ν-QUASI-ORDINARY POWER SERIES: FACTORISATION, NEWTON TREES AND RESULTANTS

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Dedicated to Anatoly Libgober on the Occasion of His Sixteenth Birthday

Abstract. A generalization of quasi-ordinary power series is studied. This class, called ν-quasi-ordinary, was introduced by H. Hironaka and iy is defined by a very mild condition on its (projected) Newton polygon. I. Luengo used ν-quasiordinary power series to give a proof of the Jung-Abhyankar theorem for quasiordinary power series. In this paper, a factorization theorem for ν-quasi-ordinary power series is given. Using the factorizacion theorem we codify ν-quasiordinary power series by its Newton tree, and we use it to compute the generalized intersection multiplicity of two ν-quasiordinary power series, resultant and discriminant.

1. Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. Let $h \in \mathbb{K}[[\mathbf{x}]][z]$, $\mathbf{x} := (x_1, \ldots, x_d)$ be an analytic germ vanishing at zero. If $d = 1$, we are considering singularities of germs of plane curves. Their factorisation is known via the well-known Puiseux theorem. There are several ways to codify this process (and other invariants), for instance via Enriques diagrams, embedded resolution tree or Eisenbud-Neumann diagrams. If one starts with the equation h of the germ of the plane curve, the initial tool to start with is the Newton polygon of h, and proceed by doing the Newton algorithm. This procedure can be codified in a tree called Newton tree. On the Newton trees we can compute other invariants of the singularity, in particular, the so called *topological Zeta-function* and prove the monodromy conjecture, see e.g [2]. Using bi-coloured Newton trees the intersection multiplicity of two germs of plane curves can be computed.

For $d > 1$ there is a class of functions which generalizes the case of curves, it is the class of quasi-ordinary singularities. In that class, one has a factorisation theorem given by Jung-Abhyankar, see [1]. Moreover if we take two quasi-ordinary functions without common factor, so that their product is a quasi-ordinary function then one can compute a d-uple which plays the same role as the intersection multiplicity in the case of curves. One feature of quasi-ordinary singularities is that their Newton polyhedron is a polygonal path, see e.g. [7], [8], that means that all compact faces have dimension one. This is why we could apply the Newton process as in the case of curves. Moreover, after Newton maps, the condition to be quasi-ordinary is preserved. In [2], using the Newton process it was possible to compute a set of candidate poles of the topological Zeta-function and prove the monodromy conjecture.

In this paper, we will study a generalization of quasi-ordinary singularities, called ν*-quasi-ordinary* by H. Hironaka [10]. A proof of the Jung-Abhyankar theorem

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using ν-quasiordinary power series was given in [11]. Roughly speaking, for a ν-*quasi-ordinary* singularity we only ask for the upper part of the Newton polyhedron to be a polygonal path, and we will extract all properties we can from this fact. First of all, we will show that this gives us a factorisation of our function. We show that we can factorize our function in functions whose Newton polyhedron has only one compact face, of dimension one, and eventually a non ν-quasi-ordinary function. We will apply the Newton process associated to theses faces of dimension 1 and we can iterate this process whenever the upper part of the corresponding Newton polyhedron is a polygonal path. As in the case of curves we will encode the Newton process in a tree, but the tree will bear leaves (arrows) and fruits (black boxes). In the case of curves and quasi-ordinary singularities, this tree only bears arrows. Using this tree, we can describe the condition for two ν-quasi-ordinary series to have an "intersection multiplicity" and compute it.

Now we come to the definitions.

We use the following notation. Let E be a set of points in $\mathbb{N}_+^{d+1}$.

- We denote by $\Delta(E)$ the smallest convex set containing the set $E + \mathbb{R}_+^{d+1}$.
- The smallest set E_0 such that $\Delta := \Delta(E_0)$ is called the set of vertices of Δ.
- The Newton polyhedron $\mathcal{N}(\Delta)$ of Δ is the union of the compact faces of Δ.

Let $f(\mathbf{x}, z) := \sum c_{\boldsymbol{\alpha},\beta}\mathbf{x}^{\boldsymbol{\alpha}} z^\beta \in \mathbb{K}[[\mathbf{x}]][z]$, $\boldsymbol{\alpha} \in \mathbb{N}^d$, $\beta \in \mathbb{N}$, $\mathbf{x} := (x_1, \dots, x_d)$, $\mathbb{K}$ being an algebraically closed field of characteristic zero.

- The support of f is $\operatorname{Supp}(f) := \{(\boldsymbol{\alpha}, \beta) \in \mathbb{N}^d \times \mathbb{N} | c_{\boldsymbol{\alpha},\beta} \neq 0\}$.
- Let $\Delta(f) := \Delta(\operatorname{Supp}(f))$, $\mathcal{N}(f) := \mathcal{N}(\Delta(\operatorname{Supp}(f)))$ is called the *Newton polyhedron* of f.
- If Γ is a compact face of $\Delta(f)$ then we define $f_\Gamma(\mathbf{x}, z) := \sum_{(\boldsymbol{\alpha},\beta)\in\Gamma} c_{\boldsymbol{\alpha},\beta}\mathbf{x}^\alpha z^\beta$. It is a weighted-homogeneous polynomial, called the *polynomial associated to* Γ.

Notation 1.1. We assume that $f(\mathbf{x}, z) = x_1^{n_1} \cdot \ldots \cdot x_d^{n_d} g(\mathbf{x}, z)$ where g is regular of order n, that is $g(\mathbf{0}, z) = a_0 z^n + a_1 z^{n+1} + \dots$, $a_0 \neq 0$. We denote $A := (n_1, \dots, n_d, n)$. Let π be the projection into $\mathbb{Q}^d \cap \{z = 0\}$ with centre A. We denote by $\Delta_n(f)$ the set of points in $\Delta(f)$ whose z-coordinate is smaller than n. A face $\widetilde{\Gamma}$ of $\mathcal{N}(f)$ is called *proper* if $A \in \widetilde{\Gamma}$ but $A \neq \widetilde{\Gamma}$ and $\widetilde{\Gamma} \setminus \{A\} \subset \Delta_n(f)$. Let $\mathcal{N}_0(f)$ be the set of compact faces of $\pi(\Delta(f))$.

Remark 1.2. Note that for any proper face $\widetilde{\Gamma}$ of $\mathcal{N}(f)$, $\pi(\widetilde{\Gamma})$ is a face of $\mathcal{N}_0(f)$ having dimension one less that $\widetilde{\Gamma}$. In particular, $\mathcal{N}_0(f)$ consists of exactly one vertex if and only if $\mathcal{N}(f)$ has only one proper face and this face is 1-dimensional. This edge will be denoted by Γ.

Let $\widetilde{\Gamma}$ be a 1-dimensional proper face of the Newton polyhedron $\mathcal{N}(f)$ such that $f_{\widetilde{\Gamma}} = x_1^{m_1} \cdot \ldots \cdot x_d^{m_d} (z - h(\mathbf{x}))^n$. Then the change of variables defined by $z \mapsto z + h(\mathbf{x})$ eliminates this face. We say in this case that $\widetilde{\Gamma}$ *can be eliminated*.

Definition 1.3. Let $f(\mathbf{x}, z) := \sum c_{\boldsymbol{\alpha},\beta}\mathbf{x}^{\boldsymbol{\alpha}} z^\beta \in \mathbb{K}[[\mathbf{x}]][z]$ be a power series and assume we are as in Notation 1.1. We say that f is a ν-*quasi-ordinary* power series if $\mathcal{N}_0(f)$ has exactly one vertex and Γ cannot be eliminated. The polynomial f_Γ is called the *initial form* of f.

For example $f(x, y, z) = z^5(z^2 + xy^2)^2 + z^2 x^7 y^9 + x^{29} + y^{51}$ is a ν-quasi-ordinary power series which is not quasi-ordinary (with respect the z-variable) and neither non-degenerated with respect to its Newton polyhedron (see e.g. [12]). Our aim is to deal with this kind of singularity that was introduced by H. Hironaka [10, Definition 6.1]. In comparison with quasi-ordinary, ν-quasi-ordinary is a very mild condition. However, ν-quasi-ordinary have interesting properties.

