

Superisolated Surface Singularities

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Dedicated to Gert-Martin Greuel on the occasion of his 60th birthday

Abstract

In this survey, we review part of the theory of superisolated surface singularities (SIS) and its applications including some new and recent developments. The class of SIS singularities is, in some sense, the simplest class of germs of normal surface singularities. Namely, their tangent cones are reduced curves and the geometry and topology of the SIS singularities can be deduced from them. Thus this class *contains*, in a canonical way, all the complex projective plane curve theory, which gives a series of nice examples and counterexamples. They were introduced by I. Luengo to show the non-smoothness of the μ -constant stratum and have been used to answer negatively some other interesting open questions. We review them and the new results on normal surface singularities whose link are rational homology spheres. We also discuss some positive results which have been proved for SIS singularities.

Introduction

A superisolated surface, SIS for short, singularity $(V, 0) \subset (\mathbb{C}^3, 0)$ is a generic perturbation of the cone over a (singular) reduced projective plane curve C of degree d , $C = \{f_d(x, y, z) = 0\} \subset \mathbb{P}^2$, by monomials of higher degree. The geometry, resolution and topology of $(V, 0)$ is determined by the singularities of C and the pair (\mathbb{P}^2, C) . This provides a canonical way to *embed* the classical and rich theory of complex projective plane curves into the theory of normal surface singularities of $(\mathbb{C}^3, 0)$. In this way one can use properties of plane curves to get interesting properties of SIS singularities. They were introduced by I. Luengo [45], and were used to answer several questions and

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conjectures, like the fact that the μ -constant stratum in the semiuniversal deformation space of an isolated hypersurface singularity is, in general, not smooth. Using Zariski pairs as tangent cones of SIS singularities, E. Artal [2] found also counterexamples for a S. S.-T. Yau's conjecture [91] relating the link of the singularity, the characteristic polynomial and the embedded topology. A *Zariski pair* is a set of two curves $C_1, C_2 \subset \mathbb{P}^2$ with the same combinatorial type but such that (\mathbb{P}^2, C_1) is not homeomorphic to (\mathbb{P}^2, C_2) .

In a recent paper [47], A. Némethi and the last two authors have found counterexamples to several conjectures on normal surface singularities whose link is a rational homology sphere. For doing this there were used SIS singularities whose tangent cone is a rational cuspidal curve. It was shown that the *Seiberg-Witten invariant conjecture* (of L.I. Nicolaescu and A. Némethi [56]), the *universal abelian cover conjecture* (of W. Neumann and J. Wahl [64]) and the *geometric genus conjecture* ([62, Question 3.2], see also [55, Problem 9.2]) fail (at least at that generality in which they were formulated).

On the other hand, from the positive point of view, SIS singularities have been used by Pi. Cassou-Noguès and the authors [7] to confirm the Monodromy Conjecture for the topological zeta function introduced by J. Denef and F. Loeser [15]. We review these results in Section 4.

It is interesting to point out that the relationship between plane curves and normal surface singularities can be used also in the other direction: to use results and ideas from normal surface singularities to get new results about curves. In this way, the results in [7] allow to find necessary conditions for the existence of an arrangement of rational plane curves. Even more, J. Fernández de Bobadilla, A. Némethi and the last two authors [20] have found a compatibility property for a rational cuspidal projective plane curve to exist based on a heavily study of the failure of the Seiberg-Witten invariant conjecture of the corresponding SIS singularities.

Since the class of SIS singularities continue being useful we have decided to write down this survey, dedicated to our friend Gert-Martin, where we present known results and open problems on SIS singularities.

1 Superisolated Surface Singularities

1.1 Isolated Hypersurface Singularities

Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be an analytic function and the corresponding germ $(V, 0) := (f^{-1}(0), 0) \subset (\mathbb{C}^{n+1}, 0)$ of a hypersurface singularity. The *Milnor fibration* of the holomorphic function f at 0 is the C^∞ locally trivial fibration $f| : B_\varepsilon(0) \cap f^{-1}(\mathbb{D}_\eta^*) \rightarrow \mathbb{D}_\eta^*$, where $B_\varepsilon(0)$ is the open ball of radius ε centered at 0, $\mathbb{D}_\eta = \{z \in \mathbb{C} : |z| < \eta\}$ and \mathbb{D}_η^* is the open punctured disk ($0 < \eta \ll \varepsilon$ and ε small enough). Milnor's classical result also shows that the topology of

the germ $(V, 0)$ in $(\mathbb{C}^{n+1}, 0)$ is determined by the pair $(S_\varepsilon^{2n+1}, L_V^{2n-1})$, where $S_\varepsilon^{2n+1} = \partial B_\varepsilon(0)$ and $L_V^{2n-1} := S_\varepsilon^{2n+1} \cap V$ is the *link* of the singularity.

Any fiber $F_{f,0}$ of the Milnor fibration is called the *Milnor fiber* of f at 0. The *monodromy transformation* $h : F_{f,0} \rightarrow F_{f,0}$ is the well-defined (up to isotopy) diffeomorphism of $F_{f,0}$ induced by a small loop around $0 \in \mathbb{D}_\eta$. The *complex algebraic monodromy of f at 0* is the corresponding linear transformation $h_* : H_*(F_{f,0}, \mathbb{C}) \rightarrow H_*(F_{f,0}, \mathbb{C})$ on the homology groups.

If $(V, 0)$ defines a germ of isolated hypersurface singularity then we have that $\tilde{H}_j(F_{f,0}, \mathbb{C}) = 0$ but for $j = n$. In particular the non-trivial complex algebraic monodromy will be denoted by $h : H_n(F_{f,0}, \mathbb{C}) \rightarrow H_n(F_{f,0}, \mathbb{C})$ and $\Delta_V(t)$ will denote its characteristic polynomial. The Monodromy Theorem describes the main properties of the monodromy operator, see for instance the references in [19]:

- (a) $\Delta_V(t)$ is a product of cyclotomic polynomials.
- (b) Let N be the maximal size of the Jordan blocks of h , then $N \leq n + 1$.
- (c) Let N_1 be the maximal size of the Jordan blocks of h for the eigenvalue 1, then $N_1 \leq n$.

1.2 Normal Surface Singularities

Let $(V, 0) = (\{f_1 = \dots = f_m = 0\}, 0) \subset (\mathbb{C}^N, 0)$ be a normal surface singularity with link L_V . One of the main problems is to determine which analytical properties of $(V, 0)$ can be read from the topology of the singularity, see the very nice survey paper by A. Némethi [55]. Since $V \cap B_\varepsilon$ is a cone over the link then L_V characterizes the topological type of $(V, 0)$.

The resolution graph $\Gamma(\pi)$ of a resolution $\pi : \tilde{V} \rightarrow V$ allows to relate analytical and topological properties of V . Via plumbing construction, W. Neumann [61] proved that the information carried in any resolution graph is the same as the information carried by the link L_V . Let $\pi : \tilde{V} \rightarrow V$ be a *good* resolution of the singular point $0 \in V$. Good means that $E = \pi^{-1}(0)$ is a normal crossing divisor. Let $\Gamma(\pi)$ be the dual graph of the resolution (each vertex decorated with the genus $g(E_i)$ and the self-intersection E_i^2 of E_i in \tilde{V}). Mumford proved that the intersection matrix $I = (E_i \cdot E_j)$ is negative definite and Grauert proved the converse, i.e., any such graph corresponds to the link of a normal surface singularity.

