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Zeta Functions for Germs of Meromorphic Functions, and Newton Diagrams*

S. M. Gusein-Zade, I. Luengo, and A. Melle-Hernández

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§1. Germs of Meromorphic Functions

A polynomial f of degree d in $n+1$ complex variables determines a meromorphic function f on the projective space $\mathbb{CP}^{n+1}$. If one wants to understand the behavior of f at infinity, it is natural to consider germs of the meromorphic function f at the points from the hyperplane at infinity $\mathbb{CP}^n_\infty \subset \mathbb{CP}^{n+1}$. In local analytic coordinates $z_0, z_1, \dots, z_n$ centered at a point $p \in \mathbb{CP}^n_\infty$ such that the hyperplane at infinity $\mathbb{CP}^n_\infty$ is given by the equation $\{z_0 = 0\}$, the germ of the function f at p has the form $f = P(z_0, \dots, z_n)/z_0^d$. Let us consider germs of meromorphic functions of general form.

Definition 1. A *germ of a meromorphic function on* $(\mathbb{C}^{n+1}, 0)$ is a fraction $f = P/Q$, where P and Q are germs of holomorphic functions $(\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$. Two germs of meromorphic functions $f = P/Q$ and $f' = P'/Q'$ are said to be *equal* if there exists a germ of a holomorphic function $U\colon (\mathbb{C}^{n+1}, 0) \to \mathbb{C}$ such that $U(0) \neq 0$, $P' = U \cdot P$, and $Q' = U \cdot Q$.

Remarks. 1. For convenience, we do not consider here functions of the type $1/Q(z)$ or $P(z)/1$.
2. According to the definition, we have $x/y \neq x^2/xy$, but $x/y = x\exp(x)/y\exp(x)$.

Recently, Arnold [2] classified simple germs of meromorphic functions for certain equivalence relations. In what follows, we will methodically use resolutions of germs of meromorphic functions.

Definition 2. A *resolution of the germ* f is a modification of the space $(\mathbb{C}^{n+1}, 0)$ (i.e., a proper analytic mapping $\pi\colon \mathscr{X} \to \mathscr{U}$ of a smooth analytic manifold $\mathscr{X}$ onto a neighborhood $\mathscr{U}$ of the origin in $\mathbb{C}^{n+1}$ that is an isomorphism outside a proper analytic subspace in $\mathscr{U}$) such that the total transform $\pi^{-1}(H)$ of the hypersurface $H = \{P=0\} \cup \{Q=0\}$ is a normal crossing divisor at each point of the manifold $\mathscr{X}$.

The fact that $\pi^{-1}(H)$ is a normal crossing divisor means that, in a neighborhood of any point of it, there exists a local system of coordinates $y_0, y_1, \dots, y_n$ such that the liftings $\widetilde{P} = P \circ \pi$ and $\widetilde{Q} = Q \circ \pi$ of the functions P and Q to the space $\mathscr{X}$ of the resolution are equal to $u y_0^{k_0} y_1^{k_1} \cdot \dots \cdot y_n^{k_n}$ and $v y_0^{l_0} y_1^{l_1} \cdot \dots \cdot y_n^{l_n}$, respectively, where $u(0) \neq 0$, $v(0) \neq 0$, and k_i and l_i are nonnegative.

Let B_ε be the closed ball of radius ε with center at the origin in $\mathbb{C}^{n+1}$, where ε is sufficiently small, so that representatives of the functions P and Q are defined in B_ε and, for any positive $\varepsilon' < \varepsilon$, the sphere $S_{\varepsilon'} = \partial B_{\varepsilon'}$ intersects the analytic spaces $\{P=0\}$, $\{Q=0\}$, and $\{P=Q=0\}$ transversally (from the standpoint of stratification). We choose a sufficiently small $\delta > 0$ and consider the ball $B_\delta \subset \mathbb{C}^2$ of radius δ centered at the origin.

Definition 3. By the 0-*Milnor fiber of the germ* f we mean the set

$$\mathscr{M}_f^0 = \{z \in B_\varepsilon : (P(z), Q(z)) \in B_\delta \subset \mathbb{C}^2,\ f(z) = P(z)/Q(z) = c\}$$

for nonzero $c \in \mathbb{C}$ with sufficiently small modulus $\|c\|$. In the same way, by the ∞-*Milnor fiber of the germ* f we mean the set

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Moscow State University, Department of Mathematics and Mechanics; e-mail: sabir@ium.ips.ras.ru; Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid; e-mails: iluengo@eucmos.sim.ucm.es, amelle@eucmos.sim.ucm.es.
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$$\mathcal{M}_f^\infty = \{z \in B_\varepsilon : (P(z), Q(z)) \in B_\delta \subset \mathbb{C}^2,\ f(z) = P(z)/Q(z) = c\}$$

for $c \in \mathbb{C}$ with sufficiently large modulus $\|c\|$.