The initial form of f may be decomposed into irreducible factors in $\mathbb{K}[[\mathbf{x}]][z]$ as follows

$$f_\Gamma = x_1^{n_1} \cdot \ldots \cdot x_d^{n_d} \prod_{j=1}^{k} (z^p - \mu_j x_1^{q_1} \cdot \ldots \cdot x_d^{q_d})^{m_j}$$

where $\gcd(p, q_1, \ldots, q_d) = 1$ and $\mu_j \in \mathbb{K}$ with $\mu_j \neq \mu_i$. One of the goals of this work is to show that each irreducible factor of f_Γ is the initial form of a ν-quasi-ordinary factor of f (not necessarily irreducible).

Definition 1.4. A power series $f(\mathbf{x}, z) \in \mathbb{K}[[\mathbf{x}]][z]$ is called *elementary* (see [4]) if there is only one compact face of its Newton polyhedron which consists of the segment $[(0, \cdots, 0, n), (r_1, \ldots, r_d, 0)]$. More generally, if Γ is a compact one-dimensional face of a Newton diagram, we say that a power series f is Γ-*elementary* if it is elementary and its Newton diagram $\widetilde{\Gamma}$ is parallel to the face Γ. We denote $\mathrm{In}(f) := f_{\widetilde{\Gamma}}$.

The first part of the article is devoted to prove the following theorem.

Theorem 1.5. Factorisation Theorem. *Let $f \in \mathbb{K}[[\mathbf{x}]][z]$ be a ν-quasi-ordinary power series. Then there exist k different Γ-elementary power series $g_{\Gamma,j} \in \mathbb{K}[[\mathbf{x}]][z]$, for $1 \leq j \leq k$, which divide f and such that $\mathrm{In}(g_{\Gamma,j}) = (z^p - \mu_j x_1^{q_1} \cdot \ldots \cdot x_d^{q_d})^{m_j}$.*

The proof of this theorem uses Newton maps associated to the face Γ.

In the second part of the paper we will define trees to encode the Newton process for any hypersurface singularity defined by a germ $f \in \mathbb{K}[[\mathbf{x}]][z]$ as in (1.1). These trees are called Newton trees asociated to f. They end with arrows or black boxes decorated with non negative numbers, except one arrow decorated with a d-uple of non negative numbers. The numbers decorating the ends can be zero, in which case the end will be called a *dead end*. If our singularity is not ν-quasi-ordinary the tree has no vertex, only an edge and arrows or black box. In the case of curves and quasi-ordinary singularities there are no black boxes. For the case of curves, Newton trees are the algebraic version of splice diagrams defined by Eisenbud and Neumann, see [5].

It is a natural question to try to compute the Newton polyhedron of the resultant of f and g with respect to z knowing their Newton polyhedra. In our case we solve the case when the Newton polyhedron of the resultant consists in one point.

In the case of a product of two power series fg, to keep track of both functions, we will consider a bi-coloured tree (say blue and red). We will say that f and g are *separated* if the ends of the bi-coloured tree of fg which are not dead ends are either blue ends or red ends. In the case $d = 1$, two power series are always either separated or have a common component.

The main result of the §4 is the following theorem.

Theorem 1.6. *If two power series f and g in $\mathbb{K}[[\mathbf{x}]][z]$ are separated then they are comparable (i.e., their resultant with respect to z is a monomial times a unit in $\mathbb{K}[[\mathbf{x}]][z]$, see Definition 4.1).*

Remark 1.7. The converse is not true. Let

$$f = (y^5 + x^2 + z^3), \quad g = (y^5 + x^2 + z^3)^2 + x^{15} y^{17} (z^3 + y^5)^6;$$

they are not separated but they are comparable. Their resultant is $x^{81} y^{51}$.

In §5 we will decorate the Newton trees in such a way that if f and g are separated then one can compute the resultant from the decorations of the tree, see Proposition 5.11. This is a generalisation of the computation of the intersection multiplicity in the case of curves using splice diagrams of Eisenbud and Neumann (see e.g. [4]).

In a forthcoming paper [3] we will use the results of this paper to prove that a power series $f \in \mathbb{K}[[\mathbf{x}]][z]$ is a quasi-ordinary power series if and only if its Newton tree ends with arrows.

2. Factorisation

Let f be a ν-quasi-ordinary power series such that $\mathcal{N}_0(f)$ has exactly one vertex and the corresponding 1-dimensional face Γ cannot be eliminated.

In order to prove Theorem 1.5, we need to state and prove some auxiliary lemmas. The initial form f_Γ of f is written as

$$f_\Gamma = x_1^{n_1} \cdot \ldots \cdot x_d^{n_d} \prod_{j=1}^{k} (z^p - \mu_j x_1^{q_1} \cdot \ldots \cdot x_d^{q_d})^{m_j} \tag{2.1}$$

the factors being irreducible in $\mathbb{K}[[\mathbf{x}]][z]$, i.e. $\gcd(p, q_1, \ldots, q_d) = 1$ and $\mu_j \in \mathbb{K}$ with $\mu_j \neq \mu_i$, see [2, pp. 23-24].

To introduce the Newton maps we need to fix some notations. Let

$$c_i := \gcd(p, q_i), \quad p_i := \frac{p}{c_i}, \quad q_i' := \frac{q_i}{c_i}. \tag{2.2}$$

We will consider three maps. The first one depends both on the face Γ and on the root μ_j. Let $(u, u_1, \ldots, u_d) \in \mathbb{N}^{d+1}$ such that $1 + u_1 q_1 + \cdots + u_d q_d = up$. Let

$$\begin{array}{rccc} \delta_{\Gamma,j}: & \mathbb{K}[[\mathbf{x}]][z] & \longrightarrow & \mathbb{K}[[\mathbf{x}]][z] \\ & h(\mathbf{x}, z) & \mapsto & h(\mu_j^{u_1} x_1, \ldots, \mu_j^{u_d} x_d, z). \end{array}$$

Then

$$\delta_{\Gamma,j}(z^p - \mu_j x_1^{q_1} \cdot \ldots \cdot x_d^{q_d}) = z^p - \mu_j^{up} x_1^{q_1} \cdot \ldots \cdot x_d^{q_d}.$$

The second map ϵ_Γ depends only on the face Γ:

$$\begin{array}{rccc} \epsilon_\Gamma: & \mathbb{K}[[\mathbf{x}]][z] & \longrightarrow & \mathbb{K}[[\mathbf{y}]][z_1] \\ & h(\mathbf{x}, z) & \mapsto & h(y_1^{p_1}, \ldots, y_d^{p_d}, y_1^{q_1'} \cdot \ldots \cdot y_d^{q_d'} \cdot z_1). \end{array}$$

It is easily seen that

$$(\epsilon_\Gamma \circ \delta_{\Gamma,j})(z^p - \mu_j x_1^{q_1} \cdot \ldots \cdot x_d^{q_d}) = y_1^{pq_1'} \cdot \ldots \cdot y_d^{pq_d'} \cdot (z_1^p - \mu_j^{up}).$$

Now the third map τ_j will depend on the root μ_j:

$$\begin{array}{rccc} \tau_j: & \mathbb{K}[[\mathbf{y}]][z_1] & \longrightarrow & \mathbb{K}[[\mathbf{y}]][z_2] \\ & h(\mathbf{y}, z_1) & \mapsto & h(\mathbf{y}, z_2 + \mu_j^u). \end{array}$$

Note that:

$$(\tau_j \circ \epsilon_\Gamma \circ \delta_{\Gamma,j})(z^p - \mu_j x_1^{q_1} \cdot \ldots \cdot x_d^{q_d}) = y_1^{p_1 q_1} \cdot \ldots \cdot y_d^{p_d q_d} \cdot ((z_2 + \mu_j^u)^p - \mu_j^{up})$$

and the last factor is of order one in z_2. Note that $p_i q_i = p q_i'$ for $1 \leq i \leq d$.

