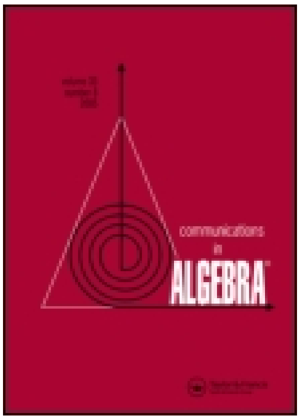

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On the topology of a generic fibre of a polynomial function

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ON THE TOPOLOGY OF A GENERIC FIBRE OF A POLYNOMIAL FUNCTION

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ABSTRACT. In this work we study the topologies of the fibres of some families of complex polynomial functions with isolated critical points. We consider polynomials with some transversality conditions at infinity and compute explicitly its global Milnor number $\mu(f)$, the invariant $\lambda(f)$ and therefore the Euler characteristic of its generic fibre. We show that under some mild transversality condition (transversal at infinity) the behavior of f at infinity is good and the topology of the generic fibre is determined by the two homogeneous parts of higher degree of f. Finally we study families of polynomials, called two-term polynomials. This polynomials may have *atypical values at infinity*. Given such a two-term polynomial f we characterize its atypical values by some invariants of f. These polynomials are a source of interesting examples.

This paper deals with the fibres of some families of complex polynomial functions which only have isolated critical points. Let $f : \mathbf{C}^n \to \mathbf{C}$ be a polynomial function and $B_f \subset \mathbf{C}$ its bifurcation set, i.e. the smallest subset such that the restriction

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of f induces a C^∞ trivial fibration $\mathbb{C}^n \setminus f^{-1}(B_f) \to \mathbb{C} \setminus B_f$, [Ph], [HL], [V]. The bifurcation set B_f contains the critical values of f and the so-called *atypical values at infinity*. In general there is not a precise definition of what is an atypical value at infinity. We have considered in [ALM] an invariant $\lambda(f)$ that helps to detect the atypical fibres at infinity.

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial function with isolated critical points. Let us denote by V_t the compactification of the fibre $F_t := f^{-1}(t)$ in $\mathbb{P}^n$ and $D = V_t \cap H_\infty$, where H_∞ is the hyperplane at infinity which may be identified with $\mathbb{P}^{n-1}$; D is the hypersurface in $\mathbb{P}^{n-1}$ defined by the homogeneous form of highest degree of f. By means of the generalized Milnor number of Parusiński we define a generically constant function $\mu_f^\infty(t) := \mu(V_t, D)$ on $\mathbb{C}$ and we set

$$\lambda_f(t) := \mu_f^\infty(t) - \mu_{gen}^\infty(f) \qquad \lambda(f) := \sum_{t\in\mathbb{C}} \lambda_f(t).$$

The main property of those invariants is that with the help of the global Milnor number $\mu(f)$ determine the Euler characteristics of the fibres of f:

Theorem. [ALM] *Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial with isolated critical points. Let $t \in \mathbb{C} \setminus B_f$.*

(*i*) *The Euler characteristic of the fibre F_t is equal to*

$$\chi(F_t) = 1 + (-1)^{n-1}(\mu(f) + \lambda(f)).$$

(*ii*) *For any value $b \in \mathbb{C}$ the Euler characteristic of the fibre F_b is equal to*

$$\chi(F_b) = \chi(F_t) + (-1)^n(\mu_f^a(b) + \lambda_f(b)).$$

where $\mu_f^a(b)$ is the sum of the Milnor numbers of the affine variety F_b at its singular points.

The main result of this work is to compute the Euler characteristic of the fibres for some families of polynomials. We obtain explicit analytic formulæ for the global Milnor number of f and for the invariant $\lambda(f)$.

Let $f = f_0 + \ldots + f_{d-k} + f_d$ be the homogeneous decomposition of f, where k is the least positive integer such that f_{d-k} is not identically zero. We will denote by D (resp. T) the divisor in $\mathbb{P}^{n-1}$ defined by $f_d = 0$ (resp. $f_{d-k} = 0$). For any hypersurface $C \subset \mathbb{P}^{n-1}$ we will denote by C_{red} the reduced hypersurface associated to C. Let us consider the decomposition $D = \sum_{j=1}^m q_j D_j$, where $q_j \geq 1$ and D_j is an irreducible hypersurface of degree d_j, $j = 1, \ldots, m$. Then $D_{red} = \sum_{j=1}^n D_j$; we will denote by $p = \sum_{j=1}^m d_j$ the degree of D_{red}.

Theorem 2.1. *If $\mathrm{Sing}(D) \cap T = \emptyset$, then D has isolated singularities, the set $\mathrm{C}(f)$ of critical points of f is finite and*

$$\mu(f) = (d-1)^n - k \sum_{P\in\mathrm{Sing}(D)} \mu(D, P).$$

polynomials considered in Theorem 2.1 are called Yomdine-at-infinity polynomials. They have isolated singularities at infinity. By results of Dimca, [D1, Theorem 2], any fibre of f has the homotopy type of a wedge of spheres of real dimension $r-1$. In [P3] Parusiński proved that the number of spheres is equal to $\mu(f)$ and that these polynomials are *good*, i.e., the topology at infinity of the fibres does not change. In particular the invariant $\lambda(f)$ is zero for these polynomials.

The other results correspond to $n=3$; we will usually take x, y, z as coordinates. We will say that f is *transversal at infinity* if $\mathrm{Sing}(D_{red}) \cap T = \emptyset$ and for any $j \in \{1, \ldots, m\}$ such that $q_j > 1$, D_j meets T at $d_j(d+k)$ points. We remark that this condition implies that if D is not reduced then T is reduced .

Theorem 3.1. *Let f be* transversal at infinity. *Then* $\mathrm{C}(f)$ *is finite and*

$$\mu(f) = (d-1)^3 - k(\chi(D) + d(2d-p-3)) + k^2(d-p).$$

In this case we remark that D may be not reduced and f has non-isolated singularities at infinity. Nevertheless, we prove that f has $\mathcal{W}$-isolated singularities at infinity, in the sense of Siersma-Tibăr, [ST]. Therefore the generic fibre of f has the homotopy type of a wedge of $\mu(f)$ 2-spheres, f is good and then $\lambda(f)=0$.

Notice that in Theorems 2.1 and 3.1, the terms of f of degree less than $d-k$ affect neither the Milnor number nor the topology of the generic fibre; this is not the situation in Theorem 4.4. We need some more notation in order to state it.

Let us consider now two germs $g, h \in \mathbb{C}\{x, y\}$ which are in the maximal ideal; and let us suppose that for any $t \in \mathbb{C}^*$, the germ $g+th$ is square-free. Then there exists a finite set $S(g,h) \subset \mathbb{C}^*$ such that the Milnor number $\mu(g+th)$ of the germ $g+th$ at the origin is constant if $t \in \mathbb{C}^* \setminus S(f,g)$; let us call $\mu(g,h)$ this constant. Then we will denote

$$\alpha^*(g,h) := \sum_{t \in S(g,h)} (\mu(g+th) - \mu(g,h)).$$

Theorem 4.4 and Proposition 4.12. *Let us suppose that $f = f_d + f_{d-k}$. The critical locus $\mathrm{C}(f)$ is finite if and only if the following conditions hold:*

(i) $\mathrm{Sing}(D) \cap \mathrm{Sing}(T) = \emptyset$ *and T and D have no common components.*

(ii) *The set of the points in $\mathbb{P}^2 \setminus (Sing(D) \cup Sing(T))$ such that the gradients of f_d and f_{d-k} are linearly dependent is finite.*

Moreover in this case

$$\mu(f) = (d-1)^3 - k(\chi(D) + d(d-3) + \sum_{P \in \mathrm{Sing}(D) \cap T} (I_P(D,T) + \alpha_P^* - 1)),$$

$$\lambda(f) = \sum_{P \in \mathrm{Sing}(D) \cap T} \Big(k\alpha_P^* + \delta\big((I_P(D,T)(d_1 - k_1) - 1)(d_1 - 1) - \mu_P\big)\Big),$$

where

- $\alpha_P^* := \alpha^*((f_{d-k}^{d_1})_P, (f_d^{d_1 - k_1})_P)$ *($(-)_P$ is the germ at P);*
- I_P *is the intersection number at P;*

- $d_1 = \frac{d}{\delta}$, $k_1 = \frac{k}{\delta}$ *where* $\delta := \gcd(d, k)$ *and*
- μ_P *is the Milnor number at* P *of the generic element of a projective pencil associated to* f.

polynomials verifying conditions (i) and (ii) of Theorem 4.4 are called two-term polynomials. We prove again that they have also $\mathcal{W}$-isolated singularities at infinity. The sum $\sum_P \alpha_P^*$ gives essentially the contribution to $\lambda(f)$ by the fibres different from F_0.

