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Integration over Spaces of Nonparametrized Arcs and Motivic Versions of the Monodromy Zeta Function

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Abstract—The notions of integration of motivic type over the space of arcs factorized by the natural $\mathbb{C}^*$-action and over the space of nonparametrized arcs (branches) are developed. As an application, two motivic versions of the zeta function of the classical monodromy transformation of a germ of an analytic function on $\mathbb{C}^d$ are given that correspond to these notions. Another key ingredient in the construction of these motivic versions of the zeta function is the use of the so-called power structure, introduced by the authors, over the Grothendieck ring of varieties.

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INTRODUCTION

The notion of motivic integration invented by M. Kontsevich and developed by V. Batyrev, J. Denef, F. Loeser, *et al.* (see, e.g., [7, 8, 13]) is an analogue of p-adic integration. It can also be considered as a generalization of the notion of integration with respect to the Euler characteristic (see [15]) in two directions. First, instead of the usual Euler characteristic (with values in the ring of integers $\mathbb{Z}$), one considers a generalized (universal) Euler characteristic with values in the Grothendieck ring $K_0(\mathcal{V}_\mathbb{C})$ of complex algebraic varieties or/and in a modification (localization, completion) of it. Second, instead of integration over, say, a (finite-dimensional) algebraic variety, one integrates over the infinite-dimensional space of arcs. This notion, in particular, allows one to construct (or to define) motivic versions of some classical invariants of varieties or of singularities. The notion of a motivic version of an invariant is not well defined. There are only two obvious requirements: such a version should be an invariant itself, and it should reduce to the classical one under a corresponding additive homomorphism: Euler characteristic, Hodge–Deligne polynomial, etc. Sometimes such invariants can be defined as certain integrals with respect to the universal Euler characteristic. However, one may meet the following problem.

On the space of arcs, there is a natural $\mathbb{C}^*$-action defined by $a * \varphi(\tau) = \varphi(a\tau)$, $a \in \mathbb{C}^*$. Most natural constructible functions on the space of arcs that could participate in the definition of an invariant (say, the order of a fixed function along an arc) are invariant with respect to this action. The integral of such a function over the space of arcs with respect to the universal Euler characteristic is divisible by the class $(\mathbb{L} - 1)$ of the punctured complex line. Therefore, the specialization of this integral by the usual Euler characteristic morphism is equal to zero, and no motivic version of a usual invariant can be constructed in this way (as such an integral). For example, the Euler characteristic of the naive zeta function of J. Denef and F. Loeser (see [8]) is equal to zero.

To define motivic versions of (integer-valued) invariants of singularities, one has "to kill" this $\mathbb{C}^*$-action. One can imagine several ways to do this. One way is to consider a certain subspace of the space of arcs (that is not invariant under the $\mathbb{C}^*$-action) rather than the whole space of

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arcs. This was made in [9], where, instead of the space of arcs, the authors considered its subspace consisting of arcs with the prescribed first coefficient of the Taylor expansion of the function under consideration along an arc (see also [2, 10, 13]).

As another possibility, one can imagine the replacement of integration over the space of arcs by integration over another (infinite-dimensional) space. For example, integration over the space of functions or over its projectivization proved useful for certain problems: in [11, 3] and other papers, it was used for computing the Poincaré series of some (multi-index) filtrations. For instance, one can factorize the space of arcs by the $\mathbb{C}^*$-action. One can say that this method is inspired by the notion of projectivization. (It is not really the projectivization since this $\mathbb{C}^*$-action is not free.) For integration over the space of functions, it was used, e.g., in [3].

Instead of arcs, we can also try to consider branches, under which we understand arcs without parametrization. In other words, we consider the space of arcs (space of maps $\varphi\colon (\mathbb{C},0) \to W$) factorized by the group $\mathrm{Aut}_{\mathbb{C},0}$ of changes of coordinates $(\mathbb{C},0) \to (\mathbb{C},0)$. One can say that this notion has more geometric meaning than the space of arcs modulo the described $\mathbb{C}^*$-action, in particular, since this action itself depends on the choice of the coordinate in the source $(\mathbb{C},0)$. Moreover, it seems that integration over the set of branches should be easier to generalize to possible constructions of integration over sets of higher dimensional subspaces. The aim of the present paper is to define the notions of motivic integration over the space of arcs factorized by the $\mathbb{C}^*$-action and over the space of branches. We believe that generalizations of the notion of motivic integration to other infinite-dimensional spaces (different from the space of arcs) may be useful for applications.

As an application of these constructions, we consider possible motivic versions of the classical monodromy zeta function

$$\zeta_f(t) = \prod_{q\ge 0} \left\{\det\left[\mathrm{id} - th_*|_{H_q(V_f;\mathbb{C})}\right]\right\}^{(-1)^{q+1}}$$

of the germ of a function $f\colon (\mathbb{C}^d,0) \to (\mathbb{C},0)$ (V_f is the Milnor fiber and $h\colon V_f \to V_f$ is the classical monodromy transformation of the germ f). By the A'Campo formula [1], the zeta function $\zeta_f(t)$ can be written as the integral of the expression $1-t^m$ over the exceptional divisor of a resolution of the germ f (in the group $1+t\cdot\mathbb{Z}[[t]]$ with respect to multiplication). The arcs on $\mathbb{C}^d$ correspond to the points of the exceptional divisor. Thus, one can replace integration over the exceptional divisor by integration over the space of arcs. However, for the reasons described above, the corresponding integral over the space of arcs degenerates to 1 under the usual Euler characteristic homomorphism. To avoid this problem, we shall elaborate the notions of integration over the space of arcs factorized by the $\mathbb{C}^*$-action and over the space of branches. Here we discuss these notions only in the smooth case. An integral over the space of arcs factorized by the $\mathbb{C}^*$-action can be considered as a well-defined division by $(\mathbb{L}-1)$ of the corresponding integral over the space of arcs itself (which is otherwise defined only up to torsion; see, e.g., [6]).

Another problem that can be met in this way is to give meaning to an expression of the form $(1-t^m)^{-\chi(Z)}$ in the A'Campo formula when the usual Euler characteristic $\chi(Z)$ is replaced by the universal one (i.e., to give sense to the expression $(1-t^m)^{-[Z]}$ for $[Z]$ from the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ or from its localization $K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}]$ by the class $\mathbb{L}$ of the complex affine line). This is done with the use of the so-called power structure over the Grothendieck ring of varieties; this power structure was introduced by the authors in [12].

