–Deligne polynomials of these series.

Remark. It is also possible to define the power structure and give its geometric description in the *relative setting*, i.e., over the Grothendieck ring $K_0(\mathcal{V}_S)$ of complex quasi-projective varieties over a variety S. The ring $K_0(\mathcal{V}_S)$ is generated by classes of varieties with maps ("projections") to S. In this case the coefficient of the series $(A(\underline{t}))^{[M]}$ is the configuration space a point of which is a pair (K,φ), where $K\subset M$ is a finite subset contained in the preimage of one point of S and φ is a map that commutes with the projections to S.

2. NESTED HILBERT SCHEMES OF J. CHEAH

Let Hilb^n_X, $n\geq 1$, be the Hilbert scheme of zero-dimensional subschemes of length n of a complex quasi-projective variety X; for $x\in X$, let $\mathrm{Hilb}^n_{X,x}$ be the Hilbert scheme of subschemes of the variety X supported at the point x.

In [4], J. Cheah considered nested Hilbert schemes on a smooth d-dimensional complex quasi-projective variety X. For $\underline{n}=(n_1,\ldots,n_r)\in\mathbb{Z}^r_{\geq 0}$, the *nested Hilbert scheme* $Z^{\underline{n}}_X$ (of depth r) is the scheme that parametrizes collections of the form $(Z_1,\ldots,Z_r)$, where $Z_i\in\mathrm{Hilb}^{n_i}_X$ and Z_i is a subscheme of Z_j for $i<j$. The scheme $Z^{\underline{n}}_X$ is nonempty only if $n_1\leq n_2\leq\ldots\leq n_r$. Notice that $Z^{(n)}_X=\mathrm{Hilb}^n_X\cong Z^{(n,\ldots,n)}_X$.

For $Y\subset X$, let $Z^{\underline{n}}_{X,Y}$ be the scheme that parametrizes collections $(Z_1,\ldots,Z_r)$ from $Z^{\underline{n}}_X$ with $\operatorname{supp} Z_i\subset Y$. For $Y=\{x\}$, $x\in X$, we will use the notation $Z^{\underline{n}}_{X,x}$.

For $r\geq 1$, let $\underline{t}=(t_1,\ldots,t_r)$ and

$$\mathcal{Z}^{(r)}_X(\underline{t}):=\sum_{\underline{n}\in Z^r_{\geq 0}}[Z^{\underline{n}}_X]\underline{t}^{\underline{n}},\qquad \mathcal{Z}^{(r)}_{X,x}(\underline{t}):=\sum_{\underline{n}\in Z^r_{\geq 0}}[Z^{\underline{n}}_{X,x}]\underline{t}^{\underline{n}}$$

be the generating series of classes (in $K_0(\mathcal{V}_{\mathbb{C}})$) of nested Hilbert schemes $Z^{\underline{n}}_X$ (respectively, supported at the point x) of depth r.

Theorem 1. *For a smooth quasi-projective variety X of dimension d, the following identity holds in the semiring $S_0(\mathcal{V}_{\mathbb{C}})[[\underline{t}]]$ (and therefore also in the ring $K_0(\mathcal{V}_{\mathbb{C}})[[\underline{t}]]$):*

$$\mathcal{Z}_X^{(r)}(\underline{t}) = \big(\mathcal{Z}_{\mathbb{A}^d,0}^{(r)}(\underline{t})\big)^{[X]}. \tag{5}$$

Proof. For a Zariski closed subset $Y \subset X$, one has $\mathcal{Z}_X^{(r)}(\underline{t}) = \mathcal{Z}_{X,Y}^{(r)}(\underline{t}) \cdot \mathcal{Z}_{X,X\setminus Y}^{(r)}(\underline{t})$. Therefore, it is sufficient to prove equation (5) for Zariski open subsets U of X that form a covering of X and for their intersections.