1.3 Superisolated Surface Singularities

Definition 1.1. A hypersurface surface singularity $(V, 0) \subset (\mathbb{C}^3, 0)$ defined as the zero locus of $f = f_d + f_{d+1} + \dots \in \mathbb{C}\{x, y, z\}$ (where f_j is homogeneous of degree j) is *superisolated*, SIS for short, if the singular points of the complex

projective plane curve $C := \{f_d = 0\} \subset \mathbb{P}^2$ are not situated on the projective curve $\{f_{d+1} = 0\}$, that is $\text{Sing}(C) \cap \{f_{d+1} = 0\} = \emptyset$ in \mathbb{P}^2 . Note that C must be a reduced curve.

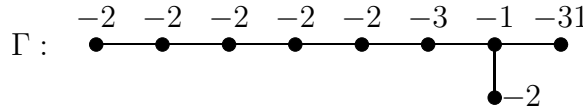
The SIS singularities were introduced by I. Luengo in [45] to study the μ -constant stratum, see Section 2. The main idea is that for a SIS singularity $(V, 0)$, the embedded topological type (and the equisingular type) of $(V, 0)$ does not depend on the choice of f_j 's (for $j > d$, as long as f_{d+1} satisfies the above requirement), e.g. one can take $f_j = 0$ for any $j > d+1$ and $f_{d+1} = l^{d+1}$ where l is a linear form not vanishing at the singular points [46].

The minimal resolution. Let $\pi : \tilde{V} \rightarrow V$ be the monoidal transformation centered at the maximal ideal $\mathfrak{m} \subset \mathcal{O}_V$ of the local ring of V at 0. Then $(V, 0)$ is a SIS singularity if and only if \tilde{V} is a non-singular surface. Thus π is the *minimal resolution* of $(V, 0)$. To construct the resolution graph $\Gamma(\pi)$ consider $C = D_1 + \dots + D_r$ the decomposition in irreducible components of the reduced curve C in \mathbb{P}^2 . Let d_i be the degree of the curve D_i in \mathbb{P}^2 . Then $\pi^{-1}(0) \cong C = D_1 + \dots + D_r$ and the self-intersection of D_i in \tilde{V} is $D_i \cdot D_i = -d_i(d - d_i + 1)$, [45, Lemma 3]. Since the link L_V can be identified with the boundary of a regular neighbourhood of $\pi^{-1}(0)$ in \tilde{V} then the topology of the tangent cone determines the topology of the abstract link L_V [45].

The minimal good resolution of $(V, 0)$ is obtained from π by doing the minimal embedded resolution of each plane curve singularity $(C, P) \subset (\mathbb{P}^2, P)$, $P \in \text{Sing}(C)$, which is not an ordinary double point whose branches belong to different global irreducible components. Let D_j be an irreducible component of C such that $P \in D_j$ and with multiplicity $n \geq 1$ at P . After blowing-up at P , the new self-intersection of the (strict transform of the) curve D_j in the (strict transform of the) surface \tilde{V} is $D_j^2 - n^2$. In this way one constructs the minimal good resolution graph Γ of $(V, 0)$.

In particular the theory of hypersurface superisolated surface singularities “contains” in a canonical way the theory of complex projective plane curves.

Example 1.2. Let $f = f_5 + z^6$ be given by the equation $f_5 = z(yz - x^2)^2 - 2xy^2(yz - x^2) + y^5$. The curve C is irreducible with unique singularity at $[0 : 0 : 1]$ (of type \mathbb{A}_{12}). The minimal good resolution graph Γ of the superisolated singularity $(V, 0)$ is



Here all the curves have genus zero.

The embedded resolution. In [1], the first author studied the Mixed Hodge Structure of the cohomology of the Milnor fibre of a SIS singularity. For that, he constructed in an effective way an embedded resolution of a SIS singularity.

The germ $(V, 0) \subset (\mathbb{C}^3, 0)$ is an isolated surface singularity. Hence, $H_0(F, \mathbb{C})$ and $H_2(F, \mathbb{C})$ are the only non-vanishing homology vector spaces on which the monodromy acts (we denote the Milnor fiber by F). The only eigenvalue of the action of the monodromy on $H_0(F, \mathbb{C})$ is equal to 1. The Jordan form of the complex monodromy on $H_2(F, \mathbb{C})$ is computed for a SIS singularity. Let $\Delta_V(t)$ be the corresponding characteristic polynomial of the complex monodromy on $H_2(F, \mathbb{C})$. Denote by $\mu(V, 0) = \deg(\Delta_V(t))$ the Milnor number of $(V, 0) \subset (\mathbb{C}^3, 0)$.

Let $\Delta^P(t)$ be the characteristic polynomial (or Alexander polynomial) of the action of the complex monodromy of the germ (C, P) on $H_1(F_{g^P}, \mathbb{C})$, (where g^P is a local equation of C at P and F_{g^P} denotes the corresponding Milnor fiber). Let μ^P be the Milnor number of g^P at P .

Let H be a \mathbb{C} -vector space and $\varphi : H \rightarrow H$ an endomorphism of H . The i -th Jordan polynomial of φ , denoted by $\Delta_i(t)$, is the monic polynomial such that for each $\zeta \in \mathbb{C}$, the multiplicity of ζ as a root of $\Delta_i(t)$ is equal to the number of Jordan blocks of size $i + 1$ with eigenvalue equal to ζ .

Let $\Delta_1(t)$ and $\Delta_2(t)$ be the first and the second Jordan polynomials of the complex monodromy on $H_2(F, \mathbb{C})$ of V and let $\Delta_1^P(t)$ be the first Jordan polynomial of the complex monodromy of the local plane singularity (C, P) . After the Monodromy Theorem these polynomials joint with $\Delta_V(t)$ and $\Delta^P(t)$, $P \in \text{Sing}(C)$, determine the corresponding Jordan form of the complex monodromy. The Alexander polynomial $\Delta_C(t)$ of the projective plane curve $C \subset \mathbb{P}^2$ was introduced by A. Libgober [37, 38] and F. Loeser and M. Vaquié [42].

Theorem 1.3 ([1]). *Let $(V, 0)$ be a SIS singularity whose tangent cone C has r irreducible components. Then the Jordan form of the complex monodromy on $H_2(F, \mathbb{C})$ is determined by the following polynomials*

(i) *The characteristic polynomial $\Delta_V(t)$ is equal to*

$$\Delta_V(t) = \frac{(t^d - 1)^{\chi(\mathbb{P}^2 \setminus C)}}{(t - 1)} \prod_{P \in \text{Sing}(C)} \Delta^P(t^{d+1}).$$

(ii) *The first Jordan polynomial is equal to*

$$\Delta_1(t) = \frac{1}{\Delta_C(t)(t - 1)^{r-1}} \prod_{P \in \text{Sing}(C)} \frac{\Delta_1^P(t^{d+1}) \Delta_{(d)}^P(t)}{\Delta_{1,(d)}^P(t)^3},$$

where

$$\Delta_{(d)}^P(t) := \gcd(\Delta^P(t), (t^d - 1)^{\mu^P}) \quad \text{and} \quad \Delta_{1,(d)}^P(t) := \gcd(\Delta_1^P(t), (t^d - 1)^{\mu^P}).$$

(iii) *The second Jordan polynomial is equal to*

$$\Delta_2(t) = \prod_{P \in \text{Sing}(C)} \Delta_{1,(d)}^P(t).$$

The first part of the theorem was stated by J. Stevens in [79]. A general formula for the zeta function of the monodromy was proved by D. Siersma [76] (see also [51],[28]). In particular the Milnor number $\mu(V, 0)$ of a SIS singularity verifies the identity

$$\mu(V, 0) = (d-1)^3 + \sum_{P \in \text{Sing } C} \mu^P.$$

Yomdin Singularities, Series of Singularities and Spectrum.