The zeta function $\zeta_f(t)$ of the monodromy transformation can be obtained from the motivic Milnor fiber defined by Denef and Loeser (see [9]). This zeta function is the limit at infinity of the Igusa motivic zeta function. The motivic Milnor fiber can be described in terms of ramified coverings of components of the exceptional divisor of a resolution of the germ f. The motivic invariants proposed here, which reduce to the zeta function $\zeta_f(t)$, are expressed in terms of the components of the exceptional divisor themselves rather than in terms of their coverings. In particular, for $d = 2$, i.e., for a function of two variables, all coefficients of the constructed series lie in the subring, of the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$, generated by the class $\mathbb{L}$ of the affine line.

The zeta function of the monodromy transformation can also be extracted from the motivic Milnor fiber with the $\mathbb{C}^*$-action considered in [10].

A motivic version of the monodromy zeta function can be constructed as the exponent of the generating series of Lefschetz numbers of iterates of the monodromy transformation of a singularity (see [9]). However, it seems that this can be done only after tensor multiplication by the field $\mathbb{Q}$ of rational numbers.

The Grothendieck semiring $S_0(\mathcal{V}_{\mathbb{C}})$ of complex quasi-projective varieties is the semigroup generated by isomorphism classes $[X]$ of such varieties modulo the relation $[X] = [X - Y] + [Y]$ for a Zariski closed subvariety $Y \subset X$; the multiplication is defined by the Cartesian product: $[X_1] \cdot [X_2] = [X_1 \times X_2]$. The Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ is the group generated by these classes with the same relation and the same multiplication. Let $\mathbb{L} \in K_0(\mathcal{V}_{\mathbb{C}})$ be the class of the complex affine line, and let $K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}]$ be the localization of the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ with respect to the class $\mathbb{L}$. The class $[X] \in S_0(\mathcal{V}_{\mathbb{C}})$ can be defined for any constructible subset X (as $\sum [X_i]$ for a partition $X = \bigcup X_i$ of the set X into a finite union of quasi-projective varieties).

In what follows, $\mathcal{R}$ will denote one of the discussed (semi)rings. There is a natural (semi)ring homomorphism $\chi\colon \mathcal{R} \to \mathbb{Z}$ that sends the class $[X]$ of a variety X to the Euler characteristic $\chi(X)$ with compact support of the set X. For a series $A(t) = \sum_{i=0}^{\infty} A_i t^i$ with coefficients A_i in $\mathcal{R}$, its specialization $\chi(A(t))$ under the Euler characteristic homomorphism is the power series $\sum_{i=0}^{\infty} \chi(A_i) t^i \in \mathbb{Z}[[t]]$.

The definition of the space of arcs on an algebraic variety and of the motivic measure on it can be found, e.g., in [7, 13] (here we use them only for smooth varieties).

1. THE SPACE OF ARCS FACTORIZED BY THE $\mathbb{C}^*$-ACTION

Let $\mathcal{L}_0$ be the space of arcs on the affine space $\mathbb{C}^d$ at the origin, i.e., the space of maps $\varphi\colon (\mathbb{C}, 0) \to (\mathbb{C}^d, 0)$ (in particular, $\mathcal{L}_0 \subset \mathfrak{m}\mathcal{O}^d_{\mathbb{C},0}$, where $\mathfrak{m}$ is the maximal ideal in the ring $\mathcal{O}_{\mathbb{C},0}$ of germs of functions). For $n \geq 0$, let $\mathcal{L}_0^n$ be the space of n-jets of arcs, i.e., the space $\mathcal{L}_0/\mathfrak{m}^n\mathcal{L}_0$ of arcs truncated at the level n. There is a natural $\mathbb{C}^*$-action on the spaces $\mathcal{L}_0$ and $\mathcal{L}_0^n$ defined by $a * \varphi(\tau) = \varphi(a\tau)$ ($a \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$). Let $\mathcal{L}_0^* := \mathcal{L}_0 \setminus \{0\}$ and $\mathcal{L}_0^{n*} := \mathcal{L}_0^n \setminus \{0\}$, and let $\mathcal{L}_0^*/\mathbb{C}^*$ and $\mathcal{L}_0^{n*}/\mathbb{C}^*$ be the corresponding spaces factorized by the $\mathbb{C}^*$-action. The space $\mathcal{L}_0^{n*}/\mathbb{C}^*$ is a (finite-dimensional) projective variety. Therefore, for a constructible subset Y of it, its generalized (universal) Euler characteristic $\chi_{\mathrm{g}}(Y) = [Y] \in K_0(\mathcal{V}_{\mathbb{C}})$ is defined. For $n \geq 0$, there exists a natural map $\pi_n\colon \mathcal{L}_0^*/\mathbb{C}^* \to (\mathcal{L}_0^{n*}/\mathbb{C}^*) \cup \{0\}$, and for $n \geq m$, there exists a natural map $\pi_{n,m}\colon (\mathcal{L}_0^{n*}/\mathbb{C}^*) \cup \{0\} \to (\mathcal{L}_0^{m*}/\mathbb{C}^*) \cup \{0\}$. The latter map is constructible. The space $\mathcal{L}_0^{m*}/\mathbb{C}^*$ can be decomposed into finitely many constructible subsets so that over each of them the map $\pi_{n,m}$ is a locally trivial fibration (in the Zariski topology) whose fiber is a vector space of dimension $d(n-m)$ factorized by a finite cyclic group action (the isotropy group of the corresponding jet). Since the class in $K_0(\mathcal{V}_{\mathbb{C}})$ of a vector space of dimension d factorized by a representation of a finite abelian group is equal to $\mathbb{L}^d$ (see [13, Lemma 5.1]), for a constructible subset Y in $\mathcal{L}_0^{m*}/\mathbb{C}^*$ one has $[\pi_{n,m}^{-1}(Y)] = \mathbb{L}^{d(n-m)} \cdot [Y]$. This inspires the following definitions.

Definition. A subset $X \subset \mathcal{L}_0^*/\mathbb{C}^*$ is called *cylindric* if there exist $n \geq 0$ and a constructible subset $Y \subset \mathcal{L}_0^{n*}/\mathbb{C}^*$ such that $X = \pi_n^{-1}(Y)$.

Definition. The *motivic measure* (or the *universal Euler characteristic*) of a cylindric subset $X \subset \mathcal{L}_0^*/\mathbb{C}^*$, $X = \pi_n^{-1}(Y)$ for $Y \subset \mathcal{L}_0^{n*}/\mathbb{C}^*$, is $\chi_{\mathrm{g}}(X) := [Y] \cdot \mathbb{L}^{-dn} \in K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}]$.