One can take U that lies in an affine chart $\mathbb{A}_{\mathbb{C}}^N$ and is such that its projection to a d-dimensional coordinate subspace (say, generated by the first d coordinates) is everywhere nondegenerate (i.e., is an étale morphism). For any point $x \in U$, this projection identifies $\underline{n}$-nested Hilbert schemes $Z_{U,x}^{\underline{n}}$ with $Z_{\mathbb{A}_{\mathbb{C}}^d,0}^{\underline{n}}$.

A nested (zero-dimensional) subscheme of U of type $\underline{n}$ is defined by a finite subset $K \subset U$ with a nested subscheme from $Z_{X,x}^{\underline{k}(x)}$ at each point $x \in K$ such that $\sum_{x\in K} \underline{k}(x) = \underline{n}$. This coincides with the description of the coefficient of $\underline{t}^{\underline{n}}$ on the right-hand side of equation (5). $\square$

Similar considerations yield a short proof of a somewhat refined version of the main result of [4]. Following J. Cheah, set

$$\mathfrak{F}_X^n = \big\{(x, Z) \in X \times \mathrm{Hilb}_X^n \colon\ x \in \operatorname{supp} Z\big\},$$

$$\mathfrak{F}_X^{n-1,n} = \big\{(x_1, x_2, Z_1, Z_2) \in X \times X \times Z_X^{(n-1,n)} \colon\ x_i \in \operatorname{supp} Z_i,\ i = 1, 2\big\},$$

$$\mathfrak{T}_X^n = \big\{(x_1, x_2, Z) \in X \times X \times \mathrm{Hilb}_X^n \colon\ x_i \in \operatorname{supp} Z,\ i = 1, 2\big\},$$

$$\mathfrak{G}_X^n = \big\{(x, Z_1, Z_2) \in X \times Z_X^{(n-1,n)} \colon\ x \in \operatorname{supp} Z_2\big\}.$$

Let series $\mathfrak{P}_X(t_0, t_1, t_2, t_3)$ and $\mathfrak{f}_d(t_0, t_1, t_2, t_3)$ in $K_0(\mathcal{V}_{\mathbb{C}})[[t_0, t_1, t_2, t_3]]$ be defined by

$$\begin{aligned}\mathfrak{P}_X(t_0, t_1, t_2, t_3) := {} & \Bigg[\sum_{n\ge 0} [\mathrm{Hilb}_X^n] t_0^n\Bigg] + \Bigg[\sum_{n\ge 1} [\mathfrak{F}_X^n] t_0^n\Bigg] t_1 + \Bigg[\sum_{n\ge 1} [\mathfrak{F}_X^n] t_0^n\Bigg] t_2 + \Bigg[\sum_{n\ge 1} [\mathfrak{T}_X^n] t_0^n\Bigg] t_1 t_2 \\ & + \Bigg[\sum_{n\ge 1} [Z_X^{(n-1,n)}] t_0^n\Bigg] t_3 + \Bigg[\sum_{n\ge 2} [Z_X^{(1,n-1,n)}] t_0^n\Bigg] t_1 t_3 + \Bigg[\sum_{n\ge 1} [\mathfrak{G}_X^n] t_0^n\Bigg] t_2 t_3 \\ & + \Bigg[\sum_{n\ge 2} [\mathfrak{F}_X^{n-1,n}] t_0^n\Bigg] t_1 t_2 t_3,\end{aligned}$$

$$\begin{aligned}\mathfrak{f}_d(t_0, t_1, t_2, t_3) := {} & \sum_{k\ge 0} [\mathrm{Hilb}_{\mathbb{A}^d,0}^k] t_0^k + \sum_{k\ge 1} [Z_{\mathbb{A}^d,0}^{(k-1,k)}] t_0^k t_3 + \sum_{k\ge 1} [\mathrm{Hilb}_{\mathbb{A}^d,0}^k] t_0^k t_2 \\ & + \sum_{k\ge 1} [Z_{\mathbb{A}^d,0}^{(k-1,k)}] t_0^k t_2 t_3 + \sum_{k\ge 1} [\mathrm{Hilb}_{\mathbb{A}^d,0}^k] t_0^k t_1 + \sum_{k\ge 1} [\mathrm{Hilb}_{\mathbb{A}^d,0}^k] t_0^k t_1 t_2 \\ & + \sum_{k\ge 2} [Z_{\mathbb{A}^d,0}^{(k-1,k)}] t_0^k t_1 t_3 + \sum_{k\ge 2} [Z_{\mathbb{A}^d,0}^{(k-1,k)}] t_0^k t_1 t_2 t_3.\end{aligned}$$