The first natural generalization of superisolated singularities are Yomdin singularities, where $d+1$ is replaced by $d+k$.

Let $(V, 0) \subset (\mathbb{C}^{n+1}, 0)$ be the germ of hypersurface defined by $f = 0$, $f = f_d + f_{d+k} + \dots \in \mathbb{C}\{x_0, \dots, x_n\}$. The singularity $(V, 0)$ is called of *Yomdin type* if $\text{Sing}(\{f_d = 0\}) \cap \{f_{d+k} = 0\} = \emptyset$ in \mathbb{P}^n .

For each $P \in \text{Sing}(\{f_d = 0\})$, let g^P be a local equation of $\{f_d = 0\} \subset \mathbb{P}^n$ at P . Formulæ for the Milnor number (see [92, 35, 46]) and for the zeta function $\zeta_f(t)$ of the complex monodromy can be written as follows [79, 76, 28, 51]:

$$\mu(V, 0) = (d-1)^{n+1} + k \sum_{P \in \text{Sing}\{f_d=0\}} \mu^P,$$

and

$$\zeta_f(t) = (1 - t^d)^{\chi(\mathbb{P}^n \setminus \{f_d=0\})} \left(\prod_{P \in \text{Sing}\{f_d=0\}} (1 - t^{d+k}) (\zeta_g^P)^k (t^{d+k}) \right)^{-1}.$$

Here, $(\zeta_g^P)^k(t)$ is the monodromy zeta function of the k -power of the corresponding monodromy $\zeta_g^P(t)$ of g at P .

Let H be a hyperplane such that $\text{Sing}(\{f_d = 0\}) \cap H = \emptyset$, H being the zero locus of a linear form g . Then the family

$$F(x_0, \dots, x_n, t) = f_d + (1-t)(f - f_d) + tg^{d+k}$$

is a μ -constant family (in fact a μ^* -constant family), see [46]. It means that to study properties of Yomdin type singularities which are preserved under μ -constant deformations is equivalent to study *series* of singularities of type $f_d + g^{d+k}$. Notice that in such a case the singular locus of f_d is 1-dimensional.

Let f be a germ of an analytic function at zero whose singular locus is 1-dimensional. Let g be a generic linear function such that $g(0) = 0$. Y. Yomdin [92] compared the vanishing cohomologies of their Milnor fibres (and then its Milnor numbers) of f and $f + g^N$, for N big enough. Later on D. Siersma [76] compared the zeta functions of their monodromies. Finally it was J. Steenbrink [78] who conjectured a relationship between the spectrum $\text{Sp}(f, 0)$ of f and the spectrum $\text{Sp}(f + g^N, 0)$ of $f + g^N$. This conjecture was proved by M. Saito [74] using his theory of mixed Hodge modules. Another proof has been given by A. Némethi and J. Steenbrink, [59]. Recently G. Guibert, F. Loeser and M. Merle [27] have proved Steenbrink's conjecture without any condition on the singular locus of f and g being any function vanishing at 0.

The notion of a spectrum $\text{Sp}(f, x)$ at x of a function f on a smooth complex algebraic variety was introduced by J. Steenbrink in [77] and by A. Varchenko in [84]. It is a fractional Laurent polynomial $\sum_{\alpha \in \mathbb{Q}} n_{\alpha} t^{\alpha}$, $n_{\alpha} \in \mathbb{Z}$ defined using the semi-simple part of the action of the monodromy on the mixed Hodge structure on the cohomology of the Milnor fibre of f at x . Here we use the convention given by M. Saito in [74] (denoted by $\text{Sp}'(f, x)$) which differs from that in [78] by multiplication by t (see Remark 2.3 in [74]).

Let $f = f_d + f_{d+k} + \dots \in \mathbb{C}\{x_0, \dots, x_n\}$ define a Yomdin type singularity $(V, 0) \subset (\mathbb{C}^{n+1}, 0)$ and, for each $P \in \text{Sing}(\{f_d = 0\})$, let g^P be a local equation of $\{f_d = 0\} \subset \mathbb{P}^n$ at P . Since the spectrum does not change under μ -constant deformations, see [82, 83], then the spectrum $\text{Sp}(f, 0)$ of $(V, 0)$ can be computed via [78, Theorem 6.1] and [74, Theorem 5.7] in terms of the spectral numbers (also called exponents) $\{\alpha_i^P\}_P$ of g^P at P .

Theorem 1.4 ([46, 74, 78]). *With the previous notations, the spectrum $\text{Sp}(f, 0)$ of a Yomdin singularity $(V, 0) \subset (\mathbb{C}^{n+1}, 0)$ defined by $f = f_d + f_{d+k} + \dots \in \mathbb{C}\{x_0, \dots, x_n\}$ is equal to*

$$\begin{aligned} \text{Sp}(f, 0) = & \left(\frac{t^{1/d} - t}{1 - t^{1/d}} \right)^n - \left(\frac{1 - t}{1 - t^{1/d}} \right) \sum_{P \in \text{Sing}(C)} \sum_{\alpha_i^P \in \text{Sp}(g^P, P)} t^{\beta_i^P} \\ & + \left(\frac{1 - t}{1 - t^{1/(d+k)}} \right) \sum_{P \in \text{Sing}(C)} \sum_{\alpha_i^P \in \text{Sp}(g^P, P)} t^{\gamma_i^{P,k}}, \end{aligned}$$

$$\text{where } \gamma_i^{P,k} := \frac{k\alpha_i^P + \lfloor d(\alpha_i^P - 1) \rfloor + d + 1}{d + k} \text{ and } \beta_i^P := \frac{\lfloor d(\alpha_i^P - 1) \rfloor + d + 1}{d}.$$

The study of non-isolated singularities defined by an analytic complex function f using perturbation $f + g^k$, g a generic linear form, has been extensively studied mostly using polar methods, Lê cycles and other methods, see [48, 49, 80] and references therein.

2 Deformations

Let $p: \mathcal{V} \rightarrow \mathcal{T}$ be a deformation of a SIS singularity $(V, 0) \subset (\mathbb{C}^3, 0)$ with a section σ such that $(V_t, \sigma(t))$ is an isolated singularity (one may assume that \mathcal{T} is one-dimensional and smooth). In general $(V_t, \sigma(t))$ is not a SIS singularity but if the corresponding multiplicities coincides, that is $\text{mult}(V_t, \sigma(t)) = \text{mult}(V, 0)$, then $(V_t, \sigma(t))$ is a SIS singularity, because one can take local coordinates such that $F(x, y, z, t) = 0$ is the equation of $(\mathcal{V}, 0) \subset (\mathbb{C}^3 \times \mathbb{C}, 0)$, $\sigma(t) = (0, t)$, and $F_d(x, y, z, t) = 0$ gives the tangent cone of $(V_t, \sigma(t))$ for all t . Thus p induces a deformation $P: \mathcal{C} \rightarrow \mathcal{T}$ of the tangent cone $C \subset \mathbb{P}^2$ and since the condition of being a SIS singularity is open then $\text{Sing}(C_t) \cap \{f_{d+1,t} = 0\} = \emptyset$ for t close to 0.