Definition. A function $\psi\colon \mathcal{L}_0^*/\mathbb{C}^* \to G$ with values in an abelian group G is *constructible* if it has countably many values and, for each $a \in G$, $a \neq 0$, the level set $\psi^{-1}(a)$ is constructible.

In the usual way (see, e.g., [7, 13]), one can define the integral of the function ψ with respect to the generalized Euler characteristic (the motivic measure) as

$$\int_{\mathcal{L}_0^*/\mathbb{C}^*} \psi\, d\chi_{\mathrm{g}} := \sum_{a\in G,\ a\neq 0} \chi_{\mathrm{g}}(\psi^{-1}(a))\cdot a.$$

Warning: not all constructible functions are integrable since the sum of a series may make no sense in the group $K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}]\otimes_{\mathbb{Z}} G$.

Let p be the factorization map $\mathcal{L}_0^* \to \mathcal{L}_0^*/\mathbb{C}^*$. For a constructible subset $X\subset \mathcal{L}_0^*/\mathbb{C}^*$, let $\widetilde{X}=p^{-1}(X)$ be the corresponding $\mathbb{C}^*$-invariant subset of the space $\mathcal{L}_0$ of arcs. One can easily see that $\chi_{\mathrm{g}}(\widetilde{X})=(\mathbb{L}-1)\chi_{\mathrm{g}}(X)$. This implies the following statement.

Proposition 1. *Let $\psi\colon \mathcal{L}_0^*/\mathbb{C}^*\to G$ be a constructible integrable function, and let $\widetilde{\psi}=\psi\circ p\colon \mathcal{L}_0^*\to G$ be the corresponding $\mathbb{C}^*$-invariant function on the space of arcs. Then the function $\widetilde{\psi}$ is integrable and*

$$\int_{\mathcal{L}_0} \widetilde{\psi}\, d\chi_{\mathrm{g}} = (\mathbb{L}-1)\int_{\mathcal{L}_0^*/\mathbb{C}^*} \psi\, d\chi_{\mathrm{g}}.$$

This means that integrals over the space $\mathcal{L}_0^*/\mathbb{C}^*$ can be considered as well-defined versions of the corresponding integrals over the space of arcs itself divided by $(\mathbb{L}-1)$. It is not quite clear whether such division is well defined in the ring $K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}]$. Usually, this division can be made formally when such an integral is computed (say, in terms of a resolution); however, either the result should be considered modulo elements from the annihilator of $(\mathbb{L}-1)$, or one should prove that the result does not depend on a resolution. For instance, in [9] it was shown that the motivic Milnor fiber (introduced in [6] up to $(\mathbb{L}-1)$-torsion) is well defined. Therefore, integration over the space $\mathcal{L}_0^*/\mathbb{C}^*$ of arcs modulo $\mathbb{C}^*$ can be considered as a formalization of this procedure.

2. THE SPACE OF BRANCHES ON $(\mathbb{C}^d,0)$

Now we adapt the construction described above to the space of arcs factorized by the group $\mathrm{Aut}_{\mathbb{C},0}$ of (formal) local changes of coordinates in $(\mathbb{C},0)$. For an arc $\varphi\colon(\mathbb{C},0)\to(\mathbb{C}^d,0)$ and $h\in\mathrm{Aut}_{\mathbb{C},0}$ ($h\colon(\mathbb{C},0)\to(\mathbb{C},0)$), let $h*\varphi(\tau):=\varphi(h^{-1}(\tau))$. This defines an action of the group $\mathrm{Aut}_{\mathbb{C},0}$ on the space of arcs.

Definition. An orbit of the described action is called a *branch* (on $(\mathbb{C}^d,0)$).

The group $\mathrm{Aut}_{\mathbb{C},0}$ also acts on the jet space $\mathcal{L}_0^n$. Moreover, such an action coincides with the induced action of the group $\mathrm{Aut}_{\mathbb{C},0}^{(n)}$ of n-jets of coordinate changes on $\mathcal{L}_0^n$, whose elements are (or can be considered as) polynomials $a_1\tau+a_2\tau^2+\ldots+a_n\tau^n$, $a_1\neq 0$. Let $\mathcal{B}_0:=\mathcal{L}_0/\mathrm{Aut}_{\mathbb{C},0}$ and $\mathcal{B}_0^n:=\mathcal{L}_0^n/\mathrm{Aut}_{\mathbb{C},0}$ be the factor spaces of this action, and let $\mathcal{B}_0^{n*}:=\mathcal{B}_0^n\setminus\{0\}=\mathcal{L}_0^{n*}/\mathrm{Aut}_{\mathbb{C},0}$. All these spaces are considered simply as sets without an additional structure.

The jet space $\mathcal{L}_0^n$ has a natural filtration defined by powers of the maximal ideal $\mathfrak{m}$ of the ring $\mathcal{O}_{\mathbb{C},0}$ of germs of functions at the origin in $\mathbb{C}$:

$$\{0\}\subset\mathfrak{m}^{n-1}\mathcal{L}_0^n\subset\mathfrak{m}^{n-2}\mathcal{L}_0^n\subset\ldots\subset\mathfrak{m}^1\mathcal{L}_0^n\subset\mathfrak{m}^0\mathcal{L}_0^n=\mathcal{L}_0^n.$$

This filtration respects the action of the group $\mathrm{Aut}_{\mathbb{C},0}$. Let

$$\mathcal{L}_0^{n*}=\bigcup_{i=0}^{n-1}\mathfrak{m}^i\mathcal{L}_0^n\setminus\mathfrak{m}^{i+1}\mathcal{L}_0^n$$

be the corresponding decomposition of the punctured jet space $\mathcal{L}_0^{n*} = \mathcal{L}_0^n \setminus \{0\}$. The space $\mathfrak{m}^i\mathcal{L}_0^n \setminus \mathfrak{m}^{i+1}\mathcal{L}_0^n$ consists of n-jets of arcs for which all d coordinate functions start at least with monomials τ^{i+1} and at least one of them starts precisely with this monomial. The isotropy group of such a jet for the action of the group $\mathrm{Aut}^{(n)}_{\mathbb{C},0}$ consists of series (coordinate changes in $(\mathbb{C},0)$) of the form $\tau + a_{n-i+1}\tau^{n-i+1} + \ldots + a_n\tau^n$ up to elements of finite order. Thus, the dimension of the isotropy group is equal to i, and the dimension of the orbit is equal to $n-i$. Therefore, the factor space $(\mathfrak{m}^i\mathcal{L}_0^n \setminus \mathfrak{m}^{i+1}\mathcal{L}_0^n)/\mathrm{Aut}^{(n)}_{\mathbb{C},0}$ is a quasi-projective variety.