Lemma 1. *The notion of* 0- (∞-)*Milnor fiber is well defined, i.e., for a sufficiently small* $\|c\|$, $0 < \|c\| \ll \delta \ll \varepsilon$ (*for sufficiently large* $\|c\|$, $\|c\|^{-1} \ll \delta \ll \varepsilon$), *the differentiable type of the manifold* $\mathcal{M}_f^0$ ($\mathcal{M}_f^\infty$) *does not depend on* ε, δ, *and* c.

Proof. Let $\pi\colon \mathcal{X} \to \mathcal{U}$ be a resolution of the germ f that is an isomorphism outside the hypersurface $H = \{P=0\} \cup \{Q=0\}$. Let $r\colon \mathbb{C}^{n+1} \to \mathbb{R}$ be the function $r(z) = \|z\|^2$ and let $\tilde{r} = r \circ \pi\colon \mathcal{X} \to \mathbb{R}$ be the lifting of the function r to the space $\mathcal{X}$ of the resolution. For a sufficiently small $\varepsilon > 0$, the hypersurface $\widetilde{S}_\varepsilon = \{\tilde{r} = \varepsilon^2\}$ (i.e., the preimage of the sphere $S_\varepsilon \subset \mathbb{C}^{n+1}$) is transversal to all strata of the total transform $\pi^{-1}(H)$ of the hypersurface H. At each point of $\pi^{-1}(H)$, in a local coordinate system, one has $P \circ \pi = u y_0^{k_0} \cdot \ldots \cdot y_n^{k_n}$ and $Q \circ \pi = v y_0^{l_0} \cdot \ldots \cdot y_n^{l_n}$ with $u(0) \neq 0$ and $v(0) \neq 0$. Thus, $f \circ \pi = w y_0^{m_0} \cdot \ldots \cdot y_n^{m_n}$ with $w(0) \neq 0$. The real hypersurface $\widetilde{S}_\varepsilon$ is transversal to all coordinate subspaces (of different dimensions). We can readily see that this implies the transversality of the hypersurface $\widetilde{S}_\varepsilon$ to the (complex) hypersurfaces $\{w y_0^{m_0} \cdot \ldots \cdot y_n^{m_n} = c\}$ for a sufficiently small $\|c\| \neq 0$ and for a sufficiently large $\|c\|$. Now the proof follows from the standard reasoning.

Remarks. 1. The definition means that $\mathcal{M}_f^0$ or $\mathcal{M}_f^\infty$ is equal to

$$\{z \in B_\varepsilon : (P(z), Q(z)) \in B_\delta \subset \mathbb{C}^2,\ P(z) = cQ(z),\ P(z) \neq 0\}$$

and thus the Milnor fibers of the functions P/Q and $RP/(RQ)$ with $R(0) = 0$ differ in general.

2. For $f = P/Q$, let $f^{-1} = Q/P$. We can readily see that $\mathcal{M}_{f^{-1}}^0 = \mathcal{M}_f^\infty$ and $\mathcal{M}_{f^{-1}}^\infty = \mathcal{M}_f^0$. Just the same properties hold for the monodromy transformations and for their zeta functions considered below.

3. It is possible (and sometimes more convenient) to define the Milnor fibers as follows:

$$\mathcal{M}_f^0 = \{z \in B_\varepsilon : \|Q(z)\| \le \delta,\ P(z) = cQ(z) \neq 0\}, \qquad 0 < \|c\| \ll \delta \ll \varepsilon,$$
$$\mathcal{M}_f^\infty = \{z \in B_\varepsilon : \|P(z)\| \le \delta,\ P(z) = cQ(z) \neq 0\}, \qquad \|c\|^{-1} \ll \delta \ll \varepsilon.$$

The meromorphic function f determines a mapping from $B_\varepsilon \setminus \{P=Q=0\}$ to the projective line $\mathbb{CP}^1$ ($z \mapsto (P(z) : Q(z))$). We denote this mapping by f again. By Lemma 1, this mapping is a locally trivial fibration over punctured neighborhoods of the points $0 = (0 : 1)$ and $\infty = (1 : 0)$ of the projective line $\mathbb{CP}^1$.

Definition 4. By the 0-*monodromy transformation* h_f^0 (∞-*monodromy transformation* h_f^∞) of the germ f we mean the monodromy transformation of the fibration f over the loop $c \cdot \exp(2\pi i t)$, $t \in [0, 1]$, with a sufficiently small (large) $\|c\| \neq 0$.