Assume now that p is a μ -constant deformation, that is $\mu(V_t, \sigma(t)) = \mu(V, 0)$ along the family. Even in this case it is not known that the multiplicity is constant. In fact the following well-known problems are still open also for SIS singularities.

Problem 2.1. Let $p: \mathcal{V} \rightarrow \mathcal{T}$ be a deformation of a SIS singularity $(V, 0) \subset (\mathbb{C}^3, 0)$ such that $\mu(V_t, \sigma(t)) = \mu(V, 0)$. Is it true that the multiplicity is constant?

Problem 2.2. Let $p: \mathcal{V} \rightarrow \mathcal{T}$ be a deformation of a SIS singularity $(V, 0) \subset (\mathbb{C}^3, 0)$ such that $\mu(V_t, \sigma(t)) = \mu(V, 0)$. Is it a topologically constant deformation?

In [45, Theorem 1], the second author gives an affirmative answer for Problem 2.1 using B. Perron results (see [70]) whose proof turned out to be incomplete. By putting together [45, Theorem 2] and the correct part of [45, Theorem 1] one gets:

Theorem 2.3. *Let $p: \mathcal{V} \rightarrow \mathcal{T}$ be a deformation of a SIS singularity $(V, 0) \subset (\mathbb{C}^3, 0)$. Then the following conditions are equivalent:*

- (a) $(V, 0)$ is topologically equivalent to $(V_t, \sigma(t))$,
- (b) $\mu(V_t, \sigma(t)) = \mu(V, 0)$ and $\text{mult}(V_t, \sigma(t)) = \text{mult}(V, 0)$,
- (c) the family $\{(V_t, \sigma(t))\}_t$ is μ^* -constant,
- (d) $\text{mult}(V_t, \sigma(t)) = \text{mult}(V, 0)$ and the induced deformation $P: \mathcal{C} \rightarrow \mathcal{T}$ of the tangent cone $C \subset \mathbb{P}^2$ of $(V, 0)$ is equisingular.

In [44], it was shown how to compute, in an effective way, equations for the equisingularity stratum Σ_C of C in the family of all projective plane curves of degree d , giving examples in which Σ_C is not smooth. Thus if one considers the SIS singularity with such tangent cone, then one gets that the μ^* -constant stratum in the versal deformation is not smooth.

The simplest example is $f = y(xy^3 + z^4)^2 + x^9 + y^{10}$. Then C has only one singular point with an A_{35} singularity, and Σ_C is singular. J. Stevens [79] using the V -filtration proved that the μ^* -constant stratum is a component of the μ -constant stratum giving the non-smoothness of the μ -constant stratum.

The nice construction by V. A. Vasil'iev and V.V. Serganova in [85], using matroids, gives another examples with non-smooth μ^* -constant stratum. The study of the properties of the equisingularity stratum Σ_C of curves is a classical subject which gained a great impulse with the work of Gert-Martin Greuel, Ch. Lossen and E. Shustin. See [25] for a detailed account of the subject and references.

Let $(V, 0) \subset (\mathbb{C}^3, 0)$ be an isolated surface singularity. B. Teissier asked whether $(V, 0)$ can be put in a μ^* -constant family such that there exists a member of the family which is defined over \mathbb{Q} (resp. \mathbb{R}). Using SIS singularities one can answer negatively to this question. Namely, it is known that there are many curves $C \subset \mathbb{P}^2$ such that no element of the equisingularity stratum Σ_C can be defined over \mathbb{Q} (or \mathbb{R}), see [93] and the end of next section. For such a curve not defined over \mathbb{R} see [4]. If one takes a SIS singularity over such a curve, Theorem 2.3 gives us that no member of a μ^* -constant deformation can be defined over \mathbb{Q} (or \mathbb{R}).

3 Zariski Pairs

Let us consider $C \subset \mathbb{P}^2$ a reduced projective curve of degree d defined by an equation $f_d(x, y, z) = 0$. If $(V, 0) \subset (\mathbb{C}^3, 0)$ is a SIS singularity with tangent cone C , then the link L_V of the singularity is completely determined by C . Let us recall, that L_V is a Waldhausen manifold and its plumbing graph is the dual graph of the good minimal resolution. In order to determine L_V we do not need the embedding of C in \mathbb{P}^2 , but only its embedding in a regular neighborhood. The needed data can be encoded in a combinatorial way.

Definition 3.1. Let $\text{Irr}(C)$ be the set of irreducible components of C . For $P \in \text{Sing}(C)$, let $B(P)$ be the set of local irreducible components of C . The *combinatorial type* of C is given by:

- A mapping $\text{deg} : \text{Irr}(C) \rightarrow \mathbb{Z}$, given by the degrees of the irreducible components of C .
- A mapping $\text{top} : \text{Sing}(C) \rightarrow \text{Top}$, where Top is the set of topological types of singular points. The image of a singular point is its topological type.
- For each $P \in \text{Sing}(C)$, a mapping $\beta_P : T(P) \rightarrow \text{Irr}(C)$ such that if γ is a branch of C at P , then $\beta_P(\gamma)$ is the global irreducible component containing γ .

Remark 3.2. There is a natural notion of isomorphism of combinatorial types. It is easily seen that combinatorial type determines and is determined by any of the following graphs (with vertices decorated with self-intersections):

- The dual graph of the preimage of C by the minimal resolution of $\text{Sing}'(C)$. The set $\text{Sing}'(C)$ is obtained from $\text{Sing}(C)$ by forgetting ordinary double points whose branches belong to distinct global irreducible components. We need to mark in the graph the r vertices corresponding to $\text{Irr}(C)$.
- The dual graph of the minimal good minimal of V . Since the minimal resolution is unique, it is not necessary to mark vertices.

Note also that the combinatorial type determine the Alexander polynomial $\Delta_V(t)$ of V (see Theorem 1.3).

Definition 3.3. A *Zariski pair* is a set of two curves $C_1, C_2 \subset \mathbb{P}^2$ with the same combinatorial type but such that (\mathbb{P}^2, C_1) is not homeomorphic to (\mathbb{P}^2, C_2) . An *Alexander-Zariski pair* $\{C_1, C_2\}$ is a Zariski pair such that the Alexander polynomials of C_1 and C_2 do not coincide.

In [1], (see here Theorem 1.3) it is shown that Jordan form of complex monodromy of a SIS singularity is determined by the combinatorial type and the Alexander polynomial of its tangent cone. The first example of Zariski pair was given by Zariski, [94, 95]; there exist sextic curves with six ordinary cusps. If these cusps are (resp. not) in a conic then the Alexander polynomial equals $t^2 - t + 1$ (resp. 1). Then, it gives an Alexander-Zariski pairs. Many other examples of Alexander-Zariski pairs have been constructed (Artal,[2], Degtyarev [13]). We state the main result in [1].

Theorem 3.4. *Let V_1, V_2 be two SIS singularities such that their tangent cones form an Alexander-Zariski pair. Then V_1 and V_2 have the same abstract topology and characteristic polynomial of the monodromy but not the same embedded topology.*

Recall that the Jordan form of the monodromy is an invariant of the embedded topology of a SIS singularity (see Theorem 1.3); since it depends on the Alexander polynomial $\Delta_C(t)$ of the tangent cone, we deduce this theorem.