Definition 2.1. The *Newton map* associated to Γ and μ_j is the composition

$$\sigma_{\Gamma,j} = \tau_j \circ \epsilon_\Gamma \circ \delta_{\Gamma,j}.$$

We have proved the following statement.

Lemma 2.2. *After a Newton map, $f_{\Gamma,j} := \sigma_{\Gamma,j}(f)$ can be written as*

$$f_{\Gamma,j}(\mathbf{y}, z_2) = y_1^{p_1 q_1 m + n_1 p_1} \cdots y_d^{p_d q_d m + n_d p_d} f_{1,\Gamma,j}(\mathbf{y}, z_2),$$

where $f_{1,\Gamma,j}(0, z)$ is a regular polynomial of order m_j, and $m := \sum_{j=1}^{k} m_j$.

Remark 2.3. If $f_{1,\Gamma,j}$ is again ν-quasi-ordinary we can iterate the process, we call this process the Newton process; note that $m_j < n$ since we cannot eliminate the face Γ.

We keep the notations of (2.1) and (2.2) and, for simplicity, we write

$$f_\Gamma = z^\ell (z^p - x_1^{q_1} \cdot \ldots \cdot x_d^{q_d})^m \prod_{j=2}^{k} (z^p - \mu_j x_1^{q_1} \cdot \ldots \cdot x_d^{q_d})^{m_j}$$

and we fix the root $\mu_1 = 1$. Note that $n = \ell + p(m + \sum_{j=2}^{k} m_j)$. The Newton map σ we are considering is

$$\begin{array}{rccl} \sigma := \sigma_{\Gamma,1} : & \mathbb{K}[[\mathbf{x}]][z] & \longrightarrow & \mathbb{K}[[\mathbf{y}]][z_1] \\ & h(\mathbf{x}, z) & \mapsto & h(y_1^{p_1}, \ldots, y_d^{p_d}, y_1^{q_1'} \cdot \ldots \cdot y_d^{q_d'} \cdot (z_1 + 1)). \end{array}$$

Notation 2.4. We denote $P(z_1) := ((z_1 + 1)^p - 1)^m$; note that P is of order m in z_1. The ideal generated by $y_1, \ldots, y_d$ will be denoted by $(\mathbf{y})$.

Remark 2.5. The line supporting Γ has equations $q_i' Z + p_i X_i = q_i' n$, $i = 1, \ldots, d$. The fact that Γ is an edge of the Newton polyhedron implies that $q_i'\beta + p_i\alpha_i \geq q_i' n$, $i = 1, \ldots, d$, if $(\boldsymbol{\alpha}, \beta) \in \operatorname{Supp}(f)$.

Definition 2.6. The *monomial set associated* to Γ is $G_{(p,q_1,\ldots,q_d)} := \bigcup_{j=0}^{p-1} G^j_{(p,q_1,\ldots,q_d)}$, where

$$G^j_{(p,q_1,\ldots,q_d)} := \{(B_1, \ldots, B_d) \in \mathbb{N}^d \ \mid \ (B_1, \ldots, B_d) \equiv j(q_1, \ldots, q_d) \bmod p\}.$$

Notation 2.7. If $B \in G_{(p,q_1,\ldots,q_d)}$, we denote

$$\mathbf{y}^{\frac{B}{c}} := y_1^{\frac{B_1}{c_1}} \cdot \ldots \cdot y_d^{\frac{B_d}{c_d}}, \quad B^q := (q_1, \ldots, q_d).$$

The d-uple c is called the gcd-uple.

Definition 2.8. The *ring associated* to Γ is the subring $\mathcal{B}$ of $\mathbb{K}[[\mathbf{y}]]$ generated by the monomials $\mathbf{y}^{\frac{B}{c}}$ where $B \in G_{(p,q_1,\ldots,q_d)}$.

Remark 2.9. Note that $G_{(p,q_1,\ldots,q_d)}$ is a semigroup, in fact an affine semigroup in the sense of toric geometry. It is easily seen that the image of σ is contained in $\mathcal{B}[z_1]$.

Lemma 2.10. *There exist $u \in \mathbb{K}[[\mathbf{y}, z_1]]$ and $r \in (\mathbf{y})\mathbb{K}[[\mathbf{y}]][z_1]$ such that*

(a) *u is unit,*
(b) $\deg_{z_1} r \leq m - 1$ *and*
(c) $\sigma(f) = \mathbf{y}^{n\frac{B^q}{c}} \cdot u(\mathbf{y}, z_1) \cdot (P(z_1) + r(\mathbf{y}, z_1))$.

Moreover, the coefficients of the polynomial $r(z_1)$ belong to $\mathcal{B}$.

Proof. We can write $\sigma(f) = \mathbf{y}^{n\frac{B^q}{c}} f_1(\mathbf{y}, z_1)$ where

$$f_1(\mathbf{y}, z_1) = P(z_1) \prod_{j=2}^{k} ((z_1 + 1)^p - \mu_j)^{m_j} + \sum c_{\boldsymbol{\alpha},\beta} \cdot \mathbf{y}^{\frac{p\boldsymbol{\alpha} + (\beta - n)B^q}{c}} (z_1 - 1)^\beta. \tag{2.3}$$

The last term is in the ideal $(\mathbf{y})$ because of the Newton polyhedron structure of f. The first term is the product of $P(z_1)$ and a unit; combining these facts we can write (2.3) as

$$f_1(\mathbf{y}, z_1) = P(z_1)\tilde{f}_1 + \overline{f_1} \tag{2.4}$$

where $\tilde{f_1} \in \mathbb{K}[[\mathbf{y}]][z_1]$ is a unit in $\mathbb{K}[[\mathbf{y}, z_1]]$ and $\overline{f_1} \in (\mathbf{y})\mathbb{K}[[\mathbf{y}]][z_1]$; note that $\deg_{z_1} \overline{f_1} \leq m-1$ and the coefficients of $\tilde{f_1}$ and $\overline{f_1}$ are in $\mathcal{B}$ (since $P(z_1) \in \mathbb{K}[z_1]$). Finally we can write

$$P(z_1) = \mathfrak{q}_1 f_1 + g_1 \tag{2.5}$$

where $\mathfrak{q}_1 := \frac{1}{\tilde{f_1}}$ is a unit and $g_1 = -\frac{\overline{f_1}}{\tilde{f_1}} \in (y_1, \dots, y_d)\mathbb{K}[[\mathbf{y}, z_1]]$. It is easily seen that the coefficients of $\mathfrak{q}_1$ and g_1 are in $\mathcal{B}$.

Claim. *For $i \geq 0$ there exist $\mathfrak{q}_i, g_i, r_i \in \mathbb{K}[[\mathbf{y}, z_1]]$ such that $P(z_1) = \mathfrak{q}_i f_1 + g_i + r_i$ and:*

(1) $\mathfrak{q}_{i+1} - \mathfrak{q}_i \in (\mathbf{y})^i\mathbb{K}[[\mathbf{y}, z_1]]$, *(in particular $\mathfrak{q}_{i+1}$ is a unit);*
(2) $g_i \in (\mathbf{y})^i\mathbb{K}[[\mathbf{y}, z_1]]$;
(3) $r_{i+1} - r_i \in (\mathbf{y})^i\mathbb{K}[[\mathbf{y}, z_1]]$, *and r_i is a polynomial of degree less than or equal to $m-1$.*

Moreover, the coefficients of $\mathfrak{q}_i$, g_i and r_i are in $\mathcal{B}$.

Let us assume the claim. Then, there exist $\mathfrak{q}(\mathbf{y}, z_1) \in \mathbb{K}[[\mathbf{y}, z_1]]$ and $r(\mathbf{y}, z_1) \in \mathbb{K}[[\mathbf{y}]][z_1]$ such that

- $\deg_{z_1} r \leq m-1$,
- $\mathfrak{q} - \mathfrak{q}_i \in (\mathbf{y})^i\mathbb{K}[[\mathbf{y}, z_1]]$ and
- $r - r_i \in (\mathbf{y})^i\mathbb{K}[[\mathbf{y}]][z_1]$.