Theorem 2 (cf. the main theorem in [4]). *Let X be a smooth quasi-projective variety of dimension d. Then*

$$\mathfrak{P}_X(t_0, t_1, t_2, t_3) = (\mathfrak{f}_d(t_0, t_1, t_2, t_3))^{[X]} \qquad \mathrm{mod}\ (t_1^2, t_2^2, t_3^2). \tag{6}$$

Proof. Using the arguments of the proof of Theorem 1, we may suppose that X lies in an affine chart $\mathbb{A}^N_{\mathbb{C}}$ and its projection to a d-dimensional coordinate subspace is nondegenerate. This identifies $\mathrm{Hilb}^s_{X,x}$ and $Z^{(s-1,s)}_{X,x}$ with $\mathrm{Hilb}^s_{\mathbb{A}^d,0}$ and $Z^{(s-1,s)}_{\mathbb{A}^d,0}$, respectively, for each point $x \in X$. To prove equation (6), one has to give an interpretation of the coefficients of the monomials $t_0^n, t_0^n t_1, \dots, t_0^n t_1 t_2 t_3$ on the right-hand side of (6). Let us make this for the coefficients of $t_0^n t_3$ and $t_0^n t_2 t_3$ (the other cases are treated in the same way).

The coefficient of the monomial $t_0^n t_3$ is represented by the space a point of which is defined by a point x_0 of X with a zero-dimensional nested subscheme from $Z^{(k(x_0)-1,k(x_0))}_{X,x_0}$ at it plus several other points x of X with a zero-dimensional subscheme from $\mathrm{Hilb}^{k(x)}_{X,x} \cong Z^{(k(x),k(x))}_{X,x}$ at each of them, such that $k(x_0) + \sum k(x) = n$. This is just the definition of a point of the space $Z^{(n-1,n)}_X$.

The monomial $t_0^n t_2 t_3$ can be obtained either as the product of two monomials of the form $t_0^* t_2$ and $t_0^* t_3$ and several monomials of the form t_0^* or as the product of a monomial of the form $t_0^* t_2 t_3$ and several monomials of the form t_0^*. Therefore, the coefficient of the monomial $t_0^n t_2 t_3$ is represented by a space consisting of two parts.

A point of the first part is defined by a point x_1 of X with a subscheme from $\mathrm{Hilb}^{k(x_1)}_{X,x_1} \cong Z^{(k(x_1),k(x_1))}_{X,x_1}$ at it, with $k \geq 1$ (i.e., it is not empty: x_1 belongs to its support), a point $x_2 \in X$ with a subscheme from $Z^{(k(x_2)-1,k(x_2))}_{X,x_2}$ at it, plus several points of X with a 0-dimensional subscheme from $\mathrm{Hilb}^{k(x)}_{X,x}$ at each of them, such that $k(x_1) + k(x_2) + \sum k(x) = n$.

A point of the second part is defined by a point x_1 of X with a nested subscheme (z_1, z_2) from $Z^{(k(x_1)-1,k(x_1))}_{X,x_1}$ at it (in this case z_2 is not empty: x_1 belongs to its support) plus several points x of X with a 0-dimensional subscheme from $\mathrm{Hilb}^{k(x)}_{X,x} \cong Z^{(k(x),k(x))}_{X,x}$ at each of them such that $k(x_1) + \sum k(x) = n$. Therefore, a point of the union of these two subspaces can be described by a nested subscheme (Z_1, Z_2) from $Z^{(n-1,n)}_X$ plus a point that belongs to the support of the subscheme Z_2. This is just the description of the space $\mathfrak{G}^n_X$. $\square$

Applying the Hodge–Deligne polynomial homomorphism to (6), one gets the main theorem of [4].