The way for proving the theorems is related with the results that Melle have obtained in the local case, [M]. Namely, in the local case when one considers germs of hypersurfaces defined by $f = f_d + f_{d+k} + \dots$ with similar hypothesis as in our theorems one obtains similar formulæ replacing $-k$ by $+k$. As in the proof of [M], we will use Parusiński's generalized Milnor number in the proof of theorems 2.1 and 3.1. Notice that polynomials under the hypothesis of these two theorems are *tame*, [B, Prop. 3.1], and they have not atypical values at infinity. By a theorem in [ALM], in these cases $\mu(f)$ is up to 1 the Euler characteristic of a generic fibre of f; it will not be the case in general for polynomials of Theorem 4.4, and we will need some comparison results to obtain a proof.

Finally we note that ($*$)-polynomials in [GN] are a particular case of Yomdine-at-infinity polynomials (when $k = 1$); in their paper, García and Némethi compute the monodromy at infinity and not only the Milnor number. The computation of the monodromy in the local case was performed in [A] for superisolated singularities (which is the local analog to ($*$)-polynomials in [GN]).

§1.- POLYNOMIALS WITH ISOLATED CRITICAL POINTS

We denote by $\chi(A)$ the Euler characteristic of the topological space A.

Definition 1.1. [P2] *Let* $\mathcal{E}$ *be a holomorphic vector bundle of rank* r *over a smooth compact complex manifold* M *of dimension* n. *Let* $s \in \mathbb{P}(H^0(M; \mathcal{E}))$ *and let* X *be the zero set of a representative of* s. *Then we define the Milnor number of* X *in* M, *denoted by* $\mu(M; X)$ *or* $\mu(X)$, *as*

$$\mu(M; X) := (-1)^{n-r+1}(\chi(X) - \chi(M; \mathcal{E})),$$

where $\chi(M; \mathcal{E})$ *is the Euler characteristic of the zero set of a section of* $\mathcal{E}$ *transversal to the zero section.*

In [P1], Parusiński defined the generalized Milnor number in the hypersurface case; when $r = 1$, both definitions agree. Let us recall some properties of this invariant from [P1] and [PP]:

Property 1.2. [P1] *Let* M *be a compact complex manifold of dimension* n *and let* X *and* Z *two hypersurfaces which are linearly equivalent as divisors. Then:*

$$\mu(X) - \mu(Z) = (-1)^n(\chi(X) - \chi(Z)).$$

Property 1.3. [PP] *Let* M *be a compact complex manifold of dimension* n *and let* X *be a hypersurface. Given* $x \in X$ *and* $f_x = 0$ *a local equation of* X *at* x, *we define the topological Milnor number of* X *at* x *as*

$$\mu_{top}(X,x) := (-1)^{n-1}(\chi(F_x)-1),$$

where F_x is the Milnor fibre of f_x; it is just the classical Milnor number if x is an isolated singular point of X. Then there exists a Whitney stratification $\mathcal{S}$ of X such that $\mu_{top}(X,x)$ is constant along each stratum; let us denote μ_S this constant number for each $S \in \mathcal{S}$. Let us take a smooth hypersurface Z which is linearly equivalent to X and which is transversal to $\mathcal{S}$. Then:

$$\mu(X) = \sum_{S\in\mathcal{S}} \chi(S \setminus Z)\mu_S.$$

Remark. In [P1], it is defined more generally $\mu(X,Y)$, where Y is a compact subvariety of X which admits a neighbourhood U in X such that $U \setminus Y$ is nonsingular. If Z and $\mathcal{S}$ are as in (1.3) and $\mathcal{S}$ induces a stratification $\mathcal{S}_Y$ on Y, then we get:

$$\mu(X,Y) = \sum_{S\in\mathcal{S}_Y} \chi(S \setminus Z)\mu_S.$$

Let $f \in \mathbb{C}[x_1,\ldots,x_n]$ be a polynomial and let $f = f_0 + f_1 + \cdots + f_d$ be its decomposition in homogeneous forms. We take the natural inclusion $\mathbb{C}^n \hookrightarrow \mathbb{P}^n$; we will denote by $\mathbb{P}^{n-1}$ the hyperplane at infinity and by x_0 the new variable. Let D be the divisor of $\mathbb{P}^{n-1}$ defined by f_d. We will denote by V_t the compactification of $F_t := f^{-1}(t)$ in $\mathbb{P}^n$; the equation of V_t is $\tilde{f}(x_0,x_1,\ldots,x_n) - tx_0^d = 0$, where $\tilde{f} := f_d + x_0 f_{d-1} + \cdots + x_0^{d-1} f_1 + x_0^d f_0$.

We will denote $\mathrm{C}(f)$ the set of critical points of f and when $\mathrm{C}(f)$ is a finite set we set $\mu(f)$ the sum of the local Milnor numbers of the germs of the level hypersurfaces of f at the points of $\mathrm{C}(f)$.

Definition 1.4. [ALM] *Let $f \in \mathbb{C}[x_1,\ldots,x_n]$ be a polynomial with isolated critical points. We define* the Milnor function of f at infinity *as a function $\mu_f^\infty \colon \mathbb{C} \to \mathbb{Z}$ such that $\mu_f^\infty(t) := \mu(V_t, D)$. If it is constant we will denote its constant value by μ_f^∞.*

We recall a construction of [AML]. Let us take a polynomial f of degree d with isolated singularities. This fact allows us to separate the affine singularities of each V_t from the singularities at infinity. For each $t \in \mathbb{C}$,

$$\mu(V_t) = \mu(V_t, D) + \mu(V_t, \mathrm{Sing}(F_t)).$$

Let us denote $\mu(V_t, \mathrm{Sing}(F_t))$ by $\mu_f^a(t)$; recall that $\mu(V_t, D) = \mu_f^\infty(t)$.

Since $\mathrm{C}(f)$ is finite we take any stratification $\mathcal{S}_\infty$ of $\mathbb{C}$ such that $\mu_f^\infty(t)$ is constant along each stratum S of $\mathcal{S}_\infty$. Let μ_S^∞ be this constant value over the stratum $S \in \mathcal{S}_\infty$. Let $S_{gen} := \mathbb{C} \setminus \{t_1,\ldots,t_m\}$ be the big stratum of $\mathcal{S}_\infty$ and let $\mu_{gen}^\infty(f)$ be the constant value of μ_f^∞ on S_{gen}.