By the 0- or ∞-monodromy operator we mean the action of the corresponding monodromy transformation in a homology group of the Milnor fiber. We want to apply the results for meromorphic functions to calculate the zeta function of a polynomial at infinity. Therefore, we consider the zeta functions $\zeta_f^0(t)$ and $\zeta_f^\infty(t)$ of the corresponding monodromy transformations:

$$\zeta_f^\bullet = \prod_{q \ge 0} \{\det[\mathrm{id} - t h_{f*}^\bullet|_{H_q(\mathcal{M}_f^\bullet;\mathbb{C})}]\}^{(-1)^q}$$

($\bullet = 0$ or ∞). This definition coincides with that used in [3, 5] and differs on the sign in the exponent from that used in [1].

§2. Resolution of Singularities and the A'Campo Formula for Germs of Meromorphic Functions

Let $f = P/Q$ be a germ of a meromorphic function on $(\mathbb{C}^{n+1}, 0)$ and let $\pi\colon \mathcal{X} \to \mathcal{U}$ be a resolution of the germ f. The preimage $\mathcal{D} = \pi^{-1}(0)$ of the origin of $\mathbb{C}^{n+1}$ is a normal crossing divisor. Let $S_{k,l}$ be the set of points of the divisor $\mathscr{D}$ in whose neighborhoods, in some local coordinates, the liftings $P\circ\pi$ and $Q\circ\pi$ of the functions P and Q have the form uy_0^k and vy_0^l, respectively ($u(0)\neq 0$, $v(0)\neq 0$). A slight modification of the arguments of A'Campo [1] permits one to obtain the following version of his formula for the zeta function of the monodromy of a meromorphic function.

Theorem 1. *Let a resolution $\pi\colon \mathscr{X}\to\mathscr{U}$ be an isomorphism outside the hypersurface $H=\{P=0\}\cup\{Q=0\}$. In this case,*

$$\zeta_f^0(t)=\prod_{k>l}(1-t^{k-l})^{\chi(S_{k,l})},\qquad \zeta_f^\infty(t)=\prod_{k<l}(1-t^{l-k})^{\chi(S_{k,l})}.$$

Remark. A resolution π of the germ $f'=RP/(RQ)$ is also a resolution of the germ $f=P/Q$. Moreover, the multiplicities of any component C of the exceptional divisor in the zero divisors of the liftings $(RP)\circ\pi$ and $(RQ)\circ\pi$ of the germs RP and RQ are obtained from those for the germs P and Q by adding the same integer, namely, the multiplicity $m=m(C)$ of the component C in the zero divisor of the lifting of the germ R. Nevertheless, the meromorphic functions f and f' can have different zeta functions. The formulas from the previous theorem can give different results for f and f' because if an open part of the component C belongs to $S_{k,l}(f)$, then, generally speaking, the part of this component that belongs to $S_{k+m,l+m}(f')$ can be strictly smaller.

§3. Zeta Functions of Meromorphic Functions in Terms of Partial Resolutions

Let $f=P/Q$ be a germ of a meromorphic function on $(\mathbb{C}^{n+1},0)$ and let $\pi\colon(\mathscr{X},\mathscr{D})\to(\mathbb{C}^{n+1},0)$ be an arbitrary modification of the space $(\mathbb{C}^{n+1},0)$ that is an isomorphism outside the hypersurface $H=\{P=0\}\cup\{Q=0\}$ (i.e., π need not be a resolution). Let $\varphi=f\circ\pi$ be the lifting of the germ f to the space $\mathscr{X}$ of the modification, i.e., the meromorphic function $P\circ\pi/(Q\circ\pi)$. For a point $x\in\pi^{-1}(H)$, let $\zeta^0_{\varphi,x}(t)$ and $\zeta^\infty_{\varphi,x}(t)$ be the zeta functions of the 0- and ∞-monodromies of the germ of the function φ at the point x. Let $\mathscr{S}=\{\Xi\}$ be a prestratification of the space $\mathscr{D}=\pi^{-1}(0)$ (that is, a partitioning into semi-analytic subspaces without any regularity conditions) such that, for each stratum Ξ of $\mathscr{S}$, the zeta functions $\zeta^0_{\varphi,x}(t)$ and $\zeta^\infty_{\varphi,x}(t)$ do not depend on x for $x\in\Xi$. We denote these zeta functions by $\zeta^0_\Xi(t)$ and $\zeta^\infty_\Xi(t)$, respectively. The arguments used in [5] yield the following assertion.