Definition. A subset $Y \subset \mathcal{B}_0^{n*}$ is called *constructible* if, for each $i = 0,\ldots,n-1$, the set $Y_i := Y \cap \big((\mathfrak{m}^i\mathcal{L}_0^n \setminus \mathfrak{m}^{i+1}\mathcal{L}_0^n)/\mathrm{Aut}^{(n)}_{\mathbb{C},0}\big)$ is a constructible subset of the set $(\mathfrak{m}^i\mathcal{L}_0^n \setminus \mathfrak{m}^{i+1}\mathcal{L}_0^n)/\mathrm{Aut}^{(n)}_{\mathbb{C},0}$. The *generalized Euler characteristic* $\chi_{\mathrm{g}}(Y) = [Y]$ of the set Y is the sum $\sum_{i=0}^{n}[Y_i]$ of the classes of its parts Y_i.

Let $\pi_n\colon \mathcal{B}_0 \to \mathcal{B}_0^n$ and, for $n \ge m$, $\pi_{n,m}\colon \mathcal{B}_0^n \to \mathcal{B}_0^m$ be the natural maps. For $n \ge m$, there exists a stratification of $\mathcal{B}_0^{m*}$ such that over each stratum the map $\pi_{n,m}\colon \mathcal{B}_0^n \to \mathcal{B}_0^m$ is a locally trivial fibration (in the Zariski topology) whose fiber is the factor of a $(d-1)(n-m)$-dimensional vector space by a finite cyclic group action. Therefore, for a constructible subset Y in $\mathcal{B}_0^{m*}$, $n \ge m$, one has $[\pi_{n,m}^{-1}(Y)] = \mathbb{L}^{(d-1)(n-m)}[Y]$. This inspires the following definitions.

Definition. A subset $X \subset \mathcal{B}_0$ is called *cylindric* if there exist $n \ge 0$ and a constructible subset $Y \subset \mathcal{B}_0^{n*} \subset \mathcal{B}_0^n$ such that $X = \pi_n^{-1}(Y)$.

Definition. The *motivic measure* (or the *universal Euler characteristic*) of a cylindric subset $X \subset \mathcal{B}_0$, $X = \pi_n^{-1}(Y)$ for $Y \subset \mathcal{B}_0^{n*}$, is $\chi_{\mathrm{g}}(X) = [Y]\cdot\mathbb{L}^{-(d-1)n} \in K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}]$.

This measure induces the corresponding notion of integration over the space of branches.

Let p_{b} be the factorization map $\mathcal{L}_0 \to \mathcal{L}_0/\mathrm{Aut}_{\mathbb{C},0} = \mathcal{B}_0$, and let p_{b}^n be the same map $\mathcal{L}_0^n \to \mathcal{L}_0^n/\mathrm{Aut}^{(n)}_{\mathbb{C},0} = \mathcal{B}_0^n$:

$$\begin{array}{ccc} \mathcal{L}_0 & \xrightarrow{\pi_n} & \mathcal{L}_0^n \\ {\scriptstyle p_{\mathrm{b}}}\big\downarrow & & \big\downarrow{\scriptstyle p_{\mathrm{b}}^n} \\ \mathcal{B}_0 & \xrightarrow{\pi_n} & \mathcal{B}_0^n \end{array}$$

For a cylindric subset $X \subset \mathcal{B}_0$, $X = \pi_n^{-1}(Y)$, $Y \subset \mathcal{B}_0^{n*}$, let $\widetilde{X} = p_{\mathrm{b}}^{-1}(X)$ be the corresponding $\mathrm{Aut}_{\mathbb{C},0}$-invariant set of arcs. Let $Y_i = Y \cap \big((\mathfrak{m}^i\mathcal{L}_0^n \setminus \mathfrak{m}^{i+1}\mathcal{L}_0^n)/\mathrm{Aut}_{\mathbb{C},0}\big)$ and $\widetilde{Y}_i = (p_{\mathrm{b}}^n)^{-1}(Y_i)$ for $i = 0,1,\ldots,n-1$. One has $Y = \bigcup_{i=0}^{n} Y_i$. Let $X_i := (\pi_n)^{-1}(Y_i)$ and $\widetilde{X}_i = (\pi_n \circ p_{\mathrm{b}})^{-1}(Y_i)$. Then $X = \bigcup_{i=0}^{n-1} X_i$ and $\widetilde{X} = \bigcup_{i=0}^{n-1}\widetilde{X}_i$. One has $[\widetilde{Y}_i] = (\mathbb{L}-1)\mathbb{L}^{n-i-1}[Y_i]$. Therefore,

$$\chi_{\mathrm{g}}(\widetilde{X}_i) = \mathbb{L}^{-dn}[\widetilde{Y}_i] = (\mathbb{L}-1)\,\mathbb{L}^{n-i-1-dn}[Y_i] = (\mathbb{L}-1)\,\mathbb{L}^{-i-1}\chi_{\mathrm{g}}(X_i).$$

This implies the following statement. Let ord be the order function on the space of arcs: $\mathrm{ord}(\varphi) = i+1$ if $\varphi \in \mathfrak{m}^i\mathcal{L}_0 \setminus \mathfrak{m}^{i+1}\mathcal{L}_0$, i.e., $\mathrm{ord}(\varphi) = i+1$ for $\varphi \in \widetilde{X}_i$ (ord is an $\mathrm{Aut}_{\mathbb{C},0}$-invariant function on the space of arcs). Let T^{ord} be the corresponding map $\mathcal{L}_0 \to \mathbb{Z}[[T]]$ (where $\mathbb{Z}[[T]]$ is considered as an abelian group with respect to summation). For a constructible function $\psi\colon \mathcal{B}_0 \to G$ (G is an abelian group), let $\widetilde{\psi} = \psi \circ p_{\mathrm{b}}\colon \mathcal{L}_0 \to G$ be the corresponding $\mathrm{Aut}_{\mathbb{C},0}$-invariant function on the space of arcs and let $\widetilde{\psi}^* = \widetilde{\psi} \otimes T^{\mathrm{ord}}$ be the corresponding function with values in $G \otimes_{\mathbb{Z}} \mathbb{Z}[[T]] = G[[T]]$.

Proposition 2. *If the function ψ is constructible and integrable, then the function $\widetilde{\psi}^*$ is integrable and*

$$\int\limits_{\mathcal{L}_0} \widetilde{\psi}^*\, d\chi_{\mathrm{g}}\,\big|_{T\mapsto\mathbb{L}} = (\mathbb{L}-1)\int\limits_{\mathcal{B}_0} \psi\, d\chi_{\mathrm{g}}.$$

Thus, an integral over the space of branches of a G-valued function, multiplied by $(\mathbb{L}-1)$, is defined by a certain integral over the space of arcs, but of a $G[[T]]$-valued function.

