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Generating Series of Classes of Hilbert Schemes of Points on Orbifolds

S. M. Gusein-Zade[a], **I. Luengo**[b], **and A. Melle-Hernández**[b]

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Abstract—The notion of power structure over the Grothendieck ring of complex quasi-projective varieties is used for describing generating series of classes of Hilbert schemes of zero-dimensional subschemes ("fat points") on complex orbifolds.

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Let $K_0(\mathcal{V}_{\mathbb{C}})$ be the Grothendieck ring of complex quasi-projective varieties. This is an abelian group generated by classes $[X]$ of all quasi-projective varieties X modulo the following relations:

(1) if varieties X and Y are isomorphic, then $[X]=[Y]$;

(2) if Y is a Zariski closed subvariety of X, then $[X]=[Y]+[X\setminus Y]$

(a quasi-projective variety is the difference of two (complex) projective (algebraic) varieties). The multiplication in $K_0(\mathcal{V}_{\mathbb{C}})$ is defined by the Cartesian product of varieties: $[X_1]\cdot[X_2]=[X_1\times X_2]$.

Let Hilb^n_X be the Hilbert scheme of zero-dimensional subschemes of length n of a complex quasi-projective variety X. According to [4], Hilb^n_X is a quasi-projective variety. For a point $x\in X$, let us denote by $\mathrm{Hilb}^n_{X,x}$ the Hilbert scheme of zero-dimensional subschemes of the variety X that are supported at the point x.

Let

$$\mathbb{H}_X(T):=1+\sum_{n=1}^{\infty}[\mathrm{Hilb}^n_X]\,T^n\in 1+TK_0(\mathcal{V}_{\mathbb{C}})[[T]]$$

and

$$\mathbb{H}_{X,x}(T):=1+\sum_{n=1}^{\infty}[\mathrm{Hilb}^n_{X,x}]\,T^n\in 1+TK_0(\mathcal{V}_{\mathbb{C}})[[T]]$$

be the generating series of classes of Hilbert schemes Hilb^n_X and $\mathrm{Hilb}^n_{X,x}$ in the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$.

In [5], a notion of power structure over a ring was defined. A *power structure* over a commutative ring R is a method to assign a certain meaning to expressions of the form $(1+a_1T+a_2T^2+\ldots)^m$, where a_i and m are elements of the ring R. In other words, a power structure is defined by a map $(1+T\cdot R[[T]])\times R\to 1+T\cdot R[[T]]$:

$$(A(T),m)\mapsto (A(T))^m,\qquad A(T)=1+a_1T+a_2T^2+\ldots,\quad a_i\in R,\quad m\in R,$$

such that all the usual properties of the exponential function hold. In [5], a natural power structure over the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ of complex quasi-projective varieties was described. It is closely

[a] Faculty of Mechanics and Mathematics, Moscow State University, Moscow, 119991 Russia.

[b] Departamento de Álgebra, Universidad Complutense de Madrid, Plaza de Ciencias 3, 28040 Madrid, Spain.

E-mail addresses: sabir@mccme.ru (S.M. Gusein-Zade), iluengo@mat.ucm.es (I. Luengo), amelle@mat.ucm.es (A. Melle-Hernández).

connected with the λ-structure (see, e.g., [9]) on $K_0(\mathcal{V}_{\mathbb{C}})$ defined by the Kapranov zeta function of a quasi-projective variety [8]:

$$\zeta_X(t) = 1 + [X]t + [S^2X]t^2 + [S^3X]t^3 + \ldots \in K_0(\mathcal{V}_{\mathbb{C}})[[t]].$$

The geometric description of this power structure is as follows: if $A_1, A_2, \ldots, M$ are quasi-projective varieties, then the coefficient of T^n in the series

$$(1 + [A_1]T + [A_2]T^2 + \ldots)^{[M]}$$

is represented by the configuration space of pairs (K, φ) consisting of a finite subset K of the variety M and a map φ from K to the disjoint union $\coprod_{i=1}^{\infty} A_i$ of the varieties A_i, such that $\sum_{x\in K} I(\varphi(x)) = n$, where $I\colon \coprod_{i=1}^{\infty} A_i \to \mathbb{Z}$ is the tautological function sending the component A_i of the disjoint union to i.