Example. Let S be a smooth quasi-projective surface. Consider the incidence variety $Z^{(n-1,n)}_S = \{(Z_1, Z_2) \in \mathrm{Hilb}^{n-1}_S \times \mathrm{Hilb}^n_S \colon Z_1 \subset Z_2\}$. Using the results of J. Cheah on the cellular decomposition of $Z^{(n-1,n)}_{\mathbb{A}^2_{\mathbb{C}},0}$ [5], one gets the result of L. Göttsche [7, Theorem 5.1]:

$$\sum_{n \geq 1} \bigl[Z_S^{n-1,n}\bigr] t^n = \frac{[S]\,t}{1 - \mathbb{L}t} \left(\prod_{k \geq 1} \frac{1}{1 - \mathbb{L}^{k-1} t^k} \right)^{[S]}.$$

3. ON MODULI SPACES OF CURVES AND POINTS (AFTER W.-P. LI AND ZH. QIN)

In [12], certain moduli spaces of 1-dimensional subschemes in a smooth d-dimensional projective complex variety were considered. Let X be a smooth d-dimensional projective complex variety with a Zariski locally trivial fibration $\mu\colon X \to S$, where S is smooth of dimension $d-1$ and the fibers $C_s = \mu^{-1}(s)$, $s \in S$, are smooth irreducible curves of genus g. Let $\beta \in H_2(X, \mathbb{Z})$ be the class of the fiber.

Let $\mathfrak{I}_n(X, \beta)$ be the moduli space of 1-dimensional closed subschemes Z of X such that $\chi(\mathcal{O}_Z) = n$ and $[Z] = \beta$, where $[Z]$ is the fundamental class of the subscheme Z, and let $\mathfrak{M}^n := \mathfrak{I}_{(1-g)+n}(X, \beta)$. Let $\mathfrak{M}^n_{X,C_s,x}$ be the moduli space of 1-dimensional closed subschemes Θ in X such that $I_\Theta \subset I_{C_s}$, the support $\mathrm{supp}(I_{C_s}/I_\Theta) \subset \{x\}$, and $\dim_x(I_{C_s}/I_\Theta) = n$. The number n

will be called the length of the subscheme Θ. Let $\mathfrak{M}^n_{\mathbb{C}^{d-1}\times\mathbb{C},\{0\}\times\mathbb{C},0}$ have the same meaning: it is the moduli space of 1-dimensional closed subschemes Θ in $\mathbb{C}^{d-1}\times\mathbb{C}$ such that $I_\Theta\subset I_{\{0\}\times\mathbb{C}}$, $\operatorname{supp}\big(I_{\{0\}\times\mathbb{C}}/I_\Theta\big)\subset\{0\}$, and $\dim_0\big(I_{\{0\}\times\mathbb{C}}/I_\Theta\big)=n$.

Theorem 3 (cf. Proposition 5.3, Lemma 6.1, and Proposition 6.2 in [12]). *Let X be a smooth d-dimensional projective complex variety with a Zariski locally trivial fibration $\mu\colon X\to S$, where S is smooth of dimension $d-1$ and the fibers $C_s\cong C$ are smooth irreducible curves of genus g. Then*

$$\sum_{n\ge0}[\mathfrak{M}^n]t^n=[S]\left(\sum_{n\ge0}\big[\mathrm{Hilb}^n_{\mathbb{C}^d,0}\big]t^n\right)^{[X]-[C]}\left(\sum_{n\ge0}\big[\mathfrak{M}^n_{\mathbb{C}^{d-1}\times\mathbb{C},\{0\}\times\mathbb{C},0}\big]t^n\right)^{[C]}. \tag{7}$$