Proof. Let $X_a := \psi^{-1}(a)$. Then

$$\begin{aligned}(\mathbb{L}-1)\int\limits_{\mathcal{B}_0}\psi\,d\chi_{\mathrm{g}} &= (\mathbb{L}-1)\sum_{a\in G,\ a\neq 0}\chi_{\mathrm{g}}(X_a)\cdot a = (\mathbb{L}-1)\sum_{a\in G,\ a\neq 0}\left(\sum_{i=0}^{n(a)-1}\chi_{\mathrm{g}}(X_{a,i})\right)\cdot a\\
&= \sum_{a\in G,\ a\neq 0}\left(\sum_{i=0}^{n(a)-1}\mathbb{L}^{i+1}\chi_{\mathrm{g}}(\widetilde{X}_{a,i})\right)\cdot a = \sum_{a\in G,\ a\neq 0}\left(\sum_{i=0}^{n(a)-1}T^{\mathrm{ord}}\chi_{\mathrm{g}}(\widetilde{X}_{a,i})\right)\Big|_{T\mapsto\mathbb{L}}\cdot a\\
&= \int\limits_{\mathcal{L}_0}\widetilde{\psi}^*\,d\chi_{\mathrm{g}}\,\big|_{T\mapsto\mathbb{L}}\end{aligned}$$

(since $\mathrm{ord}(\varphi) = i+1$ for $\varphi\in\widetilde{X}_{a,i}$). □

3. INTEGRALS IN TERMS OF A RESOLUTION

Let $f\colon(\mathbb{C}^d,0)\to(\mathbb{C},0)$ be a germ of an analytic function, and let $\pi\colon(\mathcal{X},\mathcal{D})\to(\mathbb{C},0)$ be a resolution of the germ f, i.e., a proper modification of $(\mathbb{C}^d,0)$ which is an isomorphism outside the zero-level set $\{f=0\}$; $\mathcal{X}$ is smooth, and the exceptional divisor $\mathcal{D}=\pi^{-1}(0)$ and the total transform $\mathcal{E}=(f\circ\pi)^{-1}(0)$ of the zero-level set of f are normal crossing divisors on $\mathcal{X}$. For $\varphi\in\mathcal{L}_0$, let $v_f(\varphi)=\mathrm{ord}_f(\varphi)$ be the order of the function f on the arc φ, $v_f=\mathrm{ord}_f\colon\mathcal{L}_0\to\mathbb{Z}\cup\{\infty\}$. The function v_f is $\mathrm{Aut}_{\mathbb{C},0}$-invariant (and therefore $\mathbb{C}^*$-invariant). The function $\int_{\mathcal{L}_0}t^{v_f}\,d\chi_{\mathrm{g}}$ is the (local) naive zeta function $Z_{\mathrm{naive}}(t)$ of Denef and Loeser (see [8]), which is a rational function (this follows from its description in terms of a resolution).

Let $\mathcal{E}=\bigcup_{i\in I_0}E_i$ be the decomposition of the total transform $\mathcal{E}$ into the union of irreducible components, and let $I_0=I_0'\cup I_0''$, where, for $i\in I_0'$ (respectively, for $i\in I_0''$), $E_i\subset\mathcal{D}$ (respectively, E_i is a component of the strict transform of $\{f=0\}$). For $i\in I_0$, let N_i be the multiplicity of the lifting $f\circ\pi$ of the function f to the space $\mathcal{X}$ of the resolution along the component E_i, and let ν_i-1 be the multiplicity of the d-form π^*dx along the corresponding component E_i ($dx=dx_1\wedge\ldots\wedge dx_d$ is the volume form on $\mathbb{C}^d$). For $i\in I_0$, let $\overset{\circ}{E}_i := E_i\setminus\bigcup_{j\neq i}E_j$ be "the smooth part" of the component E_i; for $I\subset I_0$, $I\neq\varnothing$, let $E_I:=\bigcap_{i\in I}E_i$ and $\overset{\circ}{E}_I := E_I\setminus\bigcup_{j\in I_0\setminus I}E_j$.

The arguments of [6, Theorem 2.2.1] imply the following statement.

Proposition 3.

$$Z_{\mathrm{naive},\,\mathcal{L}_0^*/\mathbb{C}^*}(t) := \int\limits_{\mathcal{L}_0^*/\mathbb{C}^*}t^{v_f}\,d\chi_{\mathrm{g}} = \sum_{I\subset I_0,\ \varnothing\neq I\not\subset I_0''}(\mathbb{L}-1)^{|I|-1}\big[\overset{\circ}{E}_I\big]\prod_{i\in I}\frac{\mathbb{L}^{-\nu_i}\,t^{N_i}}{1-\mathbb{L}^{-\nu_i}\,t^{N_i}}\,.$$

Now suppose that the resolution $\pi\colon(\mathcal{X},\mathcal{D})\to(\mathbb{C}^d,0)$ factorizes through the blowing-up $\pi_0\colon(\mathcal{X}_0,\mathbb{CP}^{d-1})\to(\mathbb{C}^d,0)$ of the origin in $(\mathbb{C}^d,0)$, i.e., $\pi=\pi_0\circ\pi'$, $\pi'\colon(\mathcal{X},\mathcal{D})\to(\mathcal{X}_0,\mathbb{CP}^{d-1})$. For $i\in I_0$, let M_i be the multiplicity of the component E_i in the divisor $\pi'^*(\mathbb{CP}^{d-1})$. The same arguments as those used in Proposition 2 give the following statement.