Definition 1.5. [ALM] *With the previous notations, we define for any $t \in \mathbb{C}$*

$$\lambda_f(t) := \mu_f^\infty(t) - \mu_{gen}^\infty(f)$$

and

$$\lambda(f) := \sum_{t\in\mathbb{C}} \lambda_f(t) = \sum_{j=1}^{m} \lambda_f(t_j).$$

It is easily seen that $\lambda_f(t)$ and $\lambda(f)$ do not depend on the choice of the stratification $\mathcal{S}_\infty$ of $\mathbb{C}$ that verifies $\mu_f^\infty(t)$ is constant along each stratum S of $\mathcal{S}_\infty$. Moreover

$$\sum_{S\in\mathcal{S}_\infty} \chi(S)\mu_S^\infty = \lambda(f) + \mu_{gen}^\infty(f),$$

and it is clear that if the function μ_f^∞ is constant then $\lambda(f)$ is equal to zero.

Let S_0 the set of regular values of f and let $S := S_0 \cap S_{gen}$; it is a non-empty open set in $\mathbb{C}$ and $S = \mathbb{C} \setminus \{s_1, \dots, s_k\}$. Let $\mathcal{S}$ be the stratification S, $\{s_i\}_{i=1}^k$. It is easily seen that for all $t \in S$ the integer $\mu(V_t)$ is the same.

Corollary 1.6. [ALM] *Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a polynomial with isolated critical points of degree d. Let $\mathcal{S}_\infty$ be as above. Then the number*

$$\gamma(d,n) := \mu(f) + \lambda(f) + \mu_{gen}^\infty(f) + (-1)^{n-1}\chi(D)$$

depends only on d and n.

We state two easy and useful consequences of the theorem in the introduction:

Corollary 1.7. [ALM] *Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be a polynomial with isolated critical points such that μ_f^∞ is constant. Then*

$$\chi(F_t) = 1 + (-1)^{n-1}\mu(f)$$

if t is a generic value of f.

Theorem 1.8. [ALM] *Let $f = f_d + f_{d-1} + \cdots \in \mathbb{C}[x_1, \dots, x_n]$ be a polynomial of degree d with isolated critical points. Let $\mathcal{S}_\infty$ be as above. Let $\mu(D)$ be the generalized Milnor number of D in $\mathbb{P}^{n-1}$. Then*

$$\mu(f) = (d-1)^n - \mu(D) - \sum_{S\in\mathcal{S}_\infty} \chi(S)\mu_S^\infty$$

or

$$\mu(f) + \lambda(f) = (d-1)^n - \mu(D) - \mu_{gen}^\infty(f)$$

Therefore if μ_f^∞ is constant then

$$\mu(f) = (d-1)^n - \mu(D) - \mu_f^\infty.$$

§2.- Yomdine-at-infinity polynomials

Let f be a polynomial in $\mathbb{C}[x_1, \dots, x_n]$ of degree $d > 0$. Let us denote f_j, $j = 0, 1, \dots, d$ the homogeneous form of degree j of f. Let k be the least positive integer such that f_{d-k} is not identically zero. We have denoted D (resp. T) the divisor in $\mathbb{P}^n$ defined by $f_d = 0$ (resp. $f_{d-k} = 0$).

Definition. *A polynomial* $f \in \mathbb{C}[x_1, \ldots, x_n]$ *is a* Yomdine-at-infinity polynomial *if* $\mathrm{Sing}(D) \cap T = \emptyset$.

Theorem 2.1. *Let* $f \in \mathbb{C}[x_1, \ldots, x_n]$ *be a Yomdine-at-infinity polynomial. Then* D *has isolated singularities,* $\mathrm{C}(f)$ *is finite and*

$$\mu(f) = (d-1)^n - k \sum_{P \in \mathrm{Sing}(D)} \mu(D, P).$$

It is easily seen that Yomdine-at-infinity polynomials have isolated singularities at infinity. By results of Dimca, [D1], or Parusiński,[P3, Theorem 1.4], the generic fibre of f has the homotopy type of a wedge of $\mu(f)$ spheres of real dimension $n-1$. These polynomials are *good*, i.e., the topology at infinity of the fibres does not change. In fact, these polynomials are tame by [B, Prop. 3.1].

Proof. The hypersurface D is irreducible with isolated singularities if $n > 2$ and it is reduced if $n = 2$.

Firstly we will show that f has isolated critical points; in fact, V_t has isolated singular points, $\forall t \in \mathbb{C}$. It is enough to verify that there exists a regular neighborhood $T(D)$ of D in $\mathbb{P}^n$ such that $\forall t \in \mathbb{C}$ there is no singular point of V_t in $T(D) \setminus D$.

Let P be a point of D; if D is smooth at P, then V_t is smooth at P (and also in a neighborhood of P).

If $P \in \mathrm{Sing}(D)$ we can choose a local system of coordinates $(u_1, \ldots, u_{n-1}, w)$ such that the local equation of V_t at P is $f(u_1, \ldots, u_{n-1}) - w^k = 0$, where $w = 0$ is the local equation of $\mathbb{P}^{n-1}$ and $f(u_1, \ldots, u_{n-1}) = 0$ is the local equation of D. Then V_t has an isolated singularity at P and we get that $\mathrm{C}(f)$ is finite.

Secondly we will prove that μ_f^∞ is constant and its value is

$$(k-1) \sum_{P \in \mathrm{Sing}(D)} \mu(D, P) = (k-1)\mu(D).$$

We have seen that the pencil (V_t, P) is analytically trivial, and the Milnor numbers are constant. In this case the generalized and classical Milnor numbers are the same. Then,

$$\mu_f^\infty(t) = \sum_{P \in \mathrm{Sing}(D)} \mu(V_t, P).$$

is a constant function. As we have separated variables, it is easily seen that $\mu(V_t, P) = (k-1)\mu(D, P)$, $\forall P \in \mathrm{Sing}(D)$.

Then after (1.8) the proof of the theorem is finished $\square$

§3.- TRANSVERSAL-AT-INFINITY POLYNOMIALS

From now on $n = 3$. Let $f = f_d + f_{d-k} + \ldots \in \mathbb{C}[x, y, z]$ be a polynomial of degree $d > 0$ and let $D = \sum_{j=1}^{m} q_j D_j$ and T be as in the introduction. We recall that p denotes the degree of D_{red}.

Definition. *A polynomial $f \in \mathbb{C}[x, y, z]$ is a* transversal-at-infinity polynomial *if* $\operatorname{Sing}(D_{red}) \cap T = \emptyset$ *and for any $j \in \{1, \ldots, m\}$ such that $q_j > 1$, D_j meets T at $d_j(d + k)$ points.*

We note that either D or T is reduced.

Theorem 3.1. *Let f be a transversal-at-infinity polynomial. Then,* $C(f)$ *is finite and*

$$\mu(f) = (d-1)^3 - k(\chi(D) + d(2d - p - 3)) + k^2(d - p).$$

Notice that D may be not reduced and f may have non-isolated singularities at infinity. Nevertheless, we will prove in this section that f has $\mathcal{W}$-isolated singularities at infinity, in the sense of Siersma and Tibăr, [ST]. So by their results the generic fibre of f has the homotopy type of a wedge of $\mu(f)$ 2-spheres and we will prove that f is good. We divide the proof of this theorem in a sequence of lemmas.