There are two natural homomorphisms from the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{C}})$ to the ring $\mathbb{Z}$ of integers and to the ring $\mathbb{Z}[u, v]$ of polynomials in two variables: the Euler characteristic (alternating sum of ranks of cohomology groups with compact support) $\chi\colon K_0(\mathcal{V}_{\mathbb{C}}) \to \mathbb{Z}$ and the Hodge–Deligne polynomial $e\colon K_0(\mathcal{V}_{\mathbb{C}}) \to \mathbb{Z}[u, v]$: $e(X)(u, v) = \sum e^{p,q}(X)u^p v^q$. These homomorphisms respect the power structures on the corresponding rings (see, e.g., [7]).

For a smooth quasi-projective variety X of dimension d, the following equation holds in $K_0(\mathcal{V}_{\mathbb{C}})[[T]]$ (see [6]):

$$\mathbb{H}_X(T) = \big(\mathbb{H}_{\mathbb{A}^d,0}(T)\big)^{[X]}, \tag{1}$$

where $\mathbb{A}^d$ is the complex affine space of dimension d. For $d = 2$, i.e., for surfaces, this equation was proved in other terms in the Grothendieck ring of motives by L. Göttsche [3]. For an arbitrary dimension d, the reduction of equation (1) for the Hodge–Deligne polynomial was proved by J. Cheah in [1].

Since, for a point x of a smooth variety X of dimension d, the Hilbert scheme $\mathrm{Hilb}^n_{X,x}$ can be identified with the Hilbert scheme $\mathrm{Hilb}^n_{\mathbb{A}^d,0}$, equation (1) can be written as an integral with respect to the universal Euler characteristic $\chi_{\mathrm{g}}(Y) = [Y] \in K_0(\mathcal{V}_{\mathbb{C}})$ as follows:

$$\mathbb{H}_X(T) := \int_X \mathbb{H}_{X,x}(T)^{d\chi_{\mathrm{g}}}. \tag{2}$$

Here $d\chi_{\mathrm{g}}$ is placed in the exponent since the group operation in $1+T\cdot K_0(\mathcal{V}_{\mathbb{C}})[[T]]$ is the multiplication. For a constructible function ψ on a quasi-projective variety X with values in the abelian group $1 + T \cdot K_0(\mathcal{V}_{\mathbb{C}})[[T]]$, constant and equal to $\psi_\Sigma(T)$ on a stratum Σ from a stratification $\mathcal{S} = \{\Sigma\}$ of the variety X, by definition one has

$$\int_X \psi^{d\chi_{\mathrm{g}}} = \prod_{\Sigma\in\mathcal{S}} (\psi_\Sigma(T))^{[\Sigma]}. \tag{3}$$

If a variety X is not smooth but has only isolated singularities, one can easily see that equation (2) holds. However, this could not be the case in general. Even the function $\mathbb{H}_{X,x}(T)$ (with values in $1 + T \cdot K_0(\mathcal{V}_{\mathbb{C}})[[T]]$) may be nonconstructible: singularities of the variety X at points of some strata may have moduli. It is interesting to understand to what extent equation (2) holds for varieties X such that the function $\mathbb{H}_{X,x}(T)$ is constructible.

The problem described does not occur for orbifolds: their singularities have finitely many local models. Here we prove equation (2) for complex (algebraic) orbifolds.

An *orbifold* of dimension d is a complex quasi-projective variety X with an atlas of uniformizing systems for Zariski open subsets of the variety X (see, e.g., [10]). For a Zariski open subset $U \subset X$, a *uniformizing system* is a triple $(\widetilde{U}, G, \varphi)$, where G is a finite group (depending on U), $\widetilde{U}$ is a smooth complex d-dimensional quasi-projective variety with an action of the group G, and φ is an isomorphism $\widetilde{U}/G \to U$ of quasi-projective varieties (considered with the reduced structures).