Then $P(z_1) - \mathfrak{q} f_1 - r$ is congruent with $P(z_1) - \mathfrak{q}_i f_1 - r_i = g_i \mod (\mathbf{y})^i$. Since by (2) $g_i = 0 \mod (\mathbf{y})^i$ then $P(z_1) - \mathfrak{q} f_1 - r \in (\mathbf{y})^i\mathbb{K}[[\mathbf{y}, z_1]]$, for every $i \geq 0$. Since $\mathfrak{q}_1$ is a unit then $\mathfrak{q}$ is a unit and we have proved the first statement of the Lemma. In order to prove the second statement, note that any monomial in $r(\mathbf{y}, z)$ is also a monomial in some r_i. □

Proof of the Claim. We show it by induction. The properties are true for $i = 1$ taking $r_1 = 0$. We continue with the general case.

We decompose $g_i = P(z_1)\tilde{g_i} + \overline{g_i}$, $\tilde{g_i} \in \mathbb{K}[[\mathbf{y}, z_1]]$ and $\overline{g_i} \in \mathbb{K}[[\mathbf{y}]][z_1]$ where $\deg_{z_1} \overline{g_i} \leq m-1$.

Since $g_i \in (\mathbf{y})^i\mathbb{K}[[\mathbf{y}, z_1]]$ then $\tilde{g_i}, \overline{g_i} \in (\mathbf{y})^i\mathbb{K}[[\mathbf{y}, z_1]]$. Also, the coefficients of $\tilde{g_i}$ and $\overline{g_i}$ are in $\mathcal{B}$. Let

$$\mathfrak{q}_{i+1} := \mathfrak{q}_i - \frac{\tilde{g_i}}{\tilde{f_1}}, \quad g_{i+1} := -\frac{\overline{f_1}\tilde{g_i}}{\tilde{f_1}}, \quad r_{i+1} = r_i + \overline{g_i}.$$

Then we have

- $\mathfrak{q}_{i+1} - \mathfrak{q}_i = -\dfrac{\tilde{g_i}}{\tilde{f_1}} \in (\mathbf{y})^i\mathbb{K}[[\mathbf{y}, z_1]]$;
- $r_{i+1} - r_i = \overline{g_i} \in (\mathbf{y})^i\mathbb{K}[[\mathbf{y}, z_1]]$;
- $g_{i+1} \in (\mathbf{y})^{i+1}\mathbb{K}[[\mathbf{y}, z_1]]$;
- r_{i+1} is a polynomial of degree less than or equal to $m-1$.
- The coefficients of $\mathfrak{q}_{i+1}, g_{i+1}, r_{i+1}$ are in $\mathcal{B}$.

We have the equality:

$$\begin{aligned} P(z_1) = \mathfrak{q}_i f_1 + g_i + r_i &= \mathfrak{q}_{i+1} f_1 - \frac{f_1\tilde{g_i}}{\tilde{f_1}} + P(z_1)\tilde{g_i} + \overline{g_i} + r_{i+1} - \overline{g_i} \\ &= \mathfrak{q}_{i+1} f_1 - \frac{(P(z_1)\tilde{f_1} + \overline{f_1})\tilde{g_i}}{\tilde{f_1}} + P(z_1)\tilde{g_i} + r_{i+1} = \mathfrak{q}_{i+1} f_1 + g_{i+1} + r_{i+1}. \end{aligned}$$ □

Remark 2.11. In particular we have proved that for a ν-quasi-ordinary series power f of degree n in z, such that $f(\mathbf{0}, z)$ is of order n, then $\mathbf{y}^{\frac{nB^q}{c}}$ is the maximal monomial dividing $\sigma(f)$, where σ is the Newton map.

We will give conditions for an element of $\mathcal{B}$ to be in the image of σ.

Lemma 2.12. *For $1 \leq j \leq p-1$, let $B \in G^j_{(p,q_1,\dots,q_d)}$ and let $s \in \mathbb{K}[z_1]$ be a polynomial of degree less than or equal to $m-1$. Then there exists $g \in \mathbb{K}[\mathbf{x}, z]$ and $u(z_1) \in \mathbb{K}[[z_1]]$ such that*

$$\sigma(g(\mathbf{x}, z)) = \mathbf{y}^{\frac{B+pmB^q}{c}}(s(z_1) + u(z_1)P(z_1)).$$

The order of g with respect to $\mathbf{x}$ is at least $\sum_{i=1}^{d} \frac{B_i+q_i}{p}$ and its degree with respect to z is at most mp.

Proof. We have:

$$\frac{B_1}{c_1} = \alpha_1 p_1 + jq_1', \dots, \frac{B_d}{c_d} = \alpha_d p_d + jq_d'.$$

Let

$$g(\mathbf{x}, z) := z^j \sum_{k=0}^{m-1} d_k x_1^{\alpha_1+(m-k)q_1} \cdot \dots \cdot x_d^{\alpha_d+(m-k)q_d} (z^p - x_1^{q_1} \cdot \dots \cdot x_d^{q_d})^k,$$

for some $d_0, \dots, d_{m-1} \in \mathbb{K}$. The order of g is as in the statement. Then

$$\sigma(g(\mathbf{x}, z)) = \mathbf{y}^{\frac{B+pmB^q}{c}} R(z_1)$$

with

$$R(z_1) := (z_1+1)^j \sum_{k=0}^{m-1} d_k((z_1+1)^p - 1)^k.$$

Let us write

$$R(z_1) = c_0 + c_1 z_1 + \dots + c_{m-1} z_1^{m-1} + z_1^m Q(z_1).$$

Then

$$\begin{pmatrix} c_0 \\ c_1 \\ . \\ . \\ c_{m-1} \end{pmatrix} = M \begin{pmatrix} d_0 \\ d_1 \\ . \\ . \\ d_{m-1} \end{pmatrix}$$

where M is an invertible triangular matrix. For $0 \leq k \leq m-1$, let d_k be determined so that $c_0 + c_1 z_1 + \dots + c_{m-1} z_1^{m-1} = s(z_1)$. Then

$$\sigma(g(\mathbf{x}, z)) = \mathbf{y}^{\frac{B+pmB^q}{c}}(s(z_1) + u(z_1)P(z_1))$$

with $u(z_1) = Q(z_1)\dfrac{z_1^m}{P(z_1)}$ and we recall that the order of $P(z_1)$ is m. □

Lemma 2.13. *There exist $g \in \mathbb{K}[[\mathbf{x}]][z]$ which is Γ-elementary and a unit $U \in \mathbb{K}[[\mathbf{y}, z_1]]$ such that*

$$\sigma(g) = \mathbf{y}^{pm\frac{B^q}{c}} \cdot U \cdot (P(z_1) + r(z_1))$$

where r is defined in Lemma 2.10. Moreover, $\mathrm{In}(g) = (z^p - x_1^{q_1} \cdot \dots \cdot x_d^{q_d})^m$.

Proof. Let $g_0(\mathbf{x}, z) := (z^p - x_1^{q_1} \cdot \dots \cdot x_d^{q_d})^m$. Then $\sigma(g_0) = \mathbf{y}^{pm\frac{B^q}{c}} P(z_1)$.

Write

$$r(z_1) = R_1(z_1) + R_2(z_1) + \dots + R_i(z_1) + \dots$$

where $R_i(z_1)$ is the homogeneous part of $r(z_1)$ of degree i in the variables $\mathbf{y}$; recall that it has degree in z_1 less than or equal to $m-1$. Let us denote $u_0 := 1$.