Theorem 2. *For $\bullet=0$ or ∞,*

$$\zeta_f^\bullet(t)=\prod_{\Xi\in\mathscr{S}}[\zeta^\bullet_\Xi(t)]^{\chi(\Xi)}.$$

§4. Zeta Functions in Terms of Newton Diagrams

By the Newton diagram $\Gamma=\Gamma(R)$ of a germ $R(x)=\sum a_kx^k$ of a holomorphic function $(\mathbb{C}^{n+1},0)\to(\mathbb{C},0)$ ($k=(k_0,k_1,\dots,k_n)$, $x^k=x_0^{k_0}x_1^{k_1}\cdot\ldots\cdot x_n^{k_n}$) we mean the union of the compact faces of the polytope $\Gamma_+=\Gamma_+(R)=$ *convex hull of the set* $\bigcup_{k\,:\,a_k\neq 0}(k+\mathbb{R}^{n+1}_+)\subset\mathbb{R}^{n+1}_+$.

Let $f=P/Q$ be a germ of a meromorphic function on $(\mathbb{C}^{n+1},0)$ and let $\Gamma_1=\Gamma(P)$ and $\Gamma_2=\Gamma(Q)$ be the Newton diagrams of the germs P and Q. The pair $\Lambda=(\Gamma_1,\Gamma_2)$ of Newton diagrams Γ_1 and Γ_2 is called the *Newton pair of the germ f*. The germ of the meromorphic function f is said to be *nondegenerate with respect to its Newton pair $\Lambda=(\Gamma_1,\Gamma_2)$* if the pair of germs (P,Q) is nondegenerate with respect to the pair $\Lambda=(\Gamma_1,\Gamma_2)$ in the sense of the definition in [8] (which is an adaptation of the definition by Khovanskii [6] for *germs* of complete intersections).

Let us define the zeta functions $\zeta^0_\Lambda(t)$ and $\zeta^\infty_\Lambda(t)$ for a Newton pair $\Lambda=(\Gamma_1,\Gamma_2)$. Let $1\le l\le n+1$ and let $\mathscr{I}$ be a subset of the set $\{0,1,\dots,n\}$ with the number of elements $\#\mathscr{I}$ equal to l. Let $L_\mathscr{I}$ be the coordinate subspace, $L_\mathscr{I}=\{k\in\mathbb{R}^{n+1}:k_i=0$ for $i\notin\mathscr{I}\}$, and let $\Gamma_{i,\mathscr{I}}=\Gamma_i\cap L_\mathscr{I}\subset L_\mathscr{I}$. Let $L^*_\mathscr{I}$ be the space dual to $L_\mathscr{I}$ and let $L^*_{\mathscr{I}+}$ be the positive octant of $L^*_\mathscr{I}$ (the set of covectors that take positive values on $L_{\mathscr{I}\ge 0}=\{k\in L_\mathscr{I}:k_i\ge 0$ for $i\in\mathscr{I}\}$). For a primitive integral covector $a\in(\mathbb{R}^{n+1})^*_+$ (i.e., for an indivisible element of the dual integral lattice), set $m(a,\Gamma)=\min_{x\in\Gamma}(a,x)$ and $\Delta(a,\Gamma)=\{x\in\Gamma:(a,x)=m(a,\Gamma)\}$. We denote by $m_\mathscr{I}$ and $\Delta_\mathscr{I}$ the corresponding objects for the diagram $\Gamma_{\mathscr{I}}$ and a primitive integer covector $a \in L^*_{\mathscr{I}+}$. Let $E_{\mathscr{I}}$ be the set of primitive integral covectors $a \in L^*_{\mathscr{I}+}$ such that $\dim(\Delta(a,\Gamma_1)+\Delta(a,\Gamma_2)) = l-1$ (the Minkowski sum $\Delta_1+\Delta_2$ of two polytopes Δ_1 and Δ_2 is the polytope $\{x = x_1+x_2 : x_1 \in \Delta_1,\ x_2 \in \Delta_2\}$). There exist only finitely many such covectors. For $a \in E_{\mathscr{I}}$, we set $\Delta_1 = \Delta(a,\Gamma_1)$, $\Delta_2 = \Delta(a,\Gamma_2)$, and

$$V_a = \sum_{s=0}^{l-1} V_{l-1}(\underbrace{\Delta_1,\dots,\Delta_1}_{s},\underbrace{\Delta_2,\dots,\Delta_2}_{l-1-s}).$$