For a point $x \in X$, let $(\widetilde{U}, G, \varphi)$ be a uniformizing system for a Zariski open neighbourhood U of the point x. Let $\pi_{\widetilde{U}}$ be the natural map $\widetilde{U} \to \widetilde{U}/G$ and let $\widetilde{x} \in (\varphi \circ \pi_{\widetilde{U}})^{-1}(x)$ be a representative of the corresponding orbit. The isotropy group $G_{\widetilde{x}} = \{g \in G\colon g\widetilde{x} = \widetilde{x}\}$ of the point $\widetilde{x}$ acts on the tangent space $T_{\widetilde{x}}\widetilde{U}$ by a representation $\alpha = \alpha_{\widetilde{x}}\colon G_{\widetilde{x}} \to \mathrm{GL}(d, \mathbb{C})$. Moreover, there exists a system of local parameters $z_1, \dots, z_d$ at the point $\widetilde{x}$ such that the action of the group $G_{\widetilde{x}}$ on the manifold $\widetilde{U}$ is given by standard linear equations corresponding to the representation:

$$g^* z_i = \sum_{j=1}^{d} \alpha_{i,j}(g) z_j, \tag{4}$$

where $(\alpha_{i,j}(g)) = \alpha(g)$.

We have the following:

(i) the Hilbert scheme $\mathrm{Hilb}^n_{X,x}$ of 0-dimensional subschemes on X supported at the point x is isomorphic to the Hilbert scheme $\mathrm{Hilb}^n_{\mathbb{A}^d/G_{\widetilde{x}},0}$;

(ii) the partition of the neighbourhood U into parts corresponding to different conjugacy classes of isotropy subgroups $G_{\widetilde{x}} \subset G$ and to different (nonisomorphic) representations of the group $G_{\widetilde{x}}$ is a stratification of the neighbourhood U.

Therefore, $\mathbb{H}_{X,x}(T)$ is a constructible function on the variety X with values in $1 + T \cdot K_0(\mathcal{V}_{\mathbb{C}})[[T]]$.

Theorem 1. *For an orbifold X, the following equation holds in the group $1 + T \cdot K_0(\mathcal{V}_{\mathbb{C}})[[T]]$:*

$$\mathbb{H}_X(T) = \int_X \mathbb{H}_{X,x}(T)^{d\chi_{\mathrm{g}}}. \tag{5}$$

Proof. For a locally closed subvariety $Y \subset X$, let us denote by $\mathrm{Hilb}^n_{X,Y}$ the Hilbert scheme of subschemes of length n of the variety X that are supported at points of the variety Y, and let

$$\mathbb{H}_{X,Y}(T) := 1 + \sum_{n=1}^{\infty} [\mathrm{Hilb}^n_{X,Y}]\, T^k$$

be the corresponding generating series. If Y is a Zariski closed subset of the variety X, then

$$\mathbb{H}_X(T) = \mathbb{H}_{X,Y}(T) \cdot \mathbb{H}_{X,X\setminus Y}(T). \tag{6}$$

Therefore, it is sufficient to prove equation (2) for the case when X is covered by one uniformizing system $(\widetilde{U}, G, \varphi)$ such that $\widetilde{U}$ lies in an affine space $\mathbb{A}^N$.

Let us fix a subgroup G' of the group G and a representation $\alpha\colon G' \to \mathrm{GL}(d, \mathbb{C})$. Let $X_{G',\alpha}$ be the image under the map $\varphi \circ \pi_G$ of the set of points $\widetilde{x} \in \widetilde{U}$ such that $G_{\widetilde{x}} = G'$ and the representation of the group G' on the tangent space $T_{\widetilde{x}}\widetilde{U}$ is isomorphic to α. Using (6) again, one can see that it is sufficient to prove the formula

$$\mathbb{H}_{X,Y}(T) = \bigl(\mathbb{H}_{\mathbb{A}^d/(G',\alpha),0}(T)\bigr)^{[Y]} \tag{7}$$

for each point $x \in X_{G',\alpha}$ and for a certain Zariski open neighbourhood Y of this point in an irreducible component of the set $X_{G',\alpha}$.