Claim. *For $i \geq 1$, there exist $g_i(\mathbf{x}, z) \in \mathbb{K}[\mathbf{x}, z]$, of order at least $\left\lfloor \frac{i}{p} \right\rfloor$ in $\mathbf{x}$, and $u_i \in (\mathbf{y})^i \mathbb{K}[[z_1]]$, such that*

$$\sigma(g_i) = \mathbf{y}^{pm\frac{B^q}{c}} \left(\sum_{j=1}^{i} u_{i-j} R_j(z_1) + u_i P_1(z_1) \right).$$

Let us assume the claim. We obtain:

$$\sigma\left(\sum_{j=0}^{i} g_j\right) = \mathbf{y}^{pm\frac{B^q}{c}} \left(\sum_{j=1}^{i} R_j(z_1) \sum_{h=0}^{i-j} u_h + P_1(z_1) \sum_{j=0}^{i} u_j \right).$$

The series $g := \sum_{i=0}^{\infty} g_i$ is well-defined and elementary. The series $U := 1 + \sum_{i=1}^{\infty} u_i$ is also well-defined and a unit. The statement follows. $\square$

Proof of the Claim. We will prove it by induction. We start with $i = 1$. We write $R_1(z_1) = \mathbf{y}^{\frac{B_1^1}{c}} P_1^1(z_1) + \cdots + \mathbf{y}^{\frac{B_1^s}{c}} P_1^s(z_1)$, where

$$P_1^j(z_1) \in \mathbb{K}[z_1], \quad \deg_{z_1}(P_1^j) \leq m - 1, \quad \deg_{\mathbf{y}} \mathbf{y}^{\frac{B_1^j}{c}} = 1.$$

Applying Lemma 2.12, there exist $g_1^1, \ldots, g_1^s$ and $u_1^1, \ldots, u_1^s$ such that

$$\sigma(g_1^j) = \mathbf{y}^{\frac{B_1^j + pmB^q}{c}} \cdot (P_1^j(z_1) + u_1^j P(z_1)), \quad j = 1, \ldots, s.$$

Let us define $g_1(\mathbf{x}, z) := g_1^1 + \cdots + g_1^s$ and $u_1 =: u_1^1 \mathbf{y}^{\frac{B_1^1}{c}} + \cdots + u_1^s \mathbf{y}^{\frac{B_1^s}{c}}$. Then

$$\sigma(g_1) = \mathbf{y}^{pm\frac{B^q}{c}} (R_1(z_1) + u_1 P(z_1)).$$

Let us assume the Claim for $i - 1$. We apply the algorithm of division in z_1 to $R_i + u_1 R_{i-1} + \cdots + u_{i-1} R_1$ and $P(z_1)$:

$$R_i + u_1 R_{i-1} + \cdots + u_{i-1} R_1 = R_i' + v_i P(z_1), \quad \deg_{z_1} R_i' \leq m - 1.$$

Using again Lemma 2.12, there exist g_i and u_i' such that

$$\sigma(g_i(\mathbf{x}, z) = R_i' + u_i' P(z_1) = R_i + u_1 R_{i-1} + \cdots + u_{i-1} R_1 + P(z_1)(u_i' - v_i).$$

If $u_i := u_i' - v_i$, we have

$$\sigma(g_i(\mathbf{x}, z)) = R_i' + u_i' P(z_1) = R_i + u_1 R_{i-1} + \ldots + u_{i-1} R_1 + P(z_1) u_i. \qquad \square$$

We have the ingredients to prove Theorem 1.5.

Proof of Theorem 1.5. By Lemma 2.10 we have

$$\sigma(f) = \mathbf{y}^{n\frac{B^q}{c}} \cdot u(\mathbf{y}, z_1) \cdot (P(z_1) + r(\mathbf{y}, z_1)).$$

By Lemma 2.13 there exists an elementary series $g(\mathbf{x}, z) \in \mathbb{K}[[\mathbf{x}]][z]$ with $\operatorname{In}(g) = (z^p - x_1^{q_1} \cdot \ldots \cdot x_d^{q_d})^m$ such that

$$\sigma(g) = \mathbf{y}^{pm\frac{B^q}{c}} \cdot U \cdot (P(z_1) + r(\mathbf{y}, z_1)).$$

There is a unit V such that

$$\sigma(f) = \mathbf{y}^{(\ell + \sum_{j=2}^{k} m_j)\frac{B^q}{c}} \cdot V(\mathbf{y}, z_1) \cdot \sigma(g).$$

We apply the division algorithm to f and g in $\mathbb{K}[[\mathbf{x}]][z]$ with respect to z and we obtain $f = qg + t$ where $\deg_z t < pm$. The equality $\sigma(f) = \sigma(q)\sigma(g) + \sigma(t)$ implies

$$\mathbf{y}^{(\ell + \sum_{j=2}^{k} m_j)\frac{B^q}{c}} \cdot V(\mathbf{y}, z_1) \cdot \sigma(g) = \sigma(q)\sigma(g) + \sigma(t).$$

Then, $\sigma(t)$ is divisible by $\sigma(q)$ and by degree reasons, the only possibility is $\sigma(t) = 0$ which implies $t = 0$. $\square$

Theorem 1.5 have the following consequences.

Corollary 2.14. *Let f be an irreducible power series which is ν-quasi-ordinary. Then it is elementary and* $\mathrm{In}(f) = f_0^k$*, where $f_0 \in \mathbb{K}[x, z]$ is irreducible.*

Corollary 2.15. *Let f be an elementary power series such that* $\mathrm{In}(f) = \psi_1 \cdot \ldots \cdot \psi_r$ *is a factorisation of* $\mathrm{In}(f)$ *where the factors are pairwise coprime. Then we can decompose $f = f_1 \cdot \ldots \cdot f_r$ where, for all $i = 1, \ldots, r$, the series f_i is an elementary power series such that* $\mathrm{In}(f_i) = \psi_i$*.*

Corollary 2.16. *Let f be a ν-quasi-ordinary power series. Then*

$$f = \mathbf{x}^a z^b f^\Gamma g$$

where f^Γ is Γ*-elementary, and* $\mathrm{In}(f^\Gamma) = f_\Gamma$*.*

Consider a ν-quasi-ordinary power series $f(\mathbf{x}, z)$ such that after a Newton map we obtain as above a ν-quasi-ordinary power series $f_1(\mathbf{y}, z_1)$. If f_1 factorises along its face Γ does it means that f factorises? In general, the answer is no. The following example has been constructed by M. González Villa.

Example 2.17. Let

$$f = (z^2 - x_1x_2^3)^4 - 2z^4x_1^2x_2^7 - 12z^2x_1^3x_2^{10} - 2x_1^4x_2^{13} + x_1^4x_2^{14}$$

This is an irreducible quasi-ordinary singularity. The Newton map σ is given by

$$\sigma(h(x_1, x_2, z)) := h(y_1^2, y_2^2, (z_1 + 1)y_1y_2^3).$$

We get $\sigma(f) = y_1^8y_2^{24}f_1(y_1, y_2, z_1)$ where

$$f_1(y_1, y_2, z_1) = (y_2 - 4 - 4z_1 - z_1^2)(y_2 - z_1^2)(y_2 + z_1^2)(y_2 + 4 + 4z_1 + z_1^2).$$

But the factorisation of f_1 does not imply a factorisation of f. It is easily seen that $\mathcal{B} = \mathbb{K}[[y_1^2, y_2^2, y_1y_2]]$. Note that the coefficients of the factors of f_1 are not in $\mathcal{B}$.

3. Newton trees

Let $f(\mathbf{x}, z) := \sum c_{\boldsymbol{\alpha},\beta}\mathbf{x}^{\boldsymbol{\alpha}}z^\beta \in \mathbb{K}[[\mathbf{x}]][z]$ be a power series and assume we are as in Notation 1.1. In this section we associte to $f(\mathbf{x}, z)$ a tree called *Newton tree* because the tree is built using Newton polyhedra.

We assume that $f(\mathbf{x}, z) = x_1^{n_1} \cdot \ldots \cdot x_d^{n_d} g(\mathbf{x}, z)$ where g is regular of order n, that is $g(\mathbf{0}, z) = a_0z^n + a_1z^{n+1} + \ldots$, $a_0 \neq 0$. We denote $A := (n_1, \ldots, n_d, n)$. Let π be the projection into $\mathbb{Q}^d \cap \{z = 0\}$ with centre A. We denote by $\Delta_n(f)$ the set of points in $\Delta(f)$ whose z-coordinate is smaller than n. Let $\mathcal{N}_0(f)$ be the set of compact faces of $\pi(\Delta(f))$.