Lemma 3.2. *If $f \in \mathbb{C}[x, y, z]$ is transversal at infinity, then f has isolated critical points.*

Proof. We are going to show that there exists a regular neighborhood $T(D)$ of D in $\mathbb{P}^3$ such that V_t is smooth in $T(D) \setminus D$, $\forall t \in \mathbb{C}$. It is enough to study V_t around any $P \in D$:

- P is a smooth point of D. In this case, V_t is smooth at P and therefore in a neighborhood of P. We have $\mu_{top}(V_t, P) = 0$.
- P is a smooth point of D_{red} which is singular at D and is not in T. In this case, P lies in an irreducible component D_j of D such that its multiplicity $q_j > 1$; we can choose local coordinates (u, v, w) at P such that the local equation of V_t is $u^{q_j} - w^k = 0$, where $w = 0$ is the local equation of $\mathbb{P}^2$ and $u = w = 0$ is the local equation of D_j; the singular part is contained in D. We have $\mu_{top}(V_t, P) = -(k-1)(q_j - 1)$.
- P is a singular point of D_{red} (and then, it is not in T). We can choose local coordinates (u, v, w) at P such that the local equation of V_t is $h(u, v) - w^k = 0$, where $w = 0$ is the local equation of $\mathbb{P}^2$ and $w = h(u, v) = 0$ is the local equation of D; the singular part is again contained in D. We have $\mu_{top}(V_t, P) = (k-1)\mu_{top}(D, P)$.
- P is a smooth point of D_{red} which is singular at D and is in T. As before P lies an irreducible component D_j of D such that its multiplicity $q_j > 1$; we can choose local coordinates (u, v, w) at P such that the local equation of V_t is $u^{q_j} - w^k v = 0$, where $w = 0$ is the local equation of $\mathbb{P}^2$, $u = 0$ is the local equation of D_j and $v = w = 0$ is tangent to T at P; the singular part is again contained in D. We have $\mu_{top}(V_t, P) = q_j - 1$.

We have proved that f has isolated critical points. □

Lemma 3.3. *If $f \in \mathbb{C}[x, y, z]$ is a transversal-at-infinity polynomial, then μ_f^∞ is a constant function, that implies $\lambda(f) = 0$.*

Proof. We are going to compute the function μ_f^∞. We will apply the property (1.3) (and the subsequent remark) of the generalized Milnor number.

Let $t \in \mathbb{C}$. We are going to consider a stratification $\mathcal{S}$ of V_t where the strata are:

(a) The smooth part of the affine part F_t (2-dimensional stratum).

(b) The connected components of $\operatorname{Sing}(F_t)$ (0-dimensional strata, if any).

(c) The connected components of the smooth part of D (1-dimensional strata).

(d) The connected components of the set of points which lie in the smooth part of $D_{red} \setminus T$ and which are not smooth in D (1-dimensional strata).

(e) The connected components of $\operatorname{Sing}(D_{red})$ (0-dimensional strata).

(f) The connected components of $\operatorname{Sing}(D) \cap T$ (0-dimensional strata).

We prove in the same way as in [M], that $\mathcal{S}$ is a Whitney stratification of V_t and by the computations in the proof of (3.2), the topological Milnor numbers of V_t are constant along each stratum. By Bertini's theorem, there exists a smooth surface W_d of degree d in $\mathbb{P}^3$ such that:

- W_d intersects $\operatorname{Sing}(V_t)$ only at D;
- the curve $W_d \cap \mathbb{P}^2$ intersects D_{red} at exactly dp points;
- the curve $W_d \cap \mathbb{P}^2$ does not intersect $D \cap T$.

It means that W_d is transversal to $\mathcal{S}$; let us denote $\mathcal{S}_D$ the set of strata which are contained in D. We can use it to compute $\mu_f^\infty(t) = \mu(V_t, D)$:

$$\mu(V_t, D) = \sum_{S \in \mathcal{S}_D} \chi(S \setminus W_d)\mu_S.$$

We have seen that for any stratum $S \in \mathcal{S}_D$ the number μ_S does not depend on $t \in \mathbb{C}$, and we have proved that the function μ_f^∞ is constant. $\square$

Lemma 3.4. *The constant value of μ_f^∞ is*

$$\mu_f^\infty = k(d-k)(d-p) + (k-1)\Big(d(d-3) + \chi(D)\Big)$$

Proof. We are going to compute the terms of the formula above. The strata in $\mathcal{S}_D$ are those from (c) to (f). We fix some notation; we denote $D = D_1 + \cdots + D_s + q_{s+1}D_{s+1} + \cdots + q_r D_r$, where $0 \le s \le r$ and $q_j > 1$ if $j = s+1, \dots, r$. Let us denote

$$\check{D}_j := D_j \setminus \Big(\operatorname{Sing}(D_j) \cup D_1 \cup \cdots \cup \widehat{D_j} \cup \cdots \cup D_r \cup W_d\Big).$$

(c) This term vanishes.

(d) We get in this term

$$- \sum_{j=s+1}^{r} (q_j - 1)(k-1)(\chi(\check{D}_j) - \chi(D_j \cap T)).$$

The transversality condition implies that $\chi(D_j \cap T) = d_j(d-k)$. We note that we can consider $j = 1, \dots, r$, because we are adding vanishing terms. Then, we get:

$$(k-1)(d-k)(d-p) - (k-1)\sum_{j=1}^{r}(q_j - 1)\chi(\check{D}_j).$$

(e) We get in this term

$$\sum_{P \in \mathrm{Sing}(D_{red})} (k-1)\mu_{top}(D,P) = (k-1) \sum_{P \in \mathrm{Sing}(D_{red})} \mu_{top}(D,P).$$

(f) By the transversality condition we get

$$\sum_{j=s+1}^{r} (q_j - 1)(d-k)d_j.$$

As before, we can take again $j = 1, \dots, r$, and we obtain $(d-k)(d-p)$.

Putting all these computations together, we have:

$$\mu_f^{\infty} = k(d-k)(d-p) + (k-1)\Big(\sum_{P \in \mathrm{Sing}(D_{red})} \mu_{top}(D,P) - \sum_{j=1}^{r}(q_j - 1)\chi(\check{D}_j)\Big).$$

We can use the stratification $\mathcal{S}_D$ of D for computing $\mu(D)$ in $\mathbb{P}^2$. Notice that the curve T does not play any role now. Moreover we can take the curve $W_d \cap \mathbb{P}^2$ and apply (1.3) to the (non-reduced) divisor D. So we obtain:

$$\mu_f^{\infty} = k(d-k)(d-p) + (k-1)\mu(D)$$

By (1.2) we get the result. □

Proof of Theorem (3.1). The proof is a consequence of the above three lemmas and (1.8). □

Proposition 3.5. *Let f be a transversal polynomial at infinity. Then, f has $\mathcal{W}$-isolated singularities at infinity.*

Proof. We will apply the criterium of Remark 2.5(c) in [ST], with a slight modification. We take the Whitney stratification $\mathcal{S}_D$ of D which appears in the proof of (3.3). We construct a stratification $\mathcal{W}'$ of the space

$$X := \{([x:y:z:w],t) \in \mathbb{P}^3 \times \mathbb{C} \mid \tilde{f}(x,y,z,w) = tw^d\},$$

where $\tilde{f}$ is the homogenization of degree d of f; we identify $\mathbb{C}^3$ with

$$\{([x:y:z:1], f(x,y,z)) \mid (x,y,z) \in \mathbb{C}^3\}.$$

The strata of $\mathcal{W}'$ are: $\mathbb{C}^3$ and $S_D \times \mathbb{C}$ for all $S_D \in \mathcal{S}_D$. In general, this stratification is not a Whitney one; let $\mathcal{W}$ the least fine Whitney stratification of X obtained by refining $\mathcal{W}'$. Then, in order to check if f has $\mathcal{W}$-isolated singularities at infinity it is enough to show that the restriction to X of the second projection has isolated singularities when restricted to any stratum of $\mathcal{W}$.

We notice that we may have only problems on strata $S \in \mathcal{W}$ which are contained in $S_D \times \mathbb{C}$ where $S_D \in \mathcal{S}_D$ is a 1-dimensional stratum. We fix a stratum S_D of this type; there exists an irreducible component D_i of D such that the stratum $S_D = D_i \setminus (T \cup \operatorname{Sing}(D_{red}))$.