The definition of the (Minkowski) mixed volume $V(\Delta_1,\dots,\Delta_m)$ can be found, e.g., in [4] or [8]; the $(l-1)$-dimensional volume in a rational $(l-1)$-dimensional affine subspace of $L_{\mathscr{I}}$ has to be normalized so that the volume of the unit cube spanned by any integral basis of the corresponding linear subspace is equal to one. Recall that $V_m(\underbrace{\Delta,\dots,\Delta}_{m})$ is the ordinary m-dimensional volume of the polytope Δ. We must assume that $V_0(\text{nothing}) = 1$ (this is necessary to define V_a for $l=1$). We write

$$\zeta^0_{\mathscr{I}}(t) = \prod_{a\in E_{\mathscr{I}}\,:\,m(a,\Gamma_1)>m(a,\Gamma_2)} (1-t^{m(a,\Gamma_1)-m(a,\Gamma_2)})^{(l-1)!V_a},$$

$$\zeta^\infty_{\mathscr{I}}(t) = \prod_{a\in E_{\mathscr{I}}\,:\,m(a,\Gamma_1)<m(a,\Gamma_2)} (1-t^{m(a,\Gamma_2)-m(a,\Gamma_1)})^{(l-1)!V_a},$$

$$\zeta^\bullet_l(t) = \prod_{\mathscr{I}\,:\,\#(\mathscr{I})=l} \zeta^\bullet_{\mathscr{I}}(t), \qquad \zeta^\bullet_\Lambda(t) = \prod_{l=1}^{n+1} (\zeta^\bullet_l(t))^{(-1)^{l-1}},$$

where $\bullet = 0$ or ∞.

Theorem 3. *Let $f = P/Q$ be a germ of a meromorphic function on $(\mathbb{C}^{n+1},0)$ that is nondegenerate with respect to its Newton pair $\Lambda = (\Gamma_1,\Gamma_2)$. In this case,*

$$\zeta^0_f(t) = \zeta^0_\Lambda(t), \qquad \zeta^\infty_f(t) = \zeta^\infty_\Lambda(t).$$

Proof. Let Σ be a unimodular simplicial subdivision of the octant $\mathbb{R}^{n+1}_{\geq 0}$ that corresponds to the pair (Γ_1,Γ_2) of Newton diagrams in the sense of [8, Sec. 4]. This subdivision is consistent with each of the Newton diagrams Γ_1 and Γ_2 in the sense of [9].

Let $\pi\colon (\mathscr{X},\mathscr{D}) \to (\mathbb{C}^{n+1},0)$ be the toroidal modification corresponding to Σ (see, e.g., [3]). Since the pair (P,Q) is nondegenerate with respect to the pair (Γ_1,Γ_2), it follows that π is a resolution of the germ $f = P/Q$ [8]. We have $S_{k,l} = S_k(P)\cap S_l(Q)$. The description of the sets $S_k(P)$ and $S_l(Q)$ can be found in [9, Sec. 7]. Each of these sets consists of open parts of certain complex tori of various dimensions.

The tori of dimension n correspond to one-dimensional cones of Σ that are positive (i.e., belong to $(\mathbb{R}^{n+1})^*_+$). The multiplicity of the function $P\circ\pi$ (respectively, $Q\circ\pi$) along such a torus is equal to $m(a,\Gamma_1)$ (respectively, $m(a,\Gamma_2)$) for the primitive integer covector a that spans the corresponding cone.

The tori of dimension $l-1$ correspond to positive simplicial $(n+2-l)$-dimensional cones of Σ that have a face which is a cone of the form

$$\mathfrak{S} = \{a \in (\mathbb{R}^{n+1})^*_{\geq 0} : a_j > 0 \text{ for } j \notin \mathscr{I},\ a_j = 0 \text{ for } j \in \mathscr{I}\}$$

with $\#(\mathscr{I}) = l$ (these faces are elements of the subdivision Σ themselves). In turn, these cones correspond to one-dimensional cones of a partition of the octant $L_{\mathscr{I}\geq 0}$ that is consistent with the Newton diagram $\Gamma_{i,\mathscr{I}} = \Gamma_i \cap L_{\mathscr{I}} \subset L_{\mathscr{I}}$. The multiplicities of the functions $P\circ\pi$ and $Q\circ\pi$ along such a torus are equal to $m_{\mathscr{I}}(a,\Gamma_{1,\mathscr{I}})$ and $m_{\mathscr{I}}(a,\Gamma_{2,\mathscr{I}})$, respectively, where a is the primitive integer covector spanning the corresponding one-dimensional cone.