Proof. A point of $\mathfrak{M}^n$ can be considered as consisting of a fiber $C_s=\mu^{-1}(s)$ of the bundle $\mu\colon X\to S$ (this fiber is determined by a point $s\in S$) and of several fixed points x, both outside C_s and on it, with a 0-dimensional subscheme (i.e., an element of $\mathrm{Hilb}^*_{X,x}$) at each point x that lies outside C_s and a subscheme of $\mathfrak{M}^*_{X,C_s,x}$ at each point x that lies on C_s, such that the sum of their lengths is equal to n. Thus, there is a natural map (projection) from $\mathfrak{M}^n$ to S. Over a point $s\in S$, there are somewhat different objects (subschemes) at points outside and on the curve C_s.

It is sufficient to prove equation (7) for the preimages of elements of a covering of S by Zariski open subsets and of their intersections. Therefore, without any loss of generality, we can suppose that $X=S\times C$. Moreover, let us choose a fixed point $s_0\in S$. A constructible map that sends $\mathfrak{M}^n_{X,C_s}$ to $\mathfrak{M}^n_{X,C_{s_0}}$ and is an isomorphism of strata can be defined as follows. One takes 0-dimensional subschemes that lie on the curve C_{s_0} and puts them to the corresponding points of the curve C_s, and vice versa, one takes the elements of $\mathfrak{M}^n_{X,C_s,x}$ and puts them to the corresponding points of the curve C_{s_0}. Thus, in the Grothendieck ring of quasi-projective varieties, one has $[\mathfrak{M}^n]=[S]\big[\mathfrak{M}^n_{X,C_{s_0}}\big]$.

Therefore, to prove (7), one should show that

$$\sum_{n\ge0}\big[\mathfrak{M}^n_{X,C_{s_0}}\big]t^n=\left(\sum_{n\ge0}\big[\mathrm{Hilb}^n_{\mathbb{C}^d,0}\big]t^n\right)^{[X]-[C_{s_0}]}\left(\sum_{n\ge0}\big[\mathfrak{M}^n_{\mathbb{C}^{d-1}\times\mathbb{C},\{0\}\times\mathbb{C},0}\big]t^n\right)^{[C_{s_0}]}. \tag{8}$$

Just as in the proofs in Section 2, we may suppose that at each point of the manifold X the space $\mathrm{Hilb}^k_{X,x}$ is identified with the space $\mathrm{Hilb}^k_{\mathbb{C}^d,0}$ and at each point of the curve $C_{s_0}\subset X$ the space $\mathfrak{M}^k_{X,C_{s_0},x}$ is identified with the space $\mathfrak{M}^k_{\mathbb{C}^{d-1}\times\mathbb{C},\{0\}\times\mathbb{C},0}$. The coefficient of the monomial t^n on the right-hand side of equation (8) is represented by the space a point of which is defined by several points x of the curve $C_{s_0}\subset X$ with a (1-dimensional) scheme from $\mathfrak{M}^{k(x)}_{X,C_{s_0},x}$ at each of them and several points x from $X\setminus C_{s_0}$ with a (0-dimensional) scheme from $\mathrm{Hilb}^{k(x)}_{X,x}$ at each of them, such that the sum of the lengths $k(x)$ over all these points is equal to n. This is just the description of a point of the space $\mathfrak{M}^n_{X,C_{s_0}}$. $\square$