In order to check if $\mathcal{W}$ is Whitney in a neighborhood N of S_D in X we must study if Whitney conditions are satisfied for $(\mathbb{C}^3, S_D \times \mathbb{C})$. It is the case if D_i is a reduced component of D as in this case X is smooth around $S_D \times \mathbb{C}$.

Let us suppose that D_i is not reduced and its multiplicity is m. Let us fix a point $P \in S_D$. In a neighborhood of P, we may choose local coordinates (u, v, w) in $\mathbb{P}^3$ at P, such that $w = 0$ is the local equation of the hyperplane at infinity, $f_d(x,y,z) = u^m$ and $f_{d-k}(x,y,z)$ does not vanish at the origin. Then we have that $N \cap (S_D \times \mathbb{C}) = \{(u,v,w,t) \mid w = u = 0\}$ and

$$N \cap \mathbb{C}^3 = \{(u,v,w,t) \mid u^m + w^k h(u,v,w) = tw^d, w \neq 0\},$$

where $h(0,0,0) \neq 0$. This last stratum is dense in N and we may identify N with a family of germs of plane curves based on $N \cap (S_D \times \mathbb{C})$; Whitney conditions are equivalent to the equisingularity of this family of plane curves, by standard arguments. In our case this family is equisingular. It means that $S_D \times \mathbb{C}$ is also a stratum in $\mathcal{W}$; the projection on t has no singularity at all. $\square$

§4.- Two-term polynomials

Definition 4.1. *A* two-term polynomial of type $(d, d-k)$ *is a polynomial of* $\mathbb{C}[x,y,z]$ *which is a linear combination of monomials of degree d and $d-k$.*

Notation. Given d, k we denote $\delta := \gcd(d,k)$ and $d_1 = \frac{d}{\delta}$, $k_1 = \frac{k}{\delta}$.

Definition 4.2. *Let $f = f_d + f_{d-k}$ be a two-term polynomial. The* projective pencil associated to f *is the pencil of curves in $\mathbb{P}^2$ generated by $f_d^{d_1-k_1}$ and $f_{d-k}^{d_1}$. We will denote C_t the element of the pencil determined by $f_{d-k}^{d_1} - t f_d^{d_1-k_1} = 0$.*

Lemma 4.3. *Let $f = f_d + f_{d-k}$ be a two-term polynomial. Then, the critical locus $C(f)$ is finite if only if the following conditions hold:*

(i) $\operatorname{Sing}(D) \cap \operatorname{Sing}(T) = \emptyset$ *and there is no common component of D and T.*

(ii) *The set of the points in $\mathbb{P}^2 \setminus (Sing(D) \cup Sing(T))$ such that the gradients of f_d and f_{d-k} are linearly dependent is finite.*

Proof. For the "only if" part we take into account the following fact: If $(x, y, z) \neq 0$ is a critical point of f, then $grad\ f_d$ and $grad\ f_{d-k}$ are linearly dependent at $[x : y : z]$. The condition $\mathrm{Sing}(D) \cap \mathrm{Sing}(T) = \emptyset$ implies that $grad\ f_d$ and $grad\ f_{d-k}$ do not vanish simultaneously. If $[x : y : z]$ is a a point such that $grad\ f_d$ and $grad\ f_{d-k}$ are linearly dependent and non-zero, one can adjust the proportionality factor in order to get a finite number of critical points of f in the line $[x : y : z]$. It is exactly the case when $[x : y : z] \in \mathbb{P}^2 \setminus (D \cup T)$ and it is a singular point of the projective pencil associated to f.

For the "if" part we argue as follows. If there exists a point P in the intersection $\mathrm{Sing}(D) \cap \mathrm{Sing}(T)$ then the complex line in $\mathbb{C}^3$ defined by P is contained in $\mathrm{C}(f)$. On the other hand suppose that the set $\bigcup_{t \in \mathbb{C}^*} \mathrm{Sing}(C_t)$ is not finite. For each of these points $P = (x_0 : y_0 : z_0)$ there exists $\lambda \in \mathbb{C}^*$ such that

$$grad\, f_d(P) + \lambda\, grad\, f_{d-k}(P) = 0.$$

Let β be a non zero complex number such that $\beta^{-k} = \lambda$ and let Q be the point in $\mathbb{C}^3$ whose coordinates are $(\beta^{-k} x_0, \beta^{-k} y_0, \beta^{-k} z_0)$. It is easy to see that Q is a critical value of f, so $\mathrm{C}(f)$ is not finite. $\square$

In order to compute the Milnor number of two-term polynomials with finite critical set we introduce some notation.

Let us consider two germs $g, h \in \mathbb{C}\{x, y\}$ which are in the maximal ideal; let us suppose that for any $t \in \mathbb{C}^*$, the germ $g + th$ is square-free. Let $(C_t, 0)$ be the germ of plane curve defined by $g + th$. Then there exists a finite set $S(g, h) \subset \mathbb{C}^*$ such that $\mu(C_t, 0)$ is constant if $t \in \mathbb{C}^* \setminus S(f, g)$; let us call $\mu(g, h)$ this constant. Then we will denote

$$\alpha^*(g, h) := \sum_{t \in S(g,h)} (\mu(C_t, 0) - \mu(g, h)).$$

Theorem 4.4. *Let $f = f_d + f_{d-k}$ be a two-term polynomial with finite critical set. Then*

$$\mu(f) = (d-1)^3 - k(\chi(D) + d(d-3) + \sum_{P \in \mathrm{Sing}(D) \cap T} (I_P(D, T) + \alpha_P^* - 1)),$$

where

- $\alpha_P^* := \alpha^*((f_{d-k}^{d_1})_P, (f_d^{d_1 - k_1})_P)$ *($(-)_P$ is the germ at P);*
- I_P *is the intersection number at P.*

Proof. Let $f = f_d + f_{d-k}$ be a two-term polynomial with isolated critical points.

First step. We compare the $\mu(f)$ with $\mu(g)$, where g is a suitable transversal polynomial at infinity

Let us rewrite the formula of (4.4) in terms of the generalized Milnor number

$$\mu(f) = (d-1)^3 - k(\mu(D) + \sum_{P \in \mathrm{Sing}(D) \cap T} (I_P(D, T) + \alpha_P^* - 1)).$$

Let $g := f_d + g_{d-k}$ be a polynomial which is transversal at infinity. Let K be the projective curve defined by $g_{d-k} = 0$. We choose g in order to verify:

(a') $\mathrm{Sing}(D_{red}) \cap \mathrm{Sing}(K) = \emptyset$.

(b') For $j = 1, \ldots, r$, D_j and K intersect at $(d-k)d_j$ distinct points.

This is a particular class of transversal-at-infinity polynomials. We deduce from (3.1) and some computations that

$$\mu(g) = (d-1)^3 - k\Big(\chi(D) + d(d-3) + \sum_{P \in \mathrm{Sing}(D) \cap K} \big(I_P(D,K) - 1\big)\Big).$$

We consider the invariant $\gamma(d,3)$ defined in (1.6). This invariant does not depend on the polynomial f or g and in our case the curve D is the same for both cases.

We construct stratifications S_f^∞ and S_g^∞ of $\mathbb{C}$ which match in the formula for $\gamma(d,3)$ and such that 0 is a stratum in both cases. It is clear that we can choose $S_g^\infty = \{\{0\}, \mathbb{C}^*\}$. We may suppose that

$$S_f^\infty = \{\{0\}, \{t_1\}, \ldots, \{t_m\}, U\},$$

where $U := \mathbb{C}^* \setminus \{t_1, \ldots, t_m\}$. We denote μ_{gen} the constant value of μ_f^∞ in U. We recall that $\chi(U) = -m$ and $\chi(\mathbb{C}^*) = 0$; then, we deduce:

$$\mu(f) + \mu_f^\infty(0) + \sum_{j=1}^{m} (\mu_f^\infty(t_j) - \mu_{gen}) = \mu(g) + \mu_g^\infty(0). \tag{4.5}$$

We are going to compare the fibres of f and g at 0.