Let $\widetilde{x} \in (\varphi \circ \pi_{\widetilde{U}})^{-1}(x)$ be a point of the manifold $\widetilde{U}$ such that $G_{\widetilde{x}} = G'$. The representation of the group G' on the tangent space $T_{\widetilde{x}}\widetilde{U}$ is isomorphic to the representation α. Let $u_1, \dots, u_d$ be a regular system of parameters at the point $\widetilde{x}$ (for example, one may suppose that $\widetilde{x} = 0$ and $u_1, \dots, u_d$ are d of the standard coordinates on the space $\mathbb{A}^N$ such that the projection of the tangent space $T_{\widetilde{x}}\widetilde{U}$ to the corresponding d-dimensional coordinate plane is nondegenerate). (In this case $u_1 - u_1^0, \dots, u_d - u_d^0$ is a regular system of parameters at each point from a Zariski open neighbourhood of the point $\widetilde{x}$.)

Suppose that the parameters $u_1, \dots, u_d$ are chosen in such a way that, in the corresponding coordinates on the tangent space $T_{\widetilde{x}}\widetilde{U}$, the action of the group G' is given by the standard equations (4). Define a new regular system of parameters $\widetilde{u}_1, \dots, \widetilde{u}_d$ at the point $\widetilde{x}$ by the equations

$$\widetilde{u}_i = \frac{1}{|G'|} \sum_{g \in G'} \sum_j \alpha_{i,j}(g^{-1}) g^* u_j.$$

We have

$$g^* \widetilde{u}_i = \sum_j \alpha_{i,j}(g) \widetilde{u}_j.$$

Therefore, $\widetilde{u}_1 - \widetilde{u}_1^0, \dots, \widetilde{u}_d - \widetilde{u}_d^0$ is a regular system of parameters at each point $\widetilde{x}'$ from a Zariski open neighbourhood of the point $\widetilde{x}$ in the corresponding irreducible component of the set $(\varphi \circ \pi_G)^{-1} X_{G',\alpha}$. This defines a map $\underline{\widetilde{u}} - \underline{\widetilde{u}}^0 \colon \widetilde{U}, \widetilde{x}' \to \mathbb{A}^d, 0$. There is a commutative diagram

$$\begin{array}{ccc} \widetilde{U}, \widetilde{x}' & \xrightarrow{\underline{\widetilde{u}} - \underline{\widetilde{u}}^0} & \mathbb{A}^d, 0 \\ \downarrow & & \downarrow \\ X, x' & \longrightarrow & \mathbb{A}^d/(G', \alpha), 0 \end{array}$$

where the lower map identifies the Hilbert scheme of points on X supported at the point x' with the Hilbert scheme of points on $\mathbb{A}^d/(G', \alpha)$ supported at the origin.

Thus, a zero-dimensional subscheme on the orbifold X supported at points of the subvariety Y is defined by a finite subset $K \subset Y$ to each point x of which there corresponds a zero-dimensional subscheme on $\mathbb{A}^d/(G', \alpha)$ supported at the origin. The length of the subscheme is equal to the sum of lengths of the corresponding subschemes of $\mathbb{A}^d/(G', \alpha)$. As follows from the geometric description of the power structure over the Grothendieck ring of quasi-projective varieties, the coefficient of T^n on the right-hand side of equation (7) is represented just by the configuration space of such objects. This proves the theorem. $\square$

Reductions of equation (2) under the Euler characteristic homomorphism and the Hodge–Deligne polynomial homomorphism give equations for the generating series of the corresponding invariants of Hilbert schemes of zero-dimensional subschemes of an orbifold. In particular,

$$1 + \sum_{n=1}^{\infty} \chi(\mathrm{Hilb}_X^n) T^n = \int_X \left(1 + \sum_{n=1}^{\infty} \chi(\mathrm{Hilb}_{X,x}^n) T^n \right)^{d\chi}. \tag{8}$$