In order to apply Theorem 1, we have to calculate the Euler characteristic of the corresponding part of an $(l-1)$-dimensional torus T, namely, of the complement to the intersection with the strict transform of the hypersurface $H = \{P=0\}\cup\{Q=0\}$. Let A (B, respectively) be the intersection of the torus T

with the strict transform of the hypersurface $\{P=0\}$ ($\{Q=0\}$, respectively). Let $\Delta_i := \Delta(a, \Gamma_{i,\mathscr{I}})$. It follows from the results of Khovanskii [7] that the Euler characteristic of A (of B) is equal to

$$(-1)^l(l-1)!\,V_{l-1}(\underbrace{\Delta_1,\dots,\Delta_1}_{l-1}) \ \text{(to } (-1)^l(l-1)!\,V_{l-1}(\underbrace{\Delta_2,\dots,\Delta_2}_{l-1})\text{)},$$

and the Euler characteristic of the hypersurface $A\cap B$ is equal to

$$(-1)^{l-1}(l-1)!\left[V_{l-1}(\underbrace{\Delta_1,\dots,\Delta_1}_{l-2},\Delta_2)+V_{l-1}(\underbrace{\Delta_1,\dots,\Delta_1}_{l-3},\Delta_2,\Delta_2)+\dots+V_{l-1}(\Delta_1,\underbrace{\Delta_2,\dots,\Delta_2}_{l-2})\right].$$

Thus, the Euler characteristic of the complement of $A\cup B$ in the torus T is equal to

$$\begin{aligned}&\chi(T)-\chi(A)-\chi(B)+\chi(A\cap B)\\ &\quad=(-1)^{l-1}(l-1)!\left[V_{l-1}(\underbrace{\Delta_1,\dots,\Delta_1}_{l-1})+V_{l-1}(\underbrace{\Delta_1,\dots,\Delta_1}_{l-2},\Delta_2)+\dots+V_{l-1}(\underbrace{\Delta_2,\dots,\Delta_2}_{l-1})\right],\end{aligned}$$

which proves the assertion. □

§5. A Varchenko-Type Formula for $f = P/z_0^d$

As we have mentioned at the beginning of the paper, in the study of the behavior of polynomials at infinity, the germs of meromorphic functions of the form $P(z_0, z_1, \dots, z_n)/z_0^d$ are of interest. In this case, the formulas for the zeta functions $\zeta^0_\Lambda(t)$ and $\zeta^\infty_\Lambda(t)$ are considerably reduced. Let us reformulate the definition of these zeta functions for the case in which the Newton diagram Γ_2 consists of a single point $(d, 0, \dots, 0)$ (in terms of the Newton diagram $\Gamma := \Gamma_1$ of the germ P). The description is as follows.

Let $1 \le l \le n+1$ and let $\mathscr{I}$ be a subset of the set $\{1,\dots,n\}$ with the number of elements $\#\mathscr{I}$ equal to $l-1$. Let $\gamma_1^{\mathscr{I}},\dots,\gamma_{j(\mathscr{I})}^{\mathscr{I}}$ be all $(l-1)$-dimensional faces of the diagram $\Gamma_{\mathscr{I}\cup\{0\}}$, let $a_{\mathscr{I},1},\dots,a_{\mathscr{I},j(\mathscr{I})}$ be the corresponding primitive covectors (normal to $\gamma_1^{\mathscr{I}},\dots,\gamma_{j(\mathscr{I})}^{\mathscr{I}}$), let $a^0_{\mathscr{I},s}$ be the zeroth coordinate of the covector $a_{\mathscr{I},s}$, and let $m_s(\mathscr{I}) = (a_{\mathscr{I},s}, k)$ for $k\in\gamma_s^{\mathscr{I}}$. In this case,

$$\zeta^0_{\mathscr{I}\cup\{0\}}(t) = \prod_{1\le s\le j(\mathscr{I})\,:\,m_s(\mathscr{I})>d\cdot a^0_{\mathscr{I},s}} \left(1-t^{m_s(\mathscr{I})-d\cdot a^0_{\mathscr{I},s}}\right)^{(l-1)!V_{l-1}(\gamma_s^{\mathscr{I}})},$$

$$\zeta^\infty_{\mathscr{I}\cup\{0\}}(t) = \prod_{1\le s\le j(\mathscr{I})\,:\,m_s(\mathscr{I})<d\cdot a^0_{\mathscr{I},s}} \left(1-t^{d\cdot a^0_{\mathscr{I},s}-m_s(\mathscr{I})}\right)^{(l-1)!V_{l-1}(\gamma_s^{\mathscr{I}})},$$

$$\zeta^\bullet_l(t) = \prod_{\mathscr{I}\subset\{1,\dots,n\}\,:\,\#\mathscr{I}=l-1} \zeta^\bullet_{\mathscr{I}\cup\{0\}}(t), \qquad \zeta^\bullet_\Lambda(t) = \prod_{l=1}^{n+1}\left(\zeta^\bullet_l(t)\right)^{(-1)^{l-1}}$$

($\bullet = 0$ or ∞), where $V_{l-1}(\gamma_s^{\mathscr{I}})$ is the (ordinary) $(l-1)$-dimensional volume of the face $\gamma_s^{\mathscr{I}}$ (in the hyperplane spanned by this face in $L_{\mathscr{I}\cup\{0\}}$).