Proposition 4.

$$Z_{\mathrm{naive},\,\mathcal{B}_0}(t) := \int\limits_{\mathcal{B}_0}t^{v_f}\,d\chi_{\mathrm{g}} = \sum_{I\subset I_0,\ \varnothing\neq I\not\subset I_0''}(\mathbb{L}-1)^{|I|-1}\big[\overset{\circ}{E}_I\big]\prod_{i\in I}\frac{\mathbb{L}^{-\nu_i-M_i}\,t^{N_i}}{1-\mathbb{L}^{-\nu_i-M_i}\,t^{N_i}}\,.$$

4. POWER STRUCTURE OVER THE GROTHENDIECK RING OF VARIETIES

In what follows, we will focus on integrals with respect to the motivic measure of a function $(1-t^{v_f})^{-1}$ whose values are considered as elements of the group $1+t\cdot K_0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{-1}][[t]]$ with the product as a group operation. Let $\mathcal{S}$ be either the space $\mathcal{L}_0$ of arcs, or $\mathcal{L}_0^*/\mathbb{C}^*$, or the space $\mathcal{B}_0$ of branches. To emphasize that the integration is with respect to the multiplicative structure, we denote such an integral by

$$\int\limits_{\mathcal{S}} (1-t^{v_f})^{-d\chi_\mathrm{g}}.$$

If $X_n=\{\varphi\in\mathcal{S}\colon v_f(\varphi)=n\}$, then

$$\int\limits_{\mathcal{S}} t^{v_f}\,d\chi_\mathrm{g}=\sum_{n=1}^{\infty}\chi_\mathrm{g}(X_n)\cdot t^n.$$

For the integral of $(1-t^{v_f})^{-1}$, we have

$$\int\limits_{\mathcal{S}} (1-t^{v_f})^{-d\chi_\mathrm{g}}=\prod_{n=1}^{\infty}(1-t^n)^{-\chi_\mathrm{g}(X_n)},$$

where the expression $(1-t^n)^a$, $a\in K_0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{-1}]$, is understood in the sense of [12]. There, we constructed a so-called power structure over a (semi)ring $\mathcal{R}$ (any of $S_0(\mathcal{V}_\mathbb{C})$, $K_0(\mathcal{V}_\mathbb{C})$, and $K_0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{-1}]$).

Definition. A *power structure* over a (semi)ring R is a map

$$(1+t\cdot R[[t]])\times R\to 1+t\cdot R[[t]]\colon\ (A(t),m)\mapsto (A(t))^m$$

that possesses the following properties:

(i) $(A(t))^0=1$,
(ii) $(A(t))^1=A(t)$,
(iii) $(A(t)\cdot B(t))^m=(A(t))^m\cdot(B(t))^m$,
(iv) $(A(t))^{m+n}=(A(t))^m\cdot(A(t))^n$,
(v) $(A(t))^{mn}=((A(t))^n)^m$.

According to the construction in [12], for $a=-[Z]$, where Z is a quasi-projective variety, we have

$$(1-t)^a=\zeta_Z(t)=1+\sum_{k=1}^{\infty}[S^kZ]\cdot t^k,$$

where $S^kZ=Z^k/S_k$ is the kth symmetric power of the space Z ($\zeta_Z(t)$ is the Kapranov zeta function of the variety Z, see [13]) and $(1-t)^{a/\mathbb{L}^i}=(1-t)^a\big|_{t\mapsto t/\mathbb{L}^i}$.

In [12], we also defined a map $\mathbf{Exp}\colon t\cdot K_0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{-1}][[t]]\to 1+t\cdot K_0(\mathcal{V}_\mathbb{C})[\mathbb{L}^{-1}][[t]]$ that is an isomorphism of abelian groups (with addition and multiplication as group operations, respectively). It is defined by the equation

$$\mathbf{Exp}\left(\sum_{i=1}^{\infty}a_it^i\right)=\prod_{i=1}^{\infty}(1-t^i)^{-a_i}.$$

One can easily see that

$$\int\limits_{\mathcal{S}} (1-t^{v_f})^{-d\chi_\mathrm{g}}=\mathbf{Exp}\left(\int\limits_{\mathcal{S}} t^{v_f}\,d\chi_\mathrm{g}\right). \tag{1}$$

By definition, a power structure defines the powers of power series in one variable. If one intends to apply the constructions of this paper to the (multivariable) Alexander invariants of a collection of functions $f_1, \dots, f_r$ on $(\mathbb{C}^d, 0)$, one can consider integrals of the form

$$\int_{\mathcal{S}} (1 - \underline{t}^{\underline{v}(\varphi)})^{d\chi_{\mathrm{g}}},$$

where $\underline{v}(\varphi) = (v_1(\varphi), \dots, v_r(\varphi))$, $v_i(\varphi) = v_{f_i}(\varphi)$, $\underline{t} = (t_1, \dots, t_r)$, and $\underline{t}^{\underline{v}} = t_1^{v_1} \cdot \ldots \cdot t_r^{v_r}$. Another reason to consider power series in several variables can be seen from Proposition 2 (if, say, $G = K_0(\mathcal{V}_{\mathbb{C}})[[t]]$ and $\psi(\varphi) = t^{v_f(\varphi)}$). This makes it reasonable to consider expressions of the form $A(\underline{t})^M$, where $A(\underline{t}) = 1 + \sum_{\underline{k}\in\mathbb{Z}^r_{\geq 0}, \underline{k}\neq 0} A_{\underline{k}}\, \underline{t}^{\underline{k}}$, $A_{\underline{k}} \in \mathcal{R}$, and $M \in \mathcal{R}$. If there exists a power structure over a *ring* R, there is a natural way to define the corresponding expression in the multivariable case as well. However, if R is a *semiring*, this does not work in general. Since the elements of the Grothendieck semiring $S_0(\mathcal{V}_{\mathbb{C}})$ of complex quasi-projective varieties have more geometric meaning (they are represented by "genuine" varieties rather than by virtual ones), it is reasonable to give a geometric definition of this operation over this semiring.

It is possible (and convenient) to give the definition in a slightly more general setting. Let S be an ordered abelian semigroup, with zero as the smallest element, such that each element $s \in S$ has only finitely many representations as a sum of elements of S. For a (semi)ring R, the corresponding (semi)group (semi)ring $R[[S]]$ is defined, which consists of formal sums (series) of the form $\sum_{s\in S} r_s s$, where $r_s \in R$, with the natural operations $\sum r'_s s + \sum r''_s s = \sum (r'_s + r''_s) s$ and $\left(\sum r'_s s\right) \cdot \left(\sum r''_s s\right) = \sum (r'_{s_1} \cdot r''_{s_2})(s_1 + s_2)$, where one should combine summands with the same $s_1 + s_2$ in the last expression. Let $R_+[[S]]$ be the set (an ideal in $R[[S]]$) of series of the form $\sum_{s\in S,\, s>0} r_s s$, $r_s \in R$. The (semi)ring $R[[\underline{t}]]$ of formal power series in r variables $\underline{t} = (t_1, \dots, t_r)$ with coefficients in R is the semigroup (semi)ring $R[[S]]$ for $S = \mathbb{Z}^r_{\geq 0}$.