Proposition 4.6.

$$\mu(g) - (\mu_f^\infty(0) - \mu_g^\infty(0)) = (d-1)^3 - k\Big(\chi(D) + d(d-3) + \sum_{P \in \mathrm{Sing}(D) \cap T} \big(I_P(D,T) - 1\big)\Big).$$

Proof. We distinguish two cases:

Case 1. *D is reduced.*

In this case, V_0 has only finitely many singular points at infinity which are concentrated in $\mathrm{Sing}(D)$. Let us denote $\mathrm{Sing}_1(D) := \mathrm{Sing}(D) \setminus T$ and $\mathrm{Sing}_2(D) := \mathrm{Sing}(D) \cap T$. Then,

$$\mu_f^\infty(0) = \sum_{P \in \mathrm{Sing}_1(D)} \mu(V_0, P) + \sum_{P \in \mathrm{Sing}_2(D)} \mu(V_0, P).$$

As in the proof of (2.1), if $P \in \mathrm{Sing}_1(D)$, we have: $\mu(V_0, P) = (k-1)\mu(D, P)$.

It has been computed in [M, Lema 3.3.7] that if $P \in \mathrm{Sing}_2(D)$ then

$$\mu(V_0, P) = (k-1)\mu(D, P) + k(I_P(D,T) - 1).$$

Warning. In the cited result of [M], it is important to remark that *his* D is *our* T, *his* T is *our* D and *his* d is *our* $d-k$.

Then:

$$\mu_f^\infty(0) = (k-1) \sum_{P\in \mathrm{Sing}(D)} \mu(D,P) + k \sum_{P\in \mathrm{Sing}_2(D)} (I_P(D,T)-1).$$

In this case, g is a Yomdine polynomial at infinity, and

$$\mu_g^\infty(0) = (k-1) \sum_{P\in \mathrm{Sing}(D)} \mu(D,P).$$

By the same reason,

$$\mu(g) = (d-1)^3 - k \sum_{P\in \mathrm{Sing}(D)} \mu(D,P) = (d-1)^3 - k\mu(D)$$

It is easily seen that $\mu(D) = \chi(D) + d(d-3)$ and this gives the formula.

Case 2. *D is not reduced.*

In this case T and K are reduced. We first compute the difference $\mu_f^\infty(0)-\mu_g^\infty(0)$. Let us denote $G_0 := g^{-1}(0)$. By (1.2), we have:

$$(\mu_f^\infty(0) - \mu_g^\infty(0)) + (\mu_f^a(0) - \mu_g^a(0)) = -(\chi(V_0) - \chi(\overline{G_0})) = -(\chi(F_0) - \chi(G_0));$$

last equality follows from the fact that D is the common intersection of the compactified fibres with $\mathbb{P}^2$.

Let us compute the singular points of F_0. The best way is to consider V_0 and to take the singular points of V_0 which are not at infinity. We get the points $[x:y:z:1]$ such that:

- $\mathrm{grad}(f_d)(x,y,z) + \mathrm{grad}(f_{d-k})(x,y,z) = 0$.
- $f_{d-k}(x,y,z) = f_d(x,y,z) = 0$.

We recall that the equation of V_0 is: $f_d(x,y,z) + w^k f_{d-k}(x,y,z) = 0$. Then, we have two kinds of singular points:

- $e := [0:0:0:1]$. We find in [M] that

$$\mu(F_0,e) = (d-k-1)^3 + k\Big(\chi(T) + (d-k)(d-k-3) + \sum_{P\in \mathrm{Sing}(T)\cap D} (I_P(D,T)-1)\Big).$$

- Any point $P \in D\cap T$ which is smooth in both curves and such that $I_P(D,T) > 1$ (i.e., gradients are linearly dependent) induces k different singular points $P_1,\dots,P_k$. We find again in [M] that $\mu(F_0,P_j) = I_P(D,T)-1$, $j=1,\dots,k$.

Putting together all these Milnor numbers we find $k(I_P(D,T)-1)$ for any point in $D\cap T$ which is smooth in D, and we can write down

$$\mu_f^a(0) = (d-k-1)^3 +$$
$$k\Big(\chi(T) + (d-k)(d-k-3) + \sum_{P\in D\cap T} (I_P(D,T)-1)\Big) - k \sum_{P\in \mathrm{Sing}(D)\cap T} (I_P(D,T)-1).$$

By Bezout theorem:

$$\mu_f^a(0) = (d-k-1)^3 +$$
$$k\Big(\chi(T) + (d-k)(d-k-3) + d(d-k) - \#(D\cap T)\Big) - k \sum_{P\in \mathrm{Sing}(D)\cap T} (I_P(D,T)-1),$$

where # denotes cardinality.

We get the same results for G_0 if we change T by K. Then

$$\mu_f^a(0) - \mu_g^a(0) = k\Big((\chi(T) - \chi(K)) - (\#(D\cap T) - \#(D\cap K))\Big) - \\ - k\Big(\sum_{P\in \mathrm{Sing}(D)\cap T} (I_P(D,T) - 1) - \sum_{P\in \mathrm{Sing}(D)\cap K} (I_P(D,K) - 1)\Big).$$

Let us compute now $\chi(F_0) - \chi(G_0)$. Let us denote

$$\widehat{D\cap T} := \{(x,y,z) \in \mathbb{C}^3 \mid [x:y:z] \in D\cap T\} \cup \{(0,0,0)\}$$

It is clear that $\widehat{D\cap T} \subset F_0$; let us denote $\check{F}_0 := F_0 \setminus \widehat{D\cap T}$. It is easily seen that $\chi(F_0) = \chi(\check{F}_0) + \chi(\widehat{D\cap T}) = 1 + \chi(\check{F}_0)$. As the origin is not in $\check{F}_0$, we can consider the projection $\check{F}_0 \to \mathbb{P}^2$. It is an unramified k-fold cyclic covering whose base is $\mathbb{P}^2 \setminus (D\cup T)$. Then:

$$\chi(F_0) = 1 + k\Big(\chi(\mathbb{P}^2) - \chi(D) - \chi(T) + \chi(D\cap T).\Big)$$

As before, we obtain $\chi(G_0)$ if we replace T by K. Then:

$$\chi(F_0) - \chi(G_0) = k\Big(- (\chi(T) - \chi(K)) + (\#(D\cap T) - \#(D\cap K))\Big).$$

We obtain:

$$\mu_f^\infty(0) - \mu_g^\infty(0) = k\Big(\sum_{P\in \mathrm{Sing}(D)\cap T} (I_P(D,T) - 1) - \sum_{P\in \mathrm{Sing}(D)\cap K} (I_P(D,K) - 1)\Big).$$

We have seen that

$$\mu(g) = (d-1)^3 - k\Big(\chi(D) + d(d-3) + \sum_{P\in \mathrm{Sing}(D)\cap K} (I_P(D,K) - 1)\Big).$$

These two formulæ give the result. □

Second step. Now, we compare the fibres of f at $t_1, \dots, t_m$ with generic fibres.

Proposition 4.7.

$$\sum_{j=1}^{m} (\mu_f^\infty(t_j) - \mu_{gen}) = k \sum_{P\in \mathrm{Sing}(D)\cap T} \alpha_P^*.$$

Proof. We begin with a result which allows to compute $\mu_f^\infty(t_j) - \mu_{gen}$ in terms of Euler characteristics.