In [2], J. Cheah considered nested Hilbert schemes on a smooth d-dimensional complex quasi-projective variety X. For $\underline{n} = (n_1, \dots, n_r) \in \mathbb{Z}_{\geq 0}^r$, the *nested Hilbert scheme* $Z_X^{\underline{n}}$ of depth r is the scheme that parametrizes collections of the form $(Z_1, \dots, Z_r)$, where $Z_i \in \mathrm{Hilb}_X^{n_i}$ and Z_i is a subscheme of Z_j for $i < j$. The scheme $Z_X^{\underline{n}}$ is nonempty only if $n_1 \leq n_2 \leq \ldots \leq n_r$; notice that $Z_X^{(n)} = \mathrm{Hilb}_X^n \cong Z_X^{(n,\dots,n)}$.

For $Y \subset X$, denote by $Z_{X,Y}^{\underline{n}}$ the scheme that parametrizes collections $(Z_1, \dots, Z_r)$ from $Z_X^{\underline{n}}$ with $\operatorname{supp} Z_i \subset Y$. For $Y = \{x\}$, $x \in X$, we will use the notation $Z_{X,x}^{\underline{n}}$.

For $r \geq 1$, let $\underline{T} = (T_1, \ldots, T_r)$ and let

$$\mathcal{Z}_X^{(r)}(\underline{T}) := 1 + \sum_{\underline{n} \in \mathbb{Z}_{\geq 0}^r \setminus \{0\}} [Z_X^{\underline{n}}]\, \underline{T}^{\underline{n}}, \qquad \mathcal{Z}_{X,x}^{(r)}(\underline{T}) := 1 + \sum_{\underline{n} \in \mathbb{Z}_{\geq 0}^r \setminus \{0\}} [Z_{X,x}^{\underline{n}}]\, \underline{T}^{\underline{n}}$$

be the generating series of classes of the nested Hilbert schemes $Z_X^{\underline{n}}$ of depth r (respectively, of those supported at the point x).

A series

$$A(\underline{T}) = 1 + \sum_{\underline{n} \in \mathbb{Z}_{\geq 0}^r \setminus \{0\}} a_{\underline{n}}\, \underline{T}^{\underline{n}}, \qquad \underline{T}^{\underline{n}} := T_1^{n_1} \cdot \ldots \cdot T_r^{n_r}, \quad a_{\underline{n}} \in K_0(\mathcal{V}_{\mathbb{C}}),$$

has a unique representation of the form

$$A(\underline{T}) = \prod_{\underline{k} \in \mathbb{Z}_{\geq 0}^r \setminus \{0\}} \left(1 - \underline{T}^{\underline{k}}\right)^{-s_{\underline{k}}}, \qquad s_{\underline{k}} \in K_0(\mathcal{V}_{\mathbb{C}}).$$

Then, for $m \in K_0(\mathcal{V}_{\mathbb{C}})$,

$$A(\underline{T})^m := \prod_{\underline{k} \in \mathbb{Z}_{\geq 0}^r \setminus \{0\}} \left(1 - \underline{T}^{\underline{k}}\right)^{-m s_{\underline{k}}}$$

(see [7]).

For a smooth quasi-projective variety X of dimension d, the following equation holds [7]:

$$\mathcal{Z}_X^{(r)}(\underline{T}) = \left(\mathcal{Z}_{\mathbb{A}^d,0}^{(r)}(\underline{T})\right)^{[X]}.$$

Using the same arguments as in the proof of Theorem 1, one can obtain the following statement:

Theorem 2. *For an orbifold X, the following equation holds*:

$$\mathcal{Z}_X^{(r)}(\underline{T}) = \int_X \left(\mathcal{Z}_{X,x}^{(r)}(\underline{T})\right)^{d\chi_{\mathrm{g}}}.$$

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