The Newton tree when $\mathcal{N}_0(f)$ is void is in Figure 1.

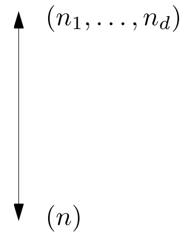

FIGURE 1.

The Newton tree when $\mathcal{N}_0(f)$ has more than one vertex is as in Figure 2.

Now we assume that $\mathcal{N}_0(f)$ has exactly one vertex and the face Γ cannot be eliminated, (i.e. f is a ν-quasiordinary power series). In such a case the corresponding face Γ is a 1-dimensional face of the Newton polyhedron of f which is the

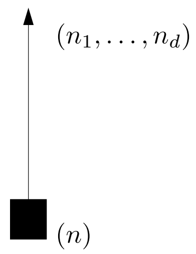

FIGURE 2.

intersection of two non-compact faces of its Newton diagram. Let us assume that the face Γ equals the segment $[A, A_1]$. If the z-coordinate of A_1 is > 0, let π_1 be the projection into $\mathbb{Q}^d$ with centre at A_1 and let $\mathcal{N}_{0,1}(f)$ be the convex hull of the image by π_1 of the set of points in $\mathcal{N}(f)$ whose last coordinate is less than n^1, if n^1 is the z-coordinate of A_1. If $\mathcal{N}_{0,1}(f)$ has only one vertex then there is another face Γ_1 of the Newton polyhedron of f which is of dimension 1 and the intersection of two non compact faces of the Newton diagram. We go on this construction with $\Gamma = [A, A_1], \Gamma_1 = [A_1, A_2], \cdots, \Gamma_j = [A_j, A_{j+1}]$ until either the z-coordinate n^{j+1} of A_{j+1} is zero, or $\mathcal{N}_{0,j+1}(f)$ is void or has more than one vertex. In fact $\Gamma \cup \Gamma_1 \cup \cdots \cup \Gamma_j$ is a *polygonal path.* To this polygonal path we associate a graph. To each face of the polygonal path, we associate a vertex. These vertices are drawn on a vertical line, from above to below in order. On the top of the line we put an arrow decorated with $(n_1, \cdots, n_d)$. On the bottom of the line, we put an arrow decorated with (n^{j+1}) if $n^{j+1} = 0$ or if $\mathcal{N}_{0,j+1}(f)$ is void or a black box decorated with (n^{j+1}) if $\mathcal{N}_{0,j+1}(f)$ has more than one vertex.

Assume the equation of the line which bears one of the edges $\Gamma, \Gamma_1, \cdots, \Gamma_j$ is

$$q_i\beta + p\alpha_i = N_i, i = 1, \cdots, d, \gcd(p, q_1, \cdots, q_d) = 1. \tag{3.1}$$

We decorate the corresponding vertex with $((N_1, \cdots, N_d))$, the extremity of the edge under the vertex near the vertex with p and the extremity of the edge above the vertex near the vertex with $(q_1, \cdots, q_d)$. Then, we can recover the polygonal path from the graph.

Then now for each edge $\Gamma, \Gamma_1, \cdots, \Gamma_j$ and for each root of the face polynomial we apply the corresponding Newton map. We get a new function and we do the same process as before (including eliminating a face if necessary). At each step, we encode the information given by the Newton polyhedron. The process stops because the z-degree decreases. We build the Newton trees by induction on the number of steps. For STEP 0, Newton tree is already defined. Now assume we consider a function with k steps. On one hand we constructed a graph associated to its Newton polyhedron. For each vertex and each root of the face polynomial, the transforms by the Newton map need a number of steps less than or equals to $k-1$. Then by assumption we can construct their Newton trees. Now we delete the top arrow of these Newton trees and we glue the edge to the corresponding vertex of the graph of the Newton polyhedron. We keep the decorations of the vertices, arrows and black boxes. By construction Newton trees are connected trees.

An *end* of the Newton tree is either an arrow or a black box. We will say that the arrows or black boxes decorated with 0 will be called *dead ends.* We will change the decorations of the edges, but we will explain how later.

Example 3.1. Consider the germ $f \in \mathbb{K}[[x_1, x_2]][z]$ defined by $f(x_1, x_2, z) := (z^3 - x_1^4 x_2^5)(z^4 - x_1^7 x_2^9) + x_1^{17} x_2^{17}$. In this case the corresponding point A has coordinantes $A = (0, 0, 7)$. There are two one-dimensional faces Γ_1 and Γ_2 of the Newton polyhedron. On each face there is only one irreducible factor.

The equations of the edges of the Newton polyhedron are:

$$\Gamma_1 \equiv \qquad 4\beta + 3\alpha_1 = 28, \quad 5\beta + 3\alpha_2 = 35;$$

$$\Gamma_2 \equiv \qquad 7\beta + 4\alpha_1 = 44, \quad 9\beta + 4\alpha_2 = 56;$$

then, the decoration of the first vertex is given by $((28, 35))$ and 3 is the decoration of the extremity of the edge under the vertex near the vertex and $(4, 5)$ is the decorations of the extremity of the edge above the vertex near the vertex.. For the second vertex, we have $((44, 56))$ and 4 and $(7, 9)$ respectively.

The first Newton map associated to Γ_1 is

$$\begin{array}{rccc} \sigma_1 : & \mathbb{K}[[\mathbf{x}]][z] & \longrightarrow & \mathbb{K}[[\mathbf{y}]][z_1] \\ & h(x_1, x_2, z) & \mapsto & h(y_1^3, y_2^3, y_1^4 y_2^5 (z_1 + 1)). \end{array}$$

And we get after local automorphisms

$$\sigma_1(f) = 3 y_1^{28} y_2^{35} z_2.$$

The second Newton map associated to Γ_2 is

$$\begin{array}{rccc} \sigma_2 : & \mathbb{K}[[\mathbf{x}]][z] & \longrightarrow & \mathbb{K}[[\mathbf{y}]][z_1] \\ & h(x_1, x_2, z) & \mapsto & h(y_1^4, y_2^4, y_1^7 y_2^9 (z_1 + 1)). \end{array}$$

And we get after local automorphisms

$$\sigma_2(f) = 4 y_1^{44} y_2^{56} z_2.$$

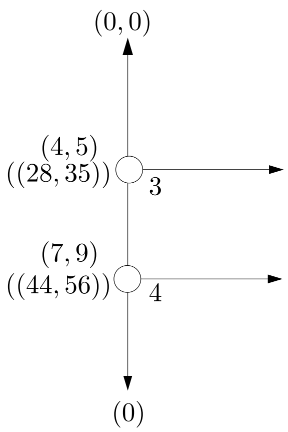

FIGURE 3.

Example 3.2. This example can be found in [9]. Consider the germ $f \in \mathbb{K}[[x_1, x_2]][z]$ defined by

$$f(x_1, x_2, z) = (z^2 - x_1^3)^2 - 2x_1^7 x_2 - 2x_1^4 x_2 z^2.$$

One has $A = (0, 0, 4)$. There is only one compact one dimensional face Γ of the Newton polyhedron with one irreducible factor. The equations of the edge of the Newton polyhedron are

$$\Gamma \equiv \qquad 3\beta + 2\alpha_1 = 12, \quad 0\beta + 2\alpha_2 = 0;$$

Then the first vertex of the Newton tree is decorated with $((12, 0))$ and the edge under the vertex with 2.