Remark 3.5. Every SIS singularity of Theorem 3.4 provides a counterexample to a Conjecture by S.S.T Yau stated in [91]: *abstract topology and characteristic polynomial of the monodromy determine embedded topology.*

There are also examples of Zariski pairs which are not Alexander-Zariski pairs (see [68, 3, 5]). Some of them are distinguished by the so-called characteristic varieties introduced by Libgober [39]. These are subtori of $(\mathbb{C}^*)^r$,

$r := \# \text{Irr}(C)$, which measure the excess of Betti numbers of finite Abelian coverings of the plane ramified on the curve (as Alexander polynomial does it for cyclic coverings).

Problem 3.6. How can one translate characteristic varieties of a projective curve in terms of invariants of the SIS singularity associated to it?

Though Alexander polynomial and characteristic varieties are topological invariants, they are in fact arithmetic invariants in the following sense. Let us suppose that a curve C is defined by a polynomial with coefficients in a number field K ; then Alexander polynomial and characteristic varieties can be computed *inside* K , i.e., they do not depend on the embedding $K \hookrightarrow \mathbb{C}$.

Definition 3.7. An *arithmetic Zariski pair* is a Zariski pair such that its elements are defined with coefficients in a number field and with conjugate equations by the action of a Galois element.

The existence of arithmetic Zariski pairs is a consequence of a work of Serre [75] and Chisini's conjecture [33]. Explicit examples have been found in [4]; moreover, there are a lot of candidates to be arithmetic Zariski pairs, for example, sextic curves with an \mathbb{A}_{19} singularity (discovered by Yoshihara [93]).

Problem 3.8. Let C_1, C_2 be an arithmetic Zariski pair and let V_1, V_2 SIS singularities such that C_1, C_2 are their respective tangent cones. Do they have the same embedded topological type?

4 Monodromy Conjecture

Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a germ of a holomorphic function and let

$$(V, 0) := (f^{-1}(0), 0) \subset (\mathbb{C}^{n+1}, 0)$$

be the germ of hypersurface singularity defined by the zero locus of f .

Let $\pi : (Y, \mathcal{D}) \rightarrow (\mathbb{C}^{n+1}, 0)$ be an *embedded resolution* of $(V, 0)$, that is, a proper analytic map on a non-singular complex manifold Y such that:

- (1) the analytic subspace $\mathcal{D} := \pi^{-1}(0)$ of Y is the union of non-singular n -dimensional manifolds in Y which are in general position;
- (2) the map $\pi|_{Y \setminus \mathcal{D}}$ is an analytic isomorphism: $Y \setminus \mathcal{D} \rightarrow \mathbb{C}^{n+1} \setminus 0$;
- (3) in a neighbourhood of any point of \mathcal{D} there exist a local system of coordinates y_0, \dots, y_n such that $f \circ \pi(y_0, \dots, y_n) = y_0^{N_0} \cdots y_n^{N_n}$.

Let $E_i, i \in I$, be the irreducible components of the divisor $\pi^{-1}(f^{-1}(0))$. For each subset $J \subset I$ we set

$$E_J := \bigcap_{j \in J} E_j, \quad \text{and} \quad \check{E}_J := E_J \setminus \bigcup_{j \notin J} E_{J \cup \{j\}}.$$

For each $j \in I$, let us denote by N_j the multiplicity of E_j in the divisor of $f \circ \pi$ and by $\nu_j - 1$ the multiplicity of E_j in the divisor of $\pi^*(\omega)$ where ω is a non-vanishing holomorphic $(n+1)$ -form in \mathbb{C}^{n+1} .

The invariant we are interested in is the *local topological zeta function* $Z_{top,0}(f, s) \in \mathbb{Q}(s)$, which is an analytic (but not topological, see [6]) subtle invariant associated with any germ of an analytic function $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$. This rational function was first introduced by J. Denef and F. Loeser as a sort of limit of the p -adic Igusa zeta function, see [15, 16]. The original definition was written in terms of an embedded resolution of its zero locus germ $(V, 0) \subset (\mathbb{C}^{n+1}, 0)$ (although it does not depend on any particular resolution). In [16], J. Denef and F. Loeser gave an intrinsic definition of $Z_{top,0}(f, s)$ using arc spaces and the motivic Igusa zeta function, –see also [17] and the Séminaire Bourbaki talk of E. Looijenga [43].

The *local topological zeta function* of f is:

$$Z_{top,0}(f, s) := \sum_{J \subset I} \chi(\check{E}_J \cap \pi^{-1}(0)) \prod_{j \in J} \frac{1}{\nu_j + N_j s} \in \mathbb{Q}(s),$$

where χ denotes the Euler-Poincaré characteristic. Each exceptional divisor E_j of an embedded resolution $\pi : (Y, \mathcal{D}) \rightarrow (\mathbb{C}^{n+1}, 0)$ of the germ $(V, 0)$ gives a candidate pole $-\nu_j/N_j$ of the rational function $Z_{top,0}(f, s)$. Nevertheless only a few of them give an actual pole of $Z_{top,0}(f, s)$. There are several conjectures related to the topological zeta functions. We focus our attention in the *Monodromy Conjecture*, see [14, 15].

Conjecture 4.1 (Local Monodromy). *If s_0 is a pole of the topological zeta function $Z_{top,0}(f, s)$ of the local singularity defined by f , then $\exp(2i\pi s_0)$ is an eigenvalue of the local monodromy at some complex point of $f^{-1}(0)$.*

If f defines an isolated hypersurface singularity, then $\exp(2i\pi s_0)$ has to be an eigenvalue of the complex algebraic monodromy of the germ $(f^{-1}(0), 0)$.

There are three general problems to consider when trying to prove (or disprove) the conjecture using resolution of singularities:

- (i) Explicit computation of an embedded resolution of the hypersurface $(V, 0) \subset (\mathbb{C}^{n+1}, 0)$.
- (ii) Determination of the poles $\{-\nu_j/N_j\}$ of $Z_{top,0}(f, s)$.

- (iii) Explicit computation of the eigenvalues of the complex algebraic monodromy (or computing the characteristic polynomials of the corresponding action of the complex algebraic monodromy) in terms of the resolution data.

The Monodromy Conjecture, which was first stated for the Igusa zeta function, has been proved for curve singularities by F. Loeser [40]. F. Loeser actually proved a stronger version of the Monodromy Conjecture: *any pole of the topological zeta function gives a root of the Bernstein polynomial of the singularity*. The behaviour of the topological zeta function for germs of curves is rather well understood once an explicit embedded resolution $\pi : (Y, \mathcal{D}) \rightarrow (\mathbb{C}^2, 0)$ of curve singularities is known, e.g. the minimal one. Basically, the poles are the $\{-\nu_j/N_j\}$ coming from rupture components in the minimal resolution, see the proof by Veys [86, 87]. The case of curves was proved in consecutive works by Strauss, Meuser, Igusa and Loeser for Igusa's local zeta function, but the same proof works for the topological zeta function. There are other recent proofs of the conjecture for the case of curves by Cassou-Noguès and the authors [8], Nicaise [67] and Rodrigues [72].

There are other classes of singularities where the embedded resolution is known. For example, for any singularity of hypersurface defined by an analytic function which is non-degenerated with respect to its Newton polytope, problems (i) and (iii) above are solved. Nevertheless, (ii) seems to be a hard combinatorial problem. This problem was partially solved by Loeser in the case where f has a non-degenerate Newton polytope and satisfies certain extra technical conditions, – [41].