4. GENERALIZED ORBIFOLD EULER CHARACTERISTIC AND THE POWER STRUCTURE

Here we rewrite some results of [15] and [16] in terms of the power structure. To this end, we need the power structure over a somewhat modified version of the Grothendieck ring $K_0(\mathcal{V}_\mathbb{C})$. For a fixed positive integer m, consider the ring $K_0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{1/m}]$. The pre-$\lambda$ structure on (and therefore the corresponding power structure over) the ring $K_0(\mathcal{V}_\mathbb{C})$ can be extended to one on $K_0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{1/m}]$ by the formula

$$\zeta_{[X]\mathbb{L}^{s/m}}(t)=\zeta_{[X]}(\mathbb{L}^{s/m}t).$$

In a similar way the corresponding pre-λ structure on the ring $\mathbb{Z}[u_1^{1/m},\dots,u_r^{1/m}]$ can be defined by the formula

$$\lambda_P(t)=\prod_{\underline{k}\in(1/m)\mathbb{Z}^r_{\geq 0}}(1-\underline{u}^{\underline{k}}t)^{-p_{\underline{k}}}$$

for a polynomial (with fractional exponents) $P=P(\underline{u})=\sum_{\underline{k}\in(1/m)\mathbb{Z}^r_{\geq 0}}p_{\underline{k}}\,\underline{u}^{\underline{k}}$. There are natural homomorphisms (χ and e) from the ring $K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{1/m}]$ to the rings $\mathbb{Z}$ and $\mathbb{Z}\big[u^{1/m},v^{1/m}\big]$ that send the element $\mathbb{L}^{1/m}$ to 1 and $(uv)^{(1/m)}$, respectively. One can easily see that these are homomorphisms of the pre-λ-rings, and therefore they respect the power structures.

Let X be a smooth quasi-projective complex algebraic variety of dimension d with an action of a finite group G of order m. For an element $g\in G$, let X^g be the set (a manifold) $\{x\in X\colon gx=x\}$ of g-invariant points of the action. If $h=vgv^{-1}$ in the group G, the element v defines an isomorphism $v\colon X^g\to X^h$. Let G_* be the set of conjugacy classes of elements of the group G. For a conjugacy class $c\in G_*$ choose its representative $g\in G$. Let $C_G(g)$ be the centralizer of the element g in the group G. The centralizer $C_G(g)$ acts on the set X^g of fixed points of the element g. Suppose that its action on the set of connected components of the manifold X^g has N_c orbits, and let $X_1^g,\dots,X_{N_c}^g$ be the unions of the components of each of these orbits. At each point $x\in X_{\alpha_c}^g$, the differential dg of the map g is an automorphism of the tangent space T_xX that acts as a diagonal matrix $\mathrm{diag}(\exp(2\pi i\theta_1),\dots,\exp(2\pi i\theta_d))$, where $0\leq\theta_i<1$, $\theta_i\in(1/m)\mathbb{Z}$. The *shift number* $F_{\alpha_c}^g$ associated with $X_{\alpha_c}^g$ is $F_{\alpha_c}^g:=\sum_{j=1}^d\theta_j\in\mathbb{Z}/m$ (it was introduced by E. Zaslow in [17]).

Definition. The *generalized orbifold Euler characteristic* $[X,G]$ of the pair (X,G) is

$$[X,G]:=\sum_{c\in G_*}\sum_{\alpha_c=1}^{N_c}\big[X_{\alpha_c}^g/C_G(g)\big]\cdot\mathbb{L}^{F_{\alpha_c}^g}\in K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{1/m}].$$

Application of the Euler characteristic morphism leads to the notion of the *orbifold Euler characteristic* invented in the study of string theory of orbifolds by L. Dixon et al. [6]:

$$\chi(X,G):=\sum_{c\in G_*}\sum_{\alpha_c=1}^{N_c}\chi\big(X_{\alpha_c}^g/C_G(g)\big)=\sum_{c\in G_*}\chi(X^g/C_G(g)).$$

Application of the Hodge–Deligne polynomial morphism leads to the notion of the *orbifold E-function* introduced by V. Batyrev in [1]:

$$E_{\mathrm{orb}}(X,G;u,v):=\sum_{c\in G_*}\sum_{\alpha_c=1}^{N_c}e\big(X_{\alpha_c}^g/C_G(g)\big)(u,v)\,(uv)^{F_{\alpha_c}^g}\in\mathbb{Z}\big[u^{1/m},v^{1/m}\big].$$