Lemma 4.8. *Fix $j = 1, \dots, m$. Let $T(D)$ be a small enough regular neighborhood of $D \subset \mathbb{P}^3$. Let $s_j \in \mathbb{C}$ close enough to t_j. Then,*

$$\mu_f^\infty(t_j) - \mu_{gen} = \chi(V_{s_j} \cap T(D)) - \chi(V_{t_j} \cap T(D)).$$

Proof. We remark that $\mu_{gen} = \mu_f^\infty(s_j) = \mu(V_{s_j})$ and

$$\mu(V_{t_j}) = \mu_f^\infty(t_j) + \sum_{P \in \Sigma(f,t_j)} \mu_{top}(V_{t_j}, P),$$

where $\Sigma(f, t_j) := \Sigma(f) \cap f^{-1}(t_j)$. By (1.3), we get:

$$\mu(V_{t_j}) - \mu(V_{s_j}) = \chi(V_{s_j}) - \chi(V_{t_j}).$$

We choose B_P Milnor fibres for V_{t_j} at $P \in \Sigma(f, t_j)$. Let us consider:

$$V_{t_j}^\infty := V_{t_j} \cap T(D), \quad V_{t_j,P} := V_{t_j} \cap B_P, \quad \check{V}_{t_j} := \overline{V_{t_j} \setminus \Big(\bigcup_{P \in \Sigma(f,t_j)} V_{t_j,P} \cup V_{t_j}^\infty \Big)}.$$

We define $V_{s_j}^\infty, V_{s_j,P}, \check{V}_{s_j}$ in the same way. If s_j is close enough to t_j, we have that $V_{s_j,P}$ is a Milnor fibre of V_{t_j} at P for every $P \in \Sigma(f, t_j)$ and the spaces $\check{V}_{s_j}$ and $\check{V}_{t_j}$ are homeomorphic. We have the formulæ:

$$\chi(V_{s_j}) = \chi(V_{s_j}^\infty) + \sum_{P \in \Sigma(f,t_j)} \chi(V_{s_j,P}) + \chi(\check{V}_{s_j}),$$

$$\chi(V_{t_j}) = \chi(V_{t_j}^\infty) + \sum_{P \in \Sigma(f,t_j)} \chi(V_{t_j,P}) + \chi(\check{V}_{t_j}).$$

We recall that the intersections of these spaces have Euler characteristics equal to zero. We know also that:

$$\chi(\check{V}_{t_j}) = \chi(\check{V}_{s_j}), \quad \chi(V_{s_j,P}) = 1 + \mu(V_{t_j}, P).$$

We get the result putting together this equalities. □

Fix now $P \in \mathrm{Sing}(D) \cap T$ and suppose that its coordinates are $(x, y, w) = (0, 0, 0)$. For a given t_j, fix a Milnor polydisk $B_P^{t_j}$ for the germ (V_{t_j}, P). We denote:

$$W_P^{t_j} := V_P^{t_j} \cap B_P^{t_j}, \quad W_P^{s_j} := V_P^{s_j} \cap B_P^{t_j}.$$

Lemma 4.9. *With the same notations as above:*

$$\mu_f^\infty(t_j) - \mu_{gen} = \sum_{P \in \mathrm{Sing}(D) \cap T} \left(\chi(W_P^{s_j}) - 1 \right).$$

Proof. It is easily seen that we can break $T(D)$ in polydisks; outside $\mathrm{Sing}(D) \cap T$ the families look in the same way and they differ only in the neighborhood of $\mathrm{Sing}(D) \cap T$. We deduce that

$$\chi(V_{t_j}\cap T(D))-\chi(V_{s_j}\cap T(D))=\sum_{P\in\mathrm{Sing}(D)\cap T}\Big(\chi(W_P^{s_j})-\chi(W_P^{t_j})\Big).$$

The result follows from the contractibility of $W_P^{t_j}$ and (4.8). □

Next lemma is very easy and important:

Lemma 4.10. *The discriminant of the polynomial $X^d - aX^k - b$ is equal to*

$$c_0 b^{k-1}(a^{d_1}-c_1 b^{d_1-k_1})^{\delta},$$

where c_0, c_1 are non-zero constants which do not depend on a, b.

Let us fix $P \in \mathrm{Sing}(D)\cap T$ with coordinates $(x,y,w)=(0,0,0)$. Let Δ_P be the common base for the Milnor polydisks $B_P^{t_j}$, $j=1,\dots,m$; Δ_P is a Milnor polydisk at the hyperplane at infinity for the germs of f_d and f_{d-k} at P. Consider the projections

$$\sigma_P^{t_j}: W_P^{t_j}\to\Delta_P \qquad \sigma_P^{s_j}: W_P^{s_j}\to\Delta_P.$$

By (4.10), the map $\sigma_P^{t_j}$ (resp. $\sigma_P^{s_j}$) is a d-fold cyclic covering over Δ_P ramified along $(D\cup C_{c_1 t_j^{k_1}})\cap\Delta_P$ (resp. $(D\cup C_{c_1 s_j^{k_1}})\cap\Delta_P$), where C_t is the curve of the projective pencil associated to f corresponding to $t\in\mathbb{C}^*$.

It is easily seen that $\Delta_P\cap D\cap C_{c_1t^{k_1}}=\{(0,0)\}$, $t\in\{t_j,s_j\}$, and ramification behaves as follows:

- Over each point of $(\Delta_P\cap D)\setminus\{0\}$ we have exactly $d-k+1$ points for $\sigma_P^{t_j}$ (resp. $\sigma_P^{s_j}$).
- Over each point of $(C_{c_1t_j^{k_1}}\cap\Delta_P)\setminus\{0\}$ (resp. $(C_{c_1s_j^{k_1}}\cap\Delta_P)\setminus\{0\}$) we have exactly $d-\delta$ points for $\sigma_P^{t_j}$ (resp. $\sigma_P^{s_j}$).
- Over 0 we have exactly 1 point for $\sigma_P^{t_j}$ (resp. $\sigma_P^{s_j}$).

Then, we have for $t\in\{t_j,s_j\}$:

$$\begin{aligned}\chi(W_P^t)=d(1-\chi(D\cap\Delta_P)-\chi(C_{c_1t^{k_1}}\cap\Delta_P)+1)+\\ (d-k+1)(\chi(D\cap\Delta_P)-1)+(d-\delta)(\chi(C_{c_1t^{k_1}}\cap\Delta_P)-1)+1.\end{aligned}$$

The contractibility of $W_P^{t_j}$ and an easy computation give:

$$\chi(W_P^{s_j})-1=\delta\Big(\chi(C_{c_1t_j^{k_1}}\cap\Delta_P)-\chi(C_{c_1s_j^{k_1}}\cap\Delta_P)\Big).$$

By hypothesis if $t\neq 0$, the Milnor number of (C_t,P) is always finite; let us denote μ_P the Milnor number of the generic element of the pencil at P. We can apply a local version of (4.8) which is a classical result for pencils of plane curves and we get

$$\chi(C_{c_1t_j^{k_1}}\cap\Delta_P)-\chi(C_{c_1s_j^{k_1}}\cap\Delta_P)=\mu(C_{c_1t_j^{k_1}},P)-\mu_P.$$

Next result is also easy:

Lemma 4.11. *Let $s \in \mathbb{C}$ be such that $\mu(C_P^s) > \mu_P$. Then the k_1-roots of $\frac{s}{c_1}$ belong to $\{t_1, \ldots, t_m\}$.*

In the introduction, we have defined

$$\alpha_P^* = \sum_{s \in \mathbb{C}^*} (\mu(C_s, P) - \mu_P).$$

Then:

$$\sum_{j=1}^{m} \left(\mu(C_{c_1 t_j^{k_1}}, P) - \mu_P\right) = k_1 \alpha_P^*.$$

Putting all these computations together, we get (4.7). □

Finally, (4.7) implies Theorem (4.4) by (4.5). □

Remark. It is easily seen that two-term polynomials with isolated critical points have only $\mathcal{W}$-isolated singularities at infinity in the sense of [ST]; the proof is similar to the case of transversal-at-infinity polynomials. Nevertheless, they are not tame in general. The topology of the generic fibre can be deduce from this remark, (4.4) and the result of the next proposition.