The Newton map associated to Γ is

$$\begin{array}{cccc}\sigma: & \mathbb{K}[[\mathbf{x}]][z] & \longrightarrow & \mathbb{K}[[\mathbf{y}]][z_1] \\ & h(x_1,x_2,z) & \mapsto & h(y_1^2,y_2,y_1^3(z_1+1)).\end{array}$$

The pullback of f is given by the equation

$$\sigma(f) = -y_1^{12}(-z_1^4 - 4z_1^3 - 4z_1^2 + 4y_1^2y_2 + 2y_1^2y_2z_1^2 + 4y_1^2y_2z_1).$$

For the new Newton polyhedron of $\sigma(f)$, the corresponding point A has coordinates $A = (0,12,2)$ and it has only one compact one dimensional face with one irreducible polynomial $-4(z_1^2 - y_1^2y_2)$.

After the Newton map, the equations of the edge of the new Newton polyhedron are

$$2\beta + 2\alpha_1 = 28, \quad \beta + 2\alpha_2 = 2;$$

Then the second vertex is decorated with $((28,2))$ and the edge under the vertex with 2, see Figure 4.

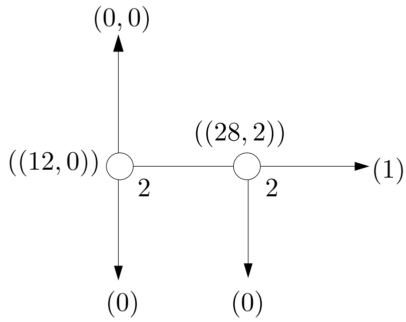

FIGURE 4.

4. Comparable power series

Definition 4.1. We say that two power series f and g in $\mathbb{K}[[\mathbf{x}]][z]$ are *comparable* if the resultant $R_z(f,g)(\mathbf{x})$ of f and g with respect to z is a monomial multiplied by a unit of $\mathbb{K}[[\mathbf{x}]]$.

First we need to define *coloured Newton trees* associated to a product of two power series f and g. We will associate the colour blue to f, the colour red to g. The coloured Newton tree can have blue parts, red parts and blue-red parts.

Let $fg(\mathbf{x},z) := \sum c_{\boldsymbol{\alpha},\beta}\mathbf{x}^{\boldsymbol{\alpha}}z^\beta \in \mathbb{K}[[\mathbf{x}]][z]$ be a product of two power series and assume we are as in Notation 1.1. We assume that

$$fg(0,z) = a_0z^n + \cdots$$

with $a_0 \neq 0$. Let $A = (0,\cdots,0,n)$. Let π be the projection into $\mathbb{Q}^d$ with centre A as before and we consider $\mathcal{N}_0(fg)$ as in §1. Three possible cases may arise:

We consider the Newton polyhedron, i.e. it corresponds to the first vertical part of the Newton tree, of fg. The part of the Newton tree corresponding to this Newton polyhedron will be blue if $\deg g(0,z) = 0$, red if $\deg_z f(0,z) = 0$ and bi-coloured blue-red otherwise. We apply the same rule for all the steps of the Newton process. In particular it has the same (bi)color for every vertical line in the Newton tree.

Example 4.2. The bi-coloured Newton tree of fg where $f = x^3 - y^2$ and $g = y^3 - x^2$ is as in Figure 5 where all vertical lines are bi-coloured, the above horizontal line is red and the below one is blue.

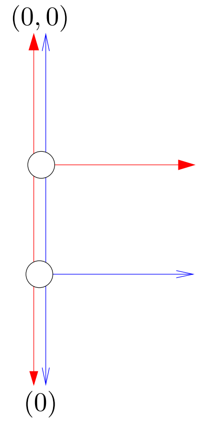

FIGURE 5.

Definition 4.3. We say that two power series f and g in $\mathbb{K}[[\mathbf{x}]][z]$ are *separated* if all the ends of the bi-coloured Newton tree of fg which are not dead ends are either red or blue.

Remark 4.4. If $d = 1$ then f and g are separated if they do not have a common component.

Remark 4.5. If two power series f and g in $\mathbb{K}[[\mathbf{x}]][z]$ are *separated* then there is, at least one, vertical line in the the bi-coloured Newton tree, which corresponds to a Newton polyhedra of an step of the Newton proccess, from which the Newton tree is not bicoloured.

Example 4.6. We illustrate the two kinds of separation: in Figure 6 there is a Newton polyhedron such that the face corresponding to f is different from the one corresponding to g. In Figure 7 they have different polynomials on the same face.

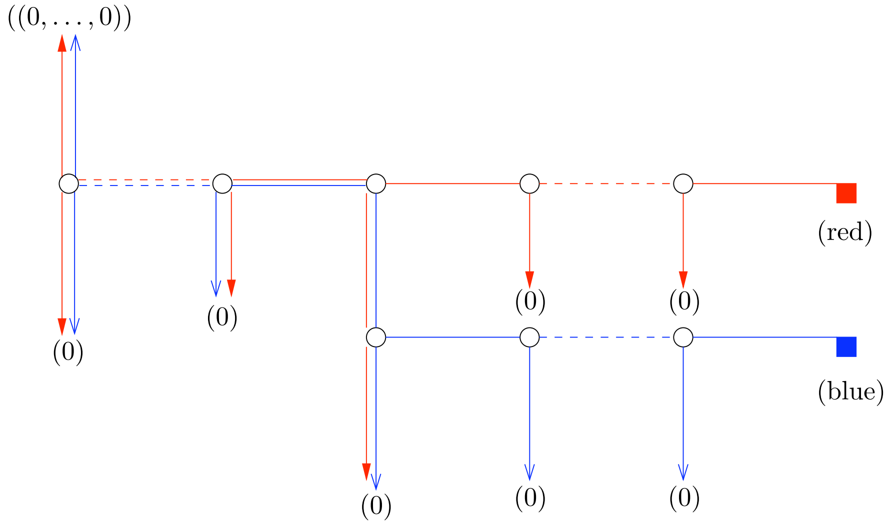

FIGURE 6.

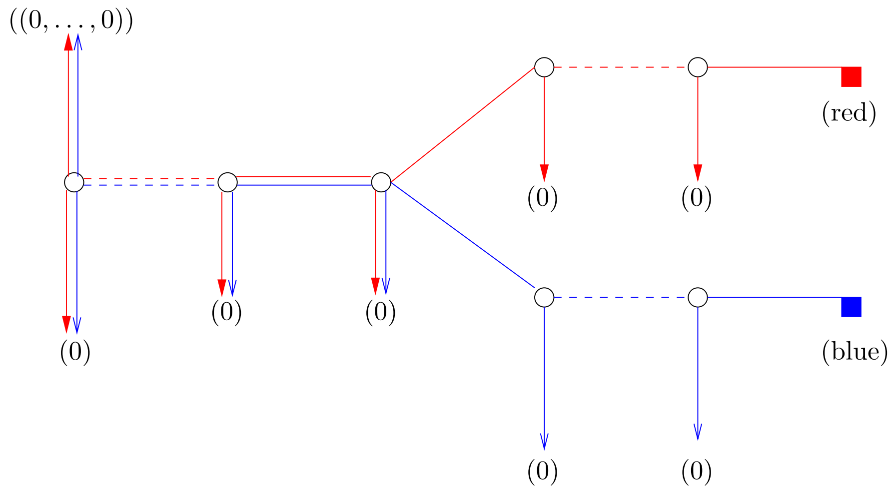

FIGURE 7.