Let $G^n=G\times\ldots\times G$ be the Cartesian power of the group G. The symmetric group S_n acts on G^n by permutation of the factors: $s(g_1,\dots,g_n)=\big(g_{s^{-1}(1)},\dots,g_{s^{-1}(n)}\big)$. The *wreath product* $G_n=G\sim S_n$ is the semidirect product of the groups G^n and S_n defined by the described action. Namely, the multiplication in the group G_n is given by the formula $(g,s)(h,t)=(g\cdot s(h),st)$, where $g,h\in G^n$ and $s,t\in S_n$. The group G^n is a normal subgroup of the group G_n via the identification of $g\in G^n$ with $(g,1)\in G_n$. For a variety X with a G-action, the corresponding action of the group G_n on the Cartesian power X^n is given by the formula

$$((g_1,\dots,g_n),s)(x_1,\dots,x_n)=\big(g_1x_{s^{-1}(1)},\dots,g_nx_{s^{-1}(n)}\big),$$

where $x_1, \dots, x_n \in X$, $g_1, \dots, g_n \in G$, and $s \in S_n$. One can see that the factor variety X^n/G_n is naturally isomorphic to the space $(X/G)^n/S_n$. In particular, $[X^n/G_n] = [(X/G)^n/S_n]$ in the Grothendieck ring $K_0(\mathcal{V}_\mathbb{C})$. Therefore,

$$\sum_{n\geq 0}[X^n/G_n]t^n = (1-t)^{-[X/G]} \in K_0(\mathcal{V}_\mathbb{C})[[t]].$$

Theorem 4 (cf. [15, 16]). *Let X be a smooth complex quasi-projective variety of dimension d with an action of a finite group G of order m. Then*

$$\sum_{n\geq 0}[X^n, G_n]t^n = \left(\prod_{r=1}^{\infty}\bigl(1-\mathbb{L}^{(r-1)d/2}t^r\bigr)\right)^{-[X,G]}. \tag{9}$$

Proof. One can say that, essentially, the proof is already contained in [16], where invariants of the G_n-action on the space X^n are related to those of the G-action on the space X (see also [15] and [14]).

Let $a = (g, s) \in G_n$, $g = (g_1, \dots, g_n)$. Let $z = (i_1, \dots, i_r)$ be one of the cycles in the permutation s. The *cycle product* of the element a corresponding to the cycle z is the product $g_{i_r}g_{i_{r-1}}\dots g_{i_1} \in G$. The conjugacy class of the cycle product is well-defined by the element g and the cycle z of the permutation s. For $c \in G_*$ and $r \geq 0$, let $m_r(c)$ be the number of r-cycles in the permutation s whose cycle products lie in c. Let $\rho(c)$ be the partition that contains $m_r(c)$ summands equal to r, and let $\rho = (\rho(c))_{c\in G_*}$ be the corresponding partition-valued function on G_*. One has

$$\|\rho\| := \sum_{c\in G_*}|\rho(c)| = \sum_{c\in G_*,\ r\geq 1} r m_r(c) = n.$$

The function ρ, or, equivalently, the data $\{m_r(c)\}_{r,c}$, is called the *type* of the element $a = (g, s) \in G_n$. Two elements of the group G_n are conjugate to each other if and only if they are of the same type.

In [16] the following was shown:

1. For a conjugacy class of elements of the group G_n containing an element a of type $\rho = \{m_r(c)\}_{r\geq 1,\, c\in G_*}$ ($\sum_{r,c} r m_r(c) = n$), the subspace $(X^n)^a$ can be naturally identified with $\prod_{c,r}(X^c)^{m_r(c)}$. The factor space $(X^n)^a/Z_{G_n}(a)$ is naturally isomorphic to the product $\prod_{c\in G_*,\, r\geq 1} S^{m_r(c)}(X^c/Z_G(c))$. The connected components of the space $(X^n)^a/Z_{G_n}(a)$ are numbered by collections of integers $(m_{r,c}(1), \dots, m_{r,c}(N_c))$ satisfying the relation $\sum_{\alpha_c=1}^{N_c} m_{r,c}(\alpha_c) = m_r(c)$. They are

$$(X^n)^a_{\{m_{r,c}(\alpha_c)\}} = \prod_{c\in G_*,\ r\geq 1}\ \prod_{\alpha_c=1}^{N_c} S^{m_{r,c}(\alpha_c)}\bigl(X^c_{\alpha_c}/Z_G(c)\bigr).$$