Proposition 4.12. *Let f be a two-term polynomial with isolated critical points. Then:*

$$\lambda(f) = \sum_{P \in \mathrm{Sing}(D) \cap T} \left(k\alpha_P^* + \delta\big((I_P(D,T)(d_1 - k_1) - 1)(d_1 - 1) - \mu_P\big)\right),$$

where μ_P is the Milnor number at P of the generic element of the projective pencil associated to f.

Proof. In (4.7) we proved that:

$$\sum_{t \in \mathbb{C}^*} \lambda_f(t) = k \sum_{P \in \mathrm{Sing}(D) \cap T} \alpha_P^*.$$

Let us take the stratification $\mathcal{S}_D$ which appears in the proof of (3.3). We compare $\mu_f^\infty(t)$, for generic value of t, and $\mu_f^\infty(0)$ in the same way we made in (3.3) and we find:

$$\mu_f^\infty(0) - \mu_{gen}^\infty(f) = \sum_{P \in \mathrm{Sing}(D) \cap T} \left(\mu_{top}(V_0, P) - \mu_{top}(V_t, P)\right).$$

From now on, we fix $P \in \mathrm{Sing}(D) \cap T$. By a linear change of coordinates, we suppose that $P := [0:0:1:0]$ in the coordinates $[x:y:z:w]$ where $w = 0$ is the equation of the plane at infinity $\mathbb{P}^2$. As P is a smooth point of T we choose an analytic system of coordinates (u, v) centered at P such that

$$f_{d-k}(x,y) = u, \qquad f_d(x,y) = v^s + uh(u,v),\ h(0,0) = 0, s := I_P(D,T).$$

The equation of the germ (V_0, P) is:

$$v^s + uh(u,v) + uw^k = 0.$$

Following [M],

$$\mu_{top}(V_0, P) = (k-1)\mu_{top}(D, P) + k(s-1).$$

Let us denote $\Phi_{t,P}$ the Milnor fibre of (V_t, P). It is the intersection of a small polydisk centered at P with the surface whose equation is:

$$v^s + uh(u,v) + uw^k - tw^d = \eta, \qquad 0 < |\eta| \lll 1.$$

Let us consider the projection $(u, v, w) \mapsto (u, v)$. By (4.10) the ramification locus of this map restricted to $\Phi_{t,P}$ is the union of the Milnor fibre of (D, P) (where we have generically $d - k + 1$ preimages) and the intersection Θ_P of a small polydisk with the curve:

$$u^{d_1} + c_1(t)(v^s + uh(u,v) - \eta)^{d_1-k_1} = 0,$$

where $c_1(t)$ is a non-constant polynomial function of t. A generic point of Θ_P has $d - \delta$ preimages. The non-generic points are the s points of the intersection of Θ_P with the Milnor fibre of (D, P) (which has only one preimage). An easy computation of Euler characteristic gives:

$$1 + \mu_{top}(V_t, P) =$$
$$d(1-\chi(\Theta_P)-(1-\mu(D,P))+s)+(d-\delta)(\chi(\Theta_P)-s)+(d-k+1)(1-\mu(D,P)-s)+s.$$

Then:

$$\mu_{top}(V_t, P) = (k-1)\mu(D,P) - (d-k)(s-1) + \delta(s - \chi(\Theta_P))$$

and

$$\mu_{top}(V_0, P) - \mu_{top}(V_t, P) = \delta((d_1 - 1)(s-1) - 1 + \chi(\Theta_P)).$$

Now, we compute $\chi(\Theta_P)$. Let us apply the map $(u, v) \mapsto (u^{d_1-k_1}, v)$; the preimage of Θ_P by this map is the union of $d_1 - k_1$ curves, each one of them homeomorphic to Θ_P. Let us denote Θ_P^1 one of them, which is the intersection of a polydisk with a curve whose equation is:

$$u^{d_1} + b_1(t)(v^s + u^{d_1-k_1}h(u^{d_1-k_1}, v) - \eta) = 0,$$

where $b_1(t)$ is a non-constant polynomial function of t. We have:

$$\chi(\Theta_P) = \chi(\Theta_P^1).$$

Let P_1 be the preimage of P by this covering and let (D_1, P_1) be the germ of the function $u^{d_1} + b_1(t)(v^s + u^{d_1-k_1}h(u_1^{d_1-k_1}, v))$; then Θ_P^1 is its Milnor fibre and we have:

$$\mu_{top}(V_0, P) - \mu_{top}(V_t, P) = \delta((d_1 - 1)(s-1) - \mu_{top}(D_1, P_1)).$$

The final step of the proof is to compare the topological Milnor numbers of the germ at P_1 of D_1 and the generic member (C_t, P) of the projective pencil associated to f, which is generated in local coordinates by $f_{d-k}^{d_1} = u^{d_1}$ and $f_d^{d_1-k_1} = (v^s + uh(u,v))^{d_1-k_1}$.

Let us consider the germ (C_t, P) and its preimage by the (d_1-k_1)-fold cyclic covering ramified along T. The local coordinates of this map are $(u,v) \mapsto (u^{d_1-k_1}, v_1)$. Its preimage $(\hat{D}_t, P_1)$ is a germ given by the equation:

$$u^{d_1(d_1-k_1)} + t(v^s + u^{d_1-k_1}h(u^{d_1-k_1}, v))^{d_1-k_1} = 0.$$

it is the union of $d_1 - k_1$ germs pairwise isomorphic and such that the intersection number between two of them is equal to sd_1. Each germ is topologically equivalent to (D_1, P_1). By applying formulæ from [M], one computes the Milnor number of $(\hat{D}_t, P_1)$ in two ways:

- As a covering of (C_t, P).

- As a decomposition into germs isomorphic to (D_1, P_1) (with no common branches).

This relationship allows to compare the Milnor numbers:

$$\mu_{top}(D_1, P_1) = \mu_P + s(d_1 - k_1 - 1)(d_1 - 1).$$

Finally:

$$\mu_{top}(V_0, P) - \mu_{top}(V_t, P) = \delta((s(d_1 - k_1) - 1)(d_1 - 1) - \mu_P.$$

We have proved the proposition. □

Next examples show some pathologies for two-term polynomials $f_d + f_{d-k}$.

1. The following two polynomials show that the conditions (*i*) and (*ii*) in (4.3) are not related.

$$f(x,y,z) = x^2z + y^3 + xy, \qquad g(x,y,z) = (y^2z + x^3)z + x^2 + 2yz.$$

For f the condition (*ii*) holds and (*i*) does not. For g the condition (*i*) holds and (*ii*) does not.

2. The principal difference between the local and the global case are the invariants α_P^*. Let f be the polynomial

$$(y^2z + x^3 + x^2z)z + x^2 + 2yz + 2xz.$$

There exist two points $P_1 = (0:1:0)$ and $P_2 = (0:0:1)$ in $\mathrm{Sing}(D) \cap T$. For them $\alpha_{P_1}^* = 1$ and $\alpha_{P_2}^* = 2$. In general α_P^* can be calculated using SINGULAR, [SIN].

3. Last example shows that (4.4) only works for polynomials with two homogeneous parts. Let f be the polynomial $x^2z + y^3 + zx + y^2$ that verifies the conditions (*i*) and (*ii*) of (4.3). Moreover if g is the polynomial $f + \frac{1}{4}z$ then $\Sigma(g)$ is not finite.

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