Even in one of the simplest cases where f has non-isolated singularities, namely the cone over a curve, problems (i) and (iii) are solved, but (ii) presents serious difficulties. B. Rodrigues and W. Veys proved in [73] the Monodromy Conjecture for any homogeneous polynomial $f_d \in \mathbb{C}[x_1, x_2, x_3]$ satisfying $\chi(\mathbb{P}^2 \setminus \{f_d = 0\}) \neq 0$. In [7] the authors complete the proof of this case studying homogeneous polynomials $f_d \in \mathbb{C}[x_1, x_2, x_3]$ satisfying $\chi(\mathbb{P}^2 \setminus \{f_d = 0\}) = 0$.

As we mentioned before, an embedded resolution is also known for superisolated surface singularities, – see [1]. This allow Pi. Cassou-Noguès and the authors to solve problems (ii) and (iii) for SIS singularities, namely the main result of [7] is to prove:

Theorem 4.2 ([7]). *The local Monodromy Conjecture is true for superisolated surface singularities.*

The local topological zeta function of a SIS singularity satisfies the following

equality, see [7, Corollary 1.12]:

$$Z_{top,0}(V, s) = \frac{\chi(\mathbb{P}^2 \setminus C)}{t-s} + \frac{\chi(\check{C})}{(t-s)(s+1)} + \sum_{P \in \text{Sing}(C)} \left(\frac{1}{t} + (t+1) \left(\frac{1}{(t-s)(s+1)} - \frac{1}{t} \right) Z_{top,P}(g^P, t) \right),$$

where g^P is a local equation of C at P , $\check{C} := C \setminus \text{Sing}(C)$ and $t := 3 + (d+1)s$.

The following properties can be easily described from the previous equalities:

Proposition 4.3. *Let \mathcal{P} be the set of poles of $Z_{top,0}(V, s)$.*

- (i) $\mathcal{P} \subset \{-1, -\frac{3}{d}\} \cup \bigcup_{P \in \text{Sing}(C)} \left\{ -\frac{3+t_0}{(d+1)} \mid t_0 \text{ pole of } Z_{top,P}(g^P, t) \right\}$.
- (ii) *If $-\frac{3}{d} \neq s_0 \in \mathcal{P}$ then $\exp(2i\pi s_0)$ is an eigenvalue of the complex algebraic monodromy of V .*
- (iii) *Let $s_0 = -\frac{3}{d}$. If s_0 is a pole of $Z_{top,P}(C, s)$ at some point $P \in \text{Sing}(C)$ and either $\chi(\mathbb{P}^2 \setminus C) > 0$ or $\chi(\mathbb{P}^2 \setminus C) = 0$, then $\exp(2i\pi s_0)$ is an eigenvalue of the complex algebraic monodromy of V .*
- (iv) *If $s_0 = -\frac{3}{d}$ is a multiple pole of $Z_{top,0}(V, s)$ then $\exp(2i\pi s_0)$ is an eigenvalue of the local algebraic monodromy at some singular point of C .*
- (v) *If $s_0 = -\frac{3}{d}$ is not a pole of $Z_{top,P}(C, s)$, the residue of $Z_{top,0}(V, s)$ at $-\frac{3}{d}$ equals $d\rho(C)$ where*

$$\rho(C) := \chi(\mathbb{P}^2 \setminus C) + \chi(\check{C}) \frac{d}{d-3} + \sum_{P \in \text{Sing}(C)} Z_{top,P} \left(C, -\frac{3}{d} \right).$$

Following Proposition 4.3, the Monodromy Conjecture for SIS singularities is proved in all but two cases:

- (N-1) $\chi(\mathbb{P}^2 \setminus C) = 0$, $s_0 = -\frac{3}{d}$ is not a pole for the local topological zeta function at any singular point in C and $\rho(C) \neq 0$.
- (N-2) $\chi(\mathbb{P}^2 \setminus C) < 0$.

The *bad divisors* are the degree d effective divisor D on \mathbb{P}^2 ($d > 3$) such that $\chi(\mathbb{P}^2 \setminus D) \leq 0$ and $s_0 = -\frac{3}{d}$ is not a pole of $Z_{top,P}(g_D^P, s)$, for any singular point P in its support D_{red} , where g_D^P is the local equation of the divisor D at P . The main part of [7, §2] is devoted to determine the bad divisors D on \mathbb{P}^2 such that $\rho(D) \neq 0$ and finally to prove the Monodromy Conjecture.

Note that the Euler-Poincaré characteristic condition on a bad divisor D implies that D has at least two irreducible components, all of them rational curves, see [31, 30, 32, 9]. In fact, the main result in [7] can be used to study arrangements $C = C_1 + \dots + C_s$ of rational plane curves such that $\chi(\mathbb{P}^2 \setminus C) \leq 0$. In particular, some necessary conditions on the combinatorial type of C (see Section 3) are obtained in order to the curve C exists.

The authors and S.M. Gusein-Zade have computed in an unpublished work the topological zeta function for Yomdin surface singularities, obtaining also a similar formula to the one for SIS singularities.

To avoid problems (i) and (ii) one can compute the so called motivic Igusa zeta function using motivic integration. In particular Pi. Cassou-Noguès and the authors in [8] have verified the conjecture (even the original conjecture by Igusa) for *quasi-ordinary hypersurface singularities* in arbitrary dimension measuring arcs and using Newton maps [8].

5 SIS Singularities with Rational Homology Sphere Links and Rational Cuspidal Curves

Superisolated surface singularities can be used to construct normal surface singularities whose link are rational homology spheres.

Let $(V, 0) = (\{f_1 = \dots = f_m = 0\}, 0) \subset (\mathbb{C}^N, 0)$ be a normal surface singularity with link L_V . One of the main problems is to determine which analytical properties of $(V, 0)$ can be read from the topology of the singularity, see [55]. Let $\pi : \tilde{V} \rightarrow V$ be a resolution of V .

The link L_V is called a *rational homology sphere* (QHS) if $H_1(L_V, \mathbb{Q}) = \{0\}$, and L_V is called an *integer homology sphere* (ZHS) if $H_1(L_V, \mathbb{Z}) = \{0\}$. In general the first Betti number $b_1(L_V) = b_1(\Gamma(\pi)) + 2 \sum_i g(E_i)$, where $b_1(\Gamma(\pi))$ is the number of independent cycles of the graph. In fact L_V is a QHS if and only if $\Gamma(\pi)$ is a tree and every E_i is a rational curve. If additionally the intersection matrix has determinant ± 1 then L_V is an ZHS.

Example 5.1. If $(V, 0) \subset (\mathbb{C}^3, 0)$ is a SIS singularity with an irreducible tangent cone $C \subset \mathbb{P}^2$ then L_V is a rational homology sphere if and only if C is a rational curve and each of its singularities (C, p) is locally irreducible, i.e a cusp.

In [47] A. Némethi and the last two authors have used SIS singularities whose link is a rational homology sphere to disprove several conjectures made during last years, see loc. cit. for a series of counterexamples. In Example 5.2 we present one of them.

For instance, it is shown that in the QHS link case the geometric genus p_g (analytical property of $(V, 0)$) does not depend only on its link L_V , even

if we work only with Gorenstein singularities (cf. [62, Question 3.2], see also [55, Problem 9.2]). Moreover for \mathbb{Q} -Gorenstein singularities (with $b_1(L_V) = 0$) analytical properties like the multiplicity, embedded dimension, Hilbert-Samuel function are not topological properties.