§6. Examples

Example 1. Let $f = (x^3 - xy)/y$. The Milnor fiber $\mathscr{M}_f^0$ (respectively, $\mathscr{M}_f^\infty$) is equal to $\{(x,y) : \|(x,y)\| < \varepsilon,\ (x^3-xy, y)\in B_\delta,\ x^3 - xy = cy\}\setminus\{(0,0)\}$, where $\|c\|\ne 0$ is sufficiently small (large). The equation $x^3 - xy = cy$ yields $y = x^3/(x+c)$, and thus the Milnor fiber $\mathscr{M}_f^0$ is diffeomorphic to the disk $\mathscr{D}$ in the x-plane with two deleted points, namely, $-c$ and the origin. In the same way, the Milnor fiber $\mathscr{M}_f^\infty$ is diffeomorphic to the punctured disk $\mathscr{D}^*$. We can readily see that the action of the monodromy transformation in the homology groups is trivial in both cases. Thus,

$$\zeta_f^0(t) = (1-t)^{-1}, \qquad \zeta_f^\infty(t) = 1.$$

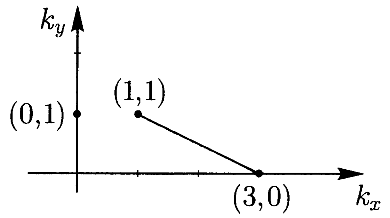

Fig. 1

Now let us calculate these zeta functions via the Newton diagrams (Fig. 1).

We have $\zeta_1^\bullet(t) = 1$ because each coordinate axis intersects only one Newton diagram. There is only one linear function (namely, $a = k_x + 2k_y$) such that $\dim \Delta(a, \Gamma_1) = 1$. The one-dimensional volume $V_1(\Delta(a,\Gamma_1))$ of the face $\Delta(a,\Gamma_1)$ is equal to 1, and $V_1(\Delta(a,\Gamma_2)) = 0$. We have $m(a,\Gamma_1) = 3$ and $m(a,\Gamma_2) = 2$. Thus, $\zeta_2^0(t) = 1 - t$, $\zeta_2^\infty(t) = 1$, $\zeta^0_{(\Gamma_1,\Gamma_2)}(t) = (1-t)^{-1}$, and $\zeta^\infty_{(\Gamma_1,\Gamma_2)}(t) = 1$, which coincides with the above formulas.

Example 2. Let $P = xyz + x^p + y^q + z^r$ be a $T_{p,q,r}$ singularity, $1/p + 1/q + 1/r < 1$, and let $Q = x^d + y^d + z^d$ be a homogeneous polynomial of degree d. Suppose that $p > q > r > d > 3$ and that p, q, and r are pairwise coprime. Let us compute the zeta functions of the germ $f = P/Q$ by using Theorems 2 and 3.

(a) It is clear that the germ f is nondegenerate with respect to its Newton pair $\Lambda = (\Gamma_1, \Gamma_2)$. Thus,

$$\zeta_f^\bullet(t) = \zeta_\Lambda^\bullet(t) = \zeta_1^\bullet(\zeta_2^\bullet)^{-1}\zeta_3^\bullet \qquad (\bullet = 0 \text{ or } \infty).$$

One has $\zeta_1^\infty = \zeta_2^\infty = 1$, and a unique covector which is necessary to compute ζ_3^∞ is $a = (1,1,1)$. In this case, $m(a,\Gamma_1) = 3$, $m(a,\Gamma_2) = d$, $\Delta(a,\Gamma_1) = \{(1,1,1)\}$, and $\Delta(a,\Gamma_2)$ is the simplex $\{k_x + k_y + k_z = d$, $k_x \geq 0$, $k_y \geq 0$, $k_z \geq 0\}$ whose two-dimensional volume is equal to $d^2/2$. Thus, $\zeta_f^\infty = (1 - t^{d-3})^{d^2}$.