It is convenient to describe the power structure over the Grothendieck semiring $S_0(\mathcal{V}_{\mathbb{C}})$ in terms of graded spaces (sets). A graded space (with grading from $S_{>0}$) is a space $\mathcal{A}$ with a function $I_{\mathcal{A}}$ on it with values in $S_{>0}$. The element $I_{\mathcal{A}}(a)$ of the semigroup S is called the weight of the point $a \in \mathcal{A}$. With a series $A \in 1 + S_0(\mathcal{V}_{\mathbb{C}})_+[[S]]$, $A = 1 + \sum_{s\in S,\, s>0} [A_s] s$, one associates the graded space $\mathcal{A} = \coprod_{s\in S,\, s>0} A_s$ with the weight function $I_{\mathcal{A}}$ that sends all points of A_s to $s \in S$. Conversely, to a graded space $(\mathcal{A}, I_{\mathcal{A}})$, there corresponds the series $A = 1 + \sum_{s\in S,\, s>0} [A_s] s$ with $A_s = I_{\mathcal{A}}^{-1}(s)$. To describe the series $A^{[M]}$, we will first describe the corresponding graded space $\mathcal{A}^M$. The space $\mathcal{A}^M$ consists of pairs (K, φ), where K is a finite subset of the variety M and φ is a map from K to the graded space $\mathcal{A}$. The weight function $I_{\mathcal{A}^M}$ on $\mathcal{A}^M$ is defined by $I_{\mathcal{A}^M}(K, \varphi) = \sum_{k\in K} I_{\mathcal{A}}(\varphi(k))$. This gives a set-theoretic description of the series $A^{[M]}$. To describe the coefficients of this series as elements of the Grothendieck semiring $S_0(\mathcal{V}_{\mathbb{C}})$, one can write the series as

$$A^{[M]} = 1 + \sum_{s_0\in S,\ s_0>0} \left\{ \sum_{\underline{k}\colon\ \sum s k_s = s_0} \left[\left(\left(\prod_s M^{k_s} \right) \setminus \Delta \right) \times \prod_s A_s^{k_s} \Big/ \prod_s S_{k_s} \right] \right\} \cdot s_0,$$

where $\underline{k} = \{k_s \colon s \in S,\ s > 0,\ k_s \in \mathbb{Z}_{\geq 0}\}$, Δ is the "large diagonal" in $M^{\Sigma k_s}$, which consists of $(\sum k_s)$-tuples of points of M with at least two coinciding ones, and the permutation group S_{k_s} acts by permuting the corresponding k_s factors in $\prod_s M^{k_s} \supset \left(\prod_s M^{k_s}\right) \setminus \Delta$ and the spaces A_s simultaneously (the connection between this formula and the description above is clear).

The same structure can be constructed over the Grothendieck (semi)ring of varieties with an action of a group.

5. MOTIVIC VERSIONS OF THE MONODROMY ZETA FUNCTION

Let

$$\eta_{\mathcal{S}}(t) := \int_{\mathcal{S}} (1 - t^{v_f})^{-d\chi_{\mathrm{g}}} = \mathbf{Exp}(Z_{\mathrm{naive},\mathcal{S}}(t)),$$

where $\mathcal{S}$ is $\mathcal{L}_0$, $\mathcal{L}_0^*/\mathbb{C}^*$, or the space of branches $\mathcal{B}_0$. Let us compute the specialization of the series $\eta_{\mathcal{S}}(t)$ under the (usual) Euler characteristic morphism.

Proposition 5. *For $\mathcal{S} = \mathcal{L}_0$, we have $\chi(\eta_{\mathcal{S}}(t)) = 1$; for $\mathcal{S} = \mathcal{L}_0^*/\mathbb{C}^*$ or $\mathcal{B}_0$, we have*

$$\eta(t) = \chi(\eta_{\mathcal{S}}(t)) = \prod_{k=1}^{\infty} \zeta_f(t^k),$$

where $\zeta_f(t)$ is the classical monodromy zeta function of the germ f.

Proof. For $\mathcal{S} = \mathcal{L}_0$, this follows from the expression for the integral $\int_{\mathcal{L}_0} t^{v_f} d\chi_{\mathrm{g}}$ in terms of a resolution since all terms in it are divisible by $(\mathbb{L} - 1)$. For $\mathcal{S} = \mathcal{L}_0^*/\mathbb{C}^*$ or $\mathcal{B}_0$, it follows from Propositions 3 and 4 that under the Euler characteristic morphism, all terms corresponding to nontrivial intersections of the components (i.e., to I with $|I| > 1$) vanish and

$$\eta(t) = \chi(\eta_{\mathcal{S}}(t)) = \prod_{i \in I_0'} \prod_{k=1}^{\infty} (1 - t^{kN_i})^{-\chi(\mathring{E}_i)} = \prod_{k=1}^{\infty} \left(\prod_{i \in I_0'} (1 - t^{kN_i})^{-\chi(\mathring{E}_i)} \right)$$

(this follows from the equation $\chi\big((A(t))^{[M]}\big) = (\chi(A(t)))^{\chi(M)}$, see [12]). According to the A'Campo formula [1], this expression is equal to

$$\eta(t) = \prod_{k=1}^{\infty} \zeta_f(t^k). \quad \square \tag{2}$$

Let $\mu(n)$ be the Möbius function:

$$\mu(n) = \begin{cases} 0 & \text{if } n \text{ has one or more repeated prime factors,} \\ 1 & \text{if } n = 1, \\ (-1)^k & \text{if } n \text{ is a product of } k \text{ distinct primes.} \end{cases}$$

The property $\mu(1) = 1$ and $\sum_{i|n} \mu(i) = 0$ for $n > 1$ implies the following statement.

Corollary 1.

$$\zeta_f(t) = \prod_{i=1}^{\infty} (\eta(t^i))^{\mu(i)}.$$

This leads to the following motivic versions of the monodromy zeta function.

Definition. For $\mathcal{S} = \mathcal{L}_0^*/\mathbb{C}^*$ or $\mathcal{B}_0$, the series

$$\zeta_{f,\mathcal{S}}(t) := \prod_{i=1}^{\infty} (\eta_{\mathcal{S}}(t^i))^{\mu(i)}$$

is called a motivic version of the monodromy zeta function for *arcs*$/\mathbb{C}^*$ or *branches*, respectively.