It is also shown that the universal abelian cover conjecture by Neumann and Wahl in [64] did not hold with the generality they stated it. The starting point of the conjecture was Neumann's result [60] that the universal abelian cover of a singularity with a good \mathbb{C}^* -action and with $b_1(L_V) = 0$ is a Brieskorn complete intersection whose weights can be determined from the Seifert invariants of the link. Their original conjecture was:

Assume that $(V, 0)$ is \mathbb{Q} -Gorenstein singularity satisfying $b_1(L_V) = 0$. Then there exists an equisingular and equivariant deformation of the universal abelian cover of $(V, 0)$ to an isolated complete intersection singularity. Moreover, the equations of this complete intersection, together with the action of $H_1(L_V, \mathbb{Z})$, can be recovered from L_V via the “splice equations”.

The semigroup condition as stated in [64] does not hold in general. Thus Neumann and Wahl restrict themselves to a very interesting class of complete intersection normal complex surface singularities called *splice type singularities*, see [65, 66]. In [66] the authors conjectured that rational singularities and \mathbb{Q} HS link minimally elliptic singularities belong to the class of *splice type singularities*. Just recently T. Okuma in [69] has given a proof of this result. See the paper by J. Wahl [89] in these proceedings.

Another conjecture that was disproved in [47] was the Seiberg-Witten invariant conjecture (SWC). A. Némethi and L. Nicolaescu [56] offered a candidate as a topological bound for the geometric genus of a rational homology sphere link of a normal surface singularity: *Let L_V be the link of a normal surface singularity.*

(a) *If L_V is a rational homology sphere then*

$$p_g \leq \mathbf{sw}(L_V) - (Z_K^2 + s)/8.$$

(b) *Additionally, if the singularity is \mathbb{Q} -Gorenstein, then in (a) the equality holds.*

Here Z_K is the *canonical cycle* associated with $\Gamma(\pi)$, and s the number of vertices in $\Gamma(\pi)$. Then $Z_K^2 + s$ does not depend on the choice of $\Gamma(\pi)$, it is a topological invariant of L_V . Set $H := H_1(L_V, \mathbb{Z})$.

The *Seiberg-Witten invariant* $\mathbf{sw}(L_V)$ of the link L_V (associated with the canonical spin^c structure) is

$$\mathbf{sw}(L_V) := -\frac{\lambda(L_V)}{|H|} + \mathcal{T}(L_V),$$

where $\mathcal{T}(M)$ is the sign-refined Reidemeister-Turaev torsion $\mathcal{T}(M)$ (associated with the canonical $spin^c$ structure) [81] and $\lambda(L_V)$ is the normalized by the Casson-Walker invariant, using the convention of [36] (cf. also with [56, 57, 58, 55]). Both invariants $\mathcal{T}(L_V)$ and $\lambda(L_V)$ can be determined from the graph (for details, see [56] or [55]).

The SWC-conjecture was verified by Némethi and Nicolaescu for quotient singularities [56], for singularities with good \mathbb{C}^* -actions [57] and hypersurface suspension singularities $g(u, v) + w^n$ with g irreducible [58].

Let $(V, 0) \subset (\mathbb{C}^3, 0)$ be a SIS singularity whose tangent cone $C \subset \mathbb{P}^2$ is an irreducible rational cuspidal curve (each singularity of C is locally irreducible).

We denote by Δ^P the characteristic polynomial of $(C, P) \subset (\mathbb{P}^2, P)$, set $\Delta(t) := \prod_{P \in \text{Sing}(C)} \Delta^P(t)$ and $2\delta := \deg \Delta(t)$. By the rationality of C one has

$$(d-1)(d-2) = 2\delta = \sum_{P \in \text{Sing}(C)} \mu^P,$$

where δ is the sum of the delta-invariants of the germs (C, P) , $P \in \text{Sing}(C)$.

The *minimal resolution* of V was described in Section 1. Since $\Delta^P(1) = 1$, this implies that $|H| = \Delta_V(1) = d$. In fact, one can verify easily that $H = \mathbb{Z}_d$, and a possible generator of H is an elementary loop in a transversal slice to C .

The other invariants which are involved in the SWC can be computed from the *minimal resolution* of V and using Laufer's formula [34]:

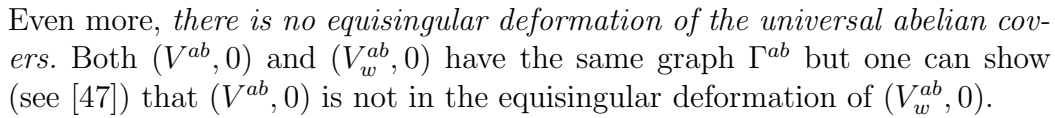
$$\left\{ \begin{array}{l} Z_K^2 + s = -(d-1)(d^2 - 3d + 1); \quad p_g = d(d-1)(d-2)/6; \quad \text{and} \\ \mathbf{sw}(L_V) = \frac{1}{d} \sum_{\xi^d=1 \neq \xi} \frac{\Delta(\xi)}{(\xi-1)^2} + \frac{1}{2d} \Delta(t)''(1) - \frac{\delta(6\delta-5)}{12d}. \end{array} \right. \quad (1)$$

Example 5.2. Let us continue with Example 1.2. The link of such SIS singularity is a rational homology sphere because the curve C is irreducible, rational and cuspidal. The plumbing graph is star-shaped, in particular it can be realized by a weighted homogeneous singularity $(V_w, 0)$.

In this case, $p_g(V, 0) = 10$ by the previous formula and $p_g(V_w, 0) = 10$ by Pinkham's formula [71]. In particular, using [62] (3.3), $(V, 0)$ is in an equisingular deformation of $(V_w, 0)$. This deformation, found with the help of J. Stevens, can be described as follows. The weights of the variables (a, \dots, f, λ) are $(62, 26, 30, 28, 93, 91, -3)$:

$$V(\lambda) = \left\{ \begin{array}{lll} ab - c^2d = \lambda f, & bc - d^2 = \lambda^2 a, & ad - c^3 = \lambda e, \\ be - df = -\lambda ac^2, & de - cf = -\lambda a^2, & af - c^2e = -\lambda b^6, \\ e^2 + a^3 + b^6c = 0, & ef + a^2c^2 + b^6d = 0, & f^2 + ac^4 + b^7 = 0 \end{array} \right\}.$$

In [60] it was proved that the universal abelian cover $(V_w^{ab}, 0)$ of $(V_w, 0)$ is $\Sigma(13, 31, 2)$, the Brieskorn hypersurface singularity $\{u^{13} + v^{31} + w^2 = 0\}$. The corresponding resolution graph Γ^{ab} (of both $(V^{ab}, 0)$ and $(V_w^{ab}, 0)$) is



Looking at the identity (1), one considers now the (a priori) rational function

J.F. Fernández de Bobadilla, A. Némethi and the last two authors in [20] proved that $R(t) \in \mathbb{Z}[t]$ and it can be written as

Moreover

In particular, the (SWC) is equivalent to $R(1) \geq 0$.

In fact it is rather curious that in all examples, based on SIS singularities, studied in [47] one gets $R(1) \leq 0$. Motivated by these examples, in [20] there were worked out many examples discovering that the coefficients of

$R(t)$ are always non-positive. This gives strong necessary conditions on the singularities of C . It is known that in the problem of classification of rational cuspidal curves one of the key points is to find necessary conditions on the singularities. We state this *compatibility property* on $R(t)$ that we have found as a conjecture.