2. The shift for the component $(X^n)^a_{\{m_{r,c}(\alpha_c)\}}$ is equal to

$$F_{\{m_{r,c}(\alpha_c)\}} = \sum_{c\in G_*,\ r\geq 1}\ \sum_{\alpha_c=1}^{N_c} m_{r,c}(\alpha_c)\bigl(F^c_{\alpha_c}(r-1)d/2\bigr).$$

These two facts imply that

$$\sum_{n\geq 0}[X^n, G_n]t^n = \sum_{n\geq 0}\left(\sum_{m_r(c)}\prod_{c,r}\prod_{\alpha_c=1}^{N_c}\bigl[S^{m_{r,c}}\bigl(X^g_{\alpha_c}/Z_G(g)\bigr)\bigr]\mathbb{L}^{m_r(c)\left(F^g_{\alpha_c}+\frac{(r-1)d}{2}\right)}\right)t^n$$

$$= \sum_{m_r(c)}\prod_{c,r}\left(\prod_{\alpha_c=1}^{N_c}\bigl[S^{m_{r,c}(\alpha_c)}\bigl(X^g_{\alpha_c}/Z_G(g)\bigr)\bigr]\mathbb{L}^{m_r(c)\left(F^g_{\alpha_c}+\frac{(r-1)d}{2}\right)}\right)t^{r m_r(c)}$$

$$= \prod_{c,r} \prod_{\alpha_c=1}^{N_c} \left(\sum_{m_{r,c}(\alpha_c)} \left[S^{m_{r,c}(\alpha_c)}\left(X^g_{\alpha_c}/Z_G(g)\right)\right] \mathbb{L}^{m_r(c)\left(F^g_{\alpha_c}+\frac{(r-1)d}{2}\right)} t^{rm_{r,c}(\alpha_c)} \right)$$

$$= \prod_{c,r} \prod_{\alpha_c=1}^{N_c} \left(1 - \mathbb{L}^{\left(F^g_{\alpha_c}+\frac{(r-1)d}{2}\right)} t^r\right)^{-[X^g_{\alpha_c}/Z_G(g)]}$$

$$= \prod_{c,r} \prod_{\alpha_c=1}^{N_c} \left(1 - \mathbb{L}^{\frac{(r-1)d}{2}} t^r\right)^{-\mathbb{L}^{F^g_{\alpha_c}}[X^g_{\alpha_c}/Z_G(g)]}$$

$$= \prod_{r\geq 1} \left(1 - \mathbb{L}^{\frac{(r-1)d}{2}} t^r\right)^{-[X,G]} = \prod_{r\geq 1} \left(1 - \left(\mathbb{L}^{\frac{d}{2}} t\right)^r\right)^{-\mathbb{L}^{-d/2}[X,G]}. \quad \square$$

Taking the Euler characteristic of both sides of equation (9), one gets Theorem 5 of [15]:

$$\sum_{n\geq 0} \chi(X^n, G_n) t^n = \prod_{r=1}^{\infty} (1 - t^r)^{-\chi(X,G)}.$$

Applying the Hodge–Deligne polynomial homomorphism, one gets the main result of [16]:

$$\sum_{n=1}^{\infty} e(X^n, G_n; u, v) t^n = \prod_{r=1}^{\infty} \prod_{p,q} \left(\frac{1}{1 - u^p v^q t^r (uv)^{(r-1)d/2}} \right)^{e^{p,q}_{(X,G)}}.$$

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