We have

$$\zeta_1^0 = (1 - t^{p-d})(1 - t^{q-d})(1 - t^{r-d}),$$
$$\zeta_2^0 = (1 - t^{r(q-d)})(1 - t^{r(p-d)})(1 - t^{q(p-d)})(1 - t^{r-d})^{2d}(1 - t^{q-d})^d.$$

To compute ζ_3^0, one has to take into account both the three covectors $(rq-q-r, r, q)$, $(r, pr-p-r, p)$, and $(q, p, qp-p-q)$ that correspond to the two-dimensional faces of the diagram Γ_1 and the three covectors $(1, r-2, 1)$, $(r-2, 1, 1)$, and $(q-2, 1, 1)$ that correspond to the pairs of the form (one-dimensional face of the diagram Γ_1, one-dimensional face of the diagram Γ_2). For instance, for $a = (1, r-2, 1)$, the face $\Delta(a,\Gamma_1)$ (the face $\Delta(a,\Gamma_2)$) is the segment between the points $(0,0,r)$ and $(1,1,1)$ (between the points $(d,0,0)$ and $(0,0,d)$, respectively). Note the "absence of symmetry": the last three covectors are not obtained from each other by permuting the coordinates and the numbers p, q, and r. Thus,

$$\zeta_3^0 = (1 - t^{r(q-d)})(1 - t^{r(p-d)})(1 - t^{q(p-d)})(1 - t^{r-d})^{2d}(1 - t^{q-d})^d,$$
$$\zeta_f^0 = (1 - t^{p-d})(1 - t^{q-d})(1 - t^{r-d}).$$

(b) To compute the zeta functions of the germ f with the help of Theorem 2, for a modification $\pi\colon (\mathscr{X}, \mathscr{D}) \to (\mathbb{C}^3, 0)$ we take blowing-up of the origin in $\mathbb{C}^3$. Let φ be the lifting $f \circ \pi$ of the germ f to the space $\mathscr{X}$. The exceptional divisor $\mathscr{D}$ of the modification is the complex projective plane $\mathbb{CP}^2$. Let H_1 and H_2 be the strict transforms of the hypersurfaces $\{P=0\}$ and $\{Q=0\}$ and let $D_i = \mathscr{D} \cap H_i$. The curve D_1 consists of three transversal lines l_1, l_2, and l_3 and has three singular points $S_1 = l_2 \cap l_3 = (0,0,1)$, $S_2 = l_1 \cap l_3 = (0,1,0)$, and $S_3 = l_1 \cap l_2 = (1,0,0)$. The curve D_2 is smooth and of degree d. It intersects D_1 transversally at $3d$ different nonsingular points $\{P_1, \ldots, P_{3d}\}$.

One has the following natural stratification of the exceptional divisor $\mathscr{D}$:

(i) the zero-dimensional strata Λ_i^0 $(i = 1,2,3)$ each of which consists of a single point S_i;

(ii) the zero-dimensional strata Ξ_i^0 each of which consists of a single point P_i $(i = 1, \ldots, 3d)$;

(iii) the one-dimensional strata $\Xi_i^1 = l_i \setminus \{D_2 \cup l_j \cup l_k\}$ $(i = 1, 2, 3)$ and $\Xi_4^1 = D_2 \setminus D_1$;
(iv) the two-dimensional strata $\Xi^2 = \mathscr{D} \setminus (D_1 \cup D_2)$.

We can readily see that $\zeta^0_{\Xi_2}(t) = 1$ and $\zeta^\infty_{\Xi_2}(t) = 1 - t^{d-3}$ and that, for each stratum Ξ from Ξ_i^0 $(1 \le i \le 3d)$ or from Ξ_i^1 $(1 \le i \le 4)$, one has $\zeta^\bullet_\Xi(t) = 1$ $(\bullet = 0 \text{ or } \infty)$.

In what follows, we assume that the exceptional divisor $\mathscr{D}$ is locally given by the equation $u = 0$. At the point S_1, the germ of the lifting φ of the function f is of the form $(u^3 x_1 y_1 + u^r + x_1^p u^p + y_1^q u^q)/(u^d x_1^d + u^d y_1^d + u^d)$. This germ has the same Newton pair as the germ $(u^3 x_1 y_1 + u^r)/u^d$. Using Theorem 3, one has $\zeta^\infty_{\Lambda_1^0} = 1$ and $\zeta^\infty_{\Lambda_1^0} = 1 - t^{r-d}$. At the point S_2, the germ of the function φ has the form $(u^3 x_1 z_1 + z_1^r u^r + x_1^p u^p + u^q)/(u^d x_1^d + u^d + z_1^d u^d)$. It has the same Newton pair as $(u^3 x_1 z_1 + z_1^r u^r + u^q)/u^d$. By Theorem 3, $\zeta^\infty_{\Lambda_2^0}(t) = 1$ and $\zeta^0_{\Lambda_2^0}(t) = 1 - t^{q-d}$. We can similarly see that $\zeta^\infty_{\Lambda_3^0}(t) = 1$ and $\zeta^0_{\Lambda_3^0}(t) = 1 - t^{p-d}$. Combining these computations, we obtain the above result (without using a partial resolution).

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