Conjecture 5.3 ((CP) [20]). *Let $(C, p_i)_{i=1}^\nu$ be a collection of local plane curve singularities, all of them locally irreducible, such that $2\delta = (d-1)(d-2)$ for some integer d . Then if $(C, p_i)_{i=1}^\nu$ can be realized as the local singularities of a degree d (automatically rational and cuspidal) projective plane curve of degree d then*

$$c_l \leq (l+1)(l+2)/2 \text{ for all } l = 0, \dots, d-3. \quad (*_l)$$

In fact the coefficients c_l can be computed from the polynomial $Q(t)$ defined in terms of $\Delta(t)$:

$$\Delta(t) = 1 + (t-1)\delta + (t-1)^2 Q(t) = \sum_{l \geq d} b_l t^l + \sum_{l=0}^{d-3} c_l t^{(d-3-l)d}.$$

The main result in [20] is the proof of the following theorem:

Theorem 5.4. *If the logarithmic Kodaira dimension $\bar{\kappa} := \bar{\kappa}(\mathbb{P}^2 \setminus C)$ is less than 2, then (CP) is true. In fact, in all these cases $c_l = \frac{(l+1)(l+2)}{2}$ for any $l = 0, \dots, d-3$.*

Corollary 5.5 ([20]). *Let $f = f_d + f_{d+1} + \dots : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$ be a hypersurface superisolated singularity with $\bar{\kappa}(\mathbb{P}^2 \setminus \{f_d = 0\}) < 2$. Then the Seiberg-Witten invariant conjecture is true for $(V, 0) = (\{f = 0\}, 0)$.*

It is even more interesting to study the compatibility property when $\nu = 1$. In this case one can prove that all the inequalities $(*_l)$ are indeed identities. These identities are equivalent (via a theorem by A. Campillo, F. Delgado and S.M. Gusein-Zade in [12]) to a very remarkable distribution of the elements of the semigroup $\Gamma_{(C,P)}$ of the singularity (C, P) in intervals of length d . It is shown there that (CP) in this case is equivalent to the following conjectural identity:

$$\sum_{k \in \Gamma_{(C,P)}} t^{[k/d]} = \frac{1-t^d}{(1-t)^2} = 1 + 2t + \dots + (d-1)t^{d-2} + d(t^{d-1} + t^d + t^{d+1} + \dots).$$

6 Final Remarks

6.1 Weighted-Yomdin Singularities

The second natural generalization of SIS singularities is obtained if one considers a weighted version of this singularities.

Definition 6.1. Let $\omega := (p_x, p_y, p_z) \in \mathbb{N}^3$ be such that $\gcd(p_x, p_y, p_z) = 1$. A polynomial f is ω -weighted homogeneous of degree d if

$$f(t^{p_x}x, t^{p_y}y, t^{p_z}z) = t^d f(x, y, z).$$

It then defines a curve in the *weighted projective plane* $\mathbb{P}_\omega^2 := \mathbb{C}^3 \setminus \{0\} / \sim$, $(x, y, z) \sim (t^{p_x}x, t^{p_y}y, t^{p_z}z)$ for all $t \in \mathbb{C}^*$. If $P \in \mathbb{P}_\omega^2$, we define its *order* ν_P as the gcd of the weights of the non-zero coordinates of P .

Definition 6.2. If $C \subset \mathbb{P}_\omega^2$ is a curve defined by a weighted homogeneous polynomial f and $P \in C$ we define the *weighted Milnor number* $\mu^\omega(C, P)$ as $\frac{\mu}{\nu_P}$ where μ is defined as follows; let us suppose that P is the equivalence class of $(x_0, y_0, 1)$ and consider the Milnor number of $f(x, y, 1) = 0$ at (x_0, y_0) . A *singular point* of C is a point such that $\mu^\omega(C, P) > 0$.

Let us consider a germ $(W, 0) \subset (\mathbb{C}^3, 0)$ defined by a power series g ; let $g = g_d + g_{d+k} + \dots$ be the weighted homogeneous decomposition of f with respect to ω and let $C_m^\omega \subset \mathbb{P}_\omega^2$ be the weighted projective locus of zeroes of g_m .

Definition 6.3. We say that $(W, 0) \subset (\mathbb{C}^3, 0)$ is a *weighted Yomdin singularity with respect to ω* if $\text{Sing}(C^\omega) \cap C_{d+k}^\omega = \emptyset$.

In a forthcoming joint work with J. Fernández de Bobadilla, we will give a proof of a formula which was suggested to us by C. Hertling.

Proposition 6.4. *The Milnor number μ of a weighted Yomdin singularity $(W, 0) \subset (\mathbb{C}^3, 0)$ with respect to ω satisfies the following equality:*

$$\mu(W, 0) = \left(\frac{d}{p_x} - 1\right) \left(\frac{d}{p_y} - 1\right) \left(\frac{d}{p_z} - 1\right) + k \sum_{P \in \text{Sing}(C^\omega)} \mu(C^\omega, P).$$

6.2 *-Polynomials

The theory of (local) SIS or Yomdin singularities has an analogous global counterpart defined by polynomials of type $f = f_d + f_{d-k} + \dots$ and the same geometric condition. For instance the formula for the global Milnor number is done by the authors in [10] and for the zeta-function of the monodromy at infinity by S.M. Gusein-Zade and the last two authors in [29]. A finer study

has been done in a series of works A. Némethi and R. García López [22, 23, 24] for $*$ -polynomials $f = f_d + f_{d-1} + \dots$. The behaviour of these polynomials at infinity imitates in some way the local behaviour of SIS singularities. They computed formulæ for the global Milnor number, monodromy at infinity, Mixed Hodge structure at infinity...

6.3 Intersection form of a SIS singularity

In the topological study of singularities, we are interested in invariants living in the complex setting (like the Jordan form of the monodromy) but also in invariants living in the integers, like monodromy over \mathbb{Z} , Seifert form or the intersection form in a distinguished basis of vanishing cycles.

It is well-known how to compute these invariants for local germs of curves. In his thesis, M. Escario computes these invariants for polynomials in two variables which are generic at infinity (in fact, for the more general concept of *tame* polynomials), using a generic polar mapping Φ and the braid monodromy of the discriminant of Φ .

Combining these techniques with Gabriélov's method (see [21]), M. Escario gives a method to compute the intersection form of the Milnor fiber in a distinguished basis of vanishing cycles for SIS singularities. In fact, this method works also for Yomdin singularities.

6.4 Durfee's conjecture for SIS singularities

A. Durfee [18] conjectured that the signature of the Milnor fibre of an hypersurface surface singularity is negative. In fact, Durfee's conjecture is the stronger inequality

$$6p_g \leq \mu. \quad (*)$$

Y. Xu and S.S.T. Yau proved $(*)$ for weighted homogeneous surface singularities, [90]. A. Némethi [52] verified the inequality $(*)$ in the case $f(x, y) + z^n$ with $f(x, y) \in \mathbb{C}\{x, y\}$ irreducible, see also T. Ashikaga [11].

Using SIS singularities, A. Melle Hernández [50] proved $(*)$ for *absolutely isolated* surface singularities. A surface hypersurface $(V, 0) \subset (\mathbb{C}^3, 0)$ is absolutely isolated if there exists a resolution $\pi: \tilde{V} \rightarrow V$ such that π is a composition of blowing-ups at points.

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