In other words,

$$\zeta_{f,\mathcal{S}}(t) = \mathbf{Exp}\left(\int\limits_{\mathcal{S}} \sum_{i=1}^{\infty} \mu(i) t^{iv_f}\, d\chi_{\mathrm{g}}\right) = \mathbf{Exp}\left(\sum_{n=1}^{\infty}\left(\sum_{k|n} \mu(k)\chi_{\mathrm{g}}\big(X_{n/k}\big)\right) t^n\right)$$
$$= \prod_{n\ge 1} (1-t^n)^{-\left(\sum_{k|n}\mu(k)\chi_{\mathrm{g}}(X_{n/k})\right)}.$$

Proposition 6. *The series $\zeta_{f,\mathcal{S}}(t) \in K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}][[t]]$ is an invariant of the germ f, and its specialization under the Euler characteristic morphism coincides with (the Taylor expansion of) the monodromy zeta function $\zeta_f(t)$.*

Propositions 3 and 4 yield the following theorem.

Theorem 1. *For a resolution $\pi\colon (\mathcal{X},\mathcal{D}) \to (\mathbb{C}^d,0)$ of the germ f, one has*

$$\zeta_{f,\mathcal{L}_0^*/\mathbb{C}^*}(t) = \prod_{m=1}^{\infty} \prod_{I\subset I_0,\ \varnothing\ne I\not\subset I_0''} \left(\prod_{\{k_i\colon i\in I\}} (1-\mathbb{L}^{-\underline{k}\,\underline{\nu}}\, t^{m\underline{k}\,\underline{N}})\right)^{-\mu(m)(\mathbb{L}-1)^{\#I-1}[\mathring{E}_I]}$$

(*here* $\underline{k} = \{k_i\colon i\in I\}$, $\underline{\nu} = \{\nu_i\colon i\in I\}$, $\underline{N} = \{N_i\colon i\in I\}$, $\underline{k}\,\underline{\nu} = \sum_{i\in I} k_i\nu_i, \dots$).

If the resolution π factorizes through the blowing-up $\pi_0\colon (\mathcal{X}_0, \mathbb{CP}^{d-1}) \to (\mathbb{C}^d,0)$ of the origin in $\mathbb{C}^d$, then

$$\zeta_{f,\mathcal{B}_0}(t) = \prod_{m=1}^{\infty} \prod_{I\subset I_0,\ \varnothing\ne I\not\subset I_0''} \left(\prod_{\{k_i\colon i\in I\}} (1-\mathbb{L}^{-\underline{k}(\underline{\nu}+\underline{M})} t^{m\underline{k}\,\underline{N}})\right)^{-\mu(m)(\mathbb{L}-1)^{\#I-1}[\mathring{E}_I]}$$

($\underline{M} = \{M_i\colon i\in I\}$).

6. FINAL REMARKS

One of the most interesting problems about the naive Denef–Loeser zeta function, the "monodromy conjecture," states that there exists a set S of pairs (ν,N) of nonnegative integers with $N>0$ such that the naive zeta function $Z_{\mathrm{naive}}(t)$ always belongs to $K_0(\mathcal{V}_{\mathbb{C}})[\mathbb{L}^{-1}][(1-\mathbb{L}^{-\nu}t^N)^{-1}]_{\{(\nu,N)\}\in S}$ and if $q = -\nu/N$ for $(\nu,N)\in S$, then $\exp(-2i\pi q)$ is an eigenvalue of the classical local monodromy operator around zero at some point $P\in f^{-1}(0)$ (see [6]).

Proposition 2 makes it clear that the monodromy conjecture for $Z_{\mathrm{naive}}(t) = \int_{\mathcal{L}_0} t^{v_f}\, d\chi_{\mathrm{g}}$ is equivalent to the monodromy conjecture for $Z_{\mathrm{naive},\,\mathcal{L}_0^*/\mathbb{C}^*}(t) = \int_{\mathcal{L}_0^*/\mathbb{C}^*} t^{v_f}\, d\chi_{\mathrm{g}}$. Because of identities (1) and (2), one has

$$\zeta_{f,\mathcal{S}}(t) = \mathbf{Exp}\left(\sum_{i=1}^{\infty} \mu(i) Z_{\mathrm{naive},\,\mathcal{L}_0^*/\mathbb{C}^*}(t^i)\right) \qquad \text{and} \qquad \chi\Big(\mathbf{Exp}\big(Z_{\mathrm{naive},\,\mathcal{L}_0^*/\mathbb{C}^*}(t)\big)\Big) = \prod_{k=1}^{\infty} \zeta_f(t^k).$$

The monodromy conjecture was originally stated in the p-adic case for the Igusa local zeta functions (see, e.g., [4]). In [5] Denef and Loeser introduced an analytic invariant, called the *local topological zeta function* of a germ f (as a kind of a limit of the local Igusa zeta function), whose initial definition was written in terms of a resolution,

$$Z_{\mathrm{top},0}(f,s) = \sum_{I\subset I_0,\ \varnothing\ne I\not\subset I_0''} \chi(\mathring{E}_I) \prod_{i\in I} \frac{1}{N_i s + \nu_i},$$

although it does not depend on a resolution. If one first replaces t by $\mathbb{L}^{-s}$ in $Z_{\mathrm{naive}}(t)$, expands $\mathbb{L}^{-s}$ and $(\mathbb{L}-1)\big(1-\mathbb{L}^{-\nu+Ns}\big)^{-1}$ in series in $\mathbb{L}-1$, and finally takes the Euler characteristic, then one gets $Z_{\mathrm{top},0}(f,s)$. The monodromy conjecture was also stated for this function in [5]. See [14] for more information about this conjecture.

In the case of integration over the space of branches, one can apply the same procedure in order to get a new analytic invariant of the germ f. The function

$$Z_{\mathcal{B},0}(f,s) := \chi\big((\mathbb{L}-1)Z_{\mathrm{naive},\,\mathcal{B}_0}(\mathbb{L}^{-s})\big)$$

is rational, and in terms of a resolution that factorizes through the blowing-up $\pi_0\colon (\mathcal{X}_0, \mathbb{CP}^{d-1}) \to (\mathbb{C}^d, 0)$ of the origin in $\mathbb{C}^d$, one has

$$Z_{\mathcal{B},0}(f,s) = \sum_{I\subset I_0,\ \varnothing\neq I\not\subset I_0''} \chi(\mathring{E}_I)\prod_{i\in I}\frac{1}{N_is+\nu_i+M_i}.$$

In particular, one may ask if the monodromy conjecture holds for $Z_{\mathcal{B},0}(f,s)$. The example $f(x,y) = (x^2+y^3)(y^2+x^3)$ shows that this is not the case. The monodromy zeta function $\zeta_f(t)$ is equal to $(1-t^{10})^2/(1-t^5)^2$, while

$$Z_{\mathcal{B},0}(f,s) = \frac{8s^2+24s+14}{(10s+7)(4s+3)(1+s)}.$$

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