To prove Theorem 1.6, we need the following lemma. Consider the morphism

$$\begin{array}{rcl}\epsilon : \mathbb{K}[[\mathbf{x}]][z_1] & \longrightarrow & \mathbb{K}[[\mathbf{y}]][z] \\ (x_1, \dots, x_d, z_1) & \mapsto & (y_1^{p_1}, \dots, y_d^{p_d}, z_1 \prod_{j=1}^{d} y_j^{q_j}).\end{array}$$

Lemma 4.7. *Let f and g be two Weierstraß polynomials in $\mathbb{K}[[\mathbf{x}]][z]$ of degree m and n respectively. Let $R_z(\mathbf{x})$ (respec. $R_{z_1}(\mathbf{y})$) be the resultant with respect to z (respec. to z_1) of f (respec. f_1) and g (respec. g_1). Assume that*

$$\epsilon(f) = y_1^{N_{f,1}} \cdot \dots \cdot y_d^{N_{f,d}} f_1(\mathbf{y}, z_1) = \mathbf{y}^{N_f} f_1(\mathbf{y}, z_1), \quad \epsilon(g) = \mathbf{y}^{N_g} g_1(\mathbf{y}, z_1)$$

such that f_1 and g_1 are not divisible by any y_j, for $1 \leq j \leq d$. Then, see Notation 2.7 and (2.2), the following identity holds

$$\epsilon(R_z) = \mathbf{y}^{nN_f + mN_g - mn\frac{Bq}{c}} R_{z_1}(\mathbf{y}).$$

Proof. The resultant $R_z(\mathbf{x})$ is the determinant of a matrix which expresses the polynomials $f, zf, \dots, z^{n-1}f, g, zg, \dots, z^{m-1}g$ in the basis $\{1, z, z^2, \dots, z^{n+m-1}\}$. Denote this matrix by $M(\mathbf{x})$.

Consider the matrix $\epsilon(M) \cdot P(\mathbf{y})$ where $P(\mathbf{y})$ is the diagonal matrix with entries $(1, \mathbf{y}^{\frac{Bq}{c}}, \cdots, (\mathbf{y}^{\frac{Bq}{c}})^{n+m-1})$. This product is also the matrix of

$$\epsilon(f), \epsilon(f) z_1 \mathbf{y}^{\frac{Bq}{c}}, \dots, \epsilon(g), \epsilon(g) z_1 \mathbf{y}^{\frac{Bq}{c}}, \dots$$

in the basis $\{1, z_1, \cdots, z_1^{n+m-1}\}$. Then

$$\mathbf{y}^{(\sum_{i=1}^{n+m-1} i))\frac{Bq}{c}} \epsilon(R_z) = \mathbf{y}^{nN_f + mN_g + (\sum_{i=1}^{n-1} i + \sum_{i=1}^{m-1} i)\frac{Bq}{c}} R_{z_1}(\mathbf{y}). \qquad \square$$

Lemma 4.8. *Let f and g be two Weierstraß polynomials in $\mathbb{K}[[\mathbf{x}]][z]$ such that $F := fg$ is a ν-quasi-ordinary power series with first face Γ. If f and g separate on the face Γ, then they are comparable.*

Proof. There is a one-dimensional face which hits the z-axis. This face is the face Γ of the Newton polyhedron of at least one of the two power series. Assume Γ is a face of the Newton polyhedron of f. Either g has another face or the face of g is Γ but the polynomials f_Γ and g_Γ are coprime. We consider the map ϵ (associated to f) and in both cases the resultant of $f_1(0, z_1)$ and $g_1(0, z_1)$ is non zero. Then $R_{z_1}(f_1, g_1)$ is a unit and, using the Lemma 4.7, $R_z(f, g)$ is a monomial multiplied by a unit. $\square$

Proof of Theorem 1.6. We have shown in Lemma 4.8 that if f and g are separated on the first Newton polyhedron then they are comparable.

Now we follow a blue-red path on the Newton tree of fg until some blue or red path separates at a vertex v. Assume that this blue path starts at v with first edge γ; consider the Newton transform associated to γ. Let f_1 be the Newton transform of f; then f_1 factorises as $f_1 = f_{1,1}f_{1,2}$, where $f_{1,1}$ corresponds to the blue edge γ and $f_{1,2}$ corresponds to the factors associated to the other blue or blue-red edges starting at v (if any). Then $R_{z_1}(f_1, g_1) = R_{z_1}(f_{1,1}, g_1)R_{z_1}(f_{1,2}, g_1)$, and for $f_{1,1}, g_1$ the separation occurs on the first Newton polyhedron. Then $R_{z_1}(f_{1,1}, g_1)$ is a monomial times a unit. Our hypothesis is that along every path not ending with dead ends we have a separation in blue and red paths. Then finally every resultant is a monomial times a unit. □

5. Computation of resultants

Let $f(\mathbf{x}, z), g(\mathbf{x}, z) \in \mathbb{K}[[\mathbf{x}]][z]$ be power series and assume we are as in Notation 1.1. We asume that we are as in Section 3.

Now we want to compute the resultant from the Newton tree when f and g are comparable. To this aim we have to explain decorations on the Newton trees.

5.1. Decorations on Newton trees.

Definition 5.1. The numerical data $(((N_1, \cdots, N_d)), p)$ associated to each vertex v are called the *global numerical data* of the vertex v of the Newton tree of f, see (3.1).

Remark 5.2. The global numerical data $(((N_1, \cdots, N_d)), p)$ of the first vertex satisfy the following property. If ϵ is a Newton map associated to this vertex, then

$$\epsilon(f) = y_1^{\frac{N_1}{c_1}} \cdot \ldots \cdot y_d^{\frac{N_d}{c_d}} f_1(\mathbf{y}, z_1) = \mathbf{y}^{\frac{N}{c}} f_1(\mathbf{y}, z_1),$$

where no y_i divides f_1, $c_i := \gcd(p, q_i)$, see Notation 2.7 and (2.2).

Now instead of decorating the vertex and the edge under the vertex, we will decorate the edge above the vertex and the edge under the vertex.

We define these decorations inductively. Consider the first Newton polyhedron. Each 1-dimensional compact face has equations (3.1). We decorate the end of the edge above the corresponding vertex with the weight $(q_1, q_2, \cdots, q_d)$ and the end of the edge under the vertex is decorated, as before with p.

Now assume that v is a vertex whose edge above is decorated with $(Q_1, \cdots, Q_d)$ and edge under is decorated with p. Consider the Newton map associated to this vertex, and assume that the new Newton polyhedron has r 1-dimensional compact faces with equations

$$q_j^i\beta + p^j\alpha_i = N_j^i, i = 1, \cdots, d, j = 1, \cdots, r.$$

Define

$$Q_j^i := \frac{pQ_i}{\gcd(p, Q_i)}p^j + q_j^i. \tag{5.1}$$

Now we decorate the end of the edge above the j-th vertex with $(Q_j^1, \cdots, Q_j^d)$ and the end of the edge under the vertex with p^j. The ends are those near the vertex.

Definition 5.3. The numerical data $((Q_j^1, \cdots, Q_j^d), p^j)$ associated to each vertex v_j are called the *local numerical data* of the vertex v_j. For a black box, its decoration is the z-degree of the associated (non ν-quasi-ordinary power series).

Remark 5.4. We notice that the local numerical data verify:

$$Q_j^i > \frac{pQ_i}{\gcd(p,Q_i)} p^j, i = 1, \cdots, d, j = 1, \cdots, r.$$

We call this condition *the growth condition on the local numerical data.*

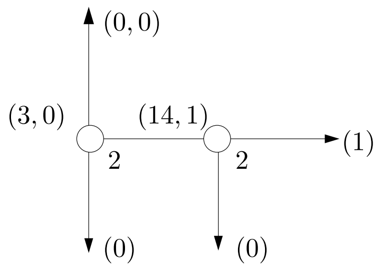

FIGURE 8. Local numerical data of Example 3.2

Remark 5.5. The local numerical data only use the slopes of the 1-dimensional compact faces of the Newton polyhedra. Hence if we consider the Newton tree of a factor of f decorated with the local numerical data then the decorations will be the same if we look at the part of the Newton tree of f corresponding to the factor. On the contrary the global numerical data use the position of each face in the Newton polyhedron of f, hence they change if we isolate a factor of f.

5.2. **Irreducible-like ν-quasi-ordinary power series.**

The Newton tree of an *irreducible-like* ν-quasi-ordinary power series f has only one end which is not a dead end. The Newton polyhedron of f has only one compact 1-dimensional face Γ and moreover, one can assume that $f_\Gamma = (z^p - x_1^{q_1} \cdot \ldots \cdot x_d^{q_d})^{p^1 \cdot \ldots \cdot p^g \cdot n_f}$. In particular the shape of its Newton tree is given in Figure 9. Let us fix its global decorations as in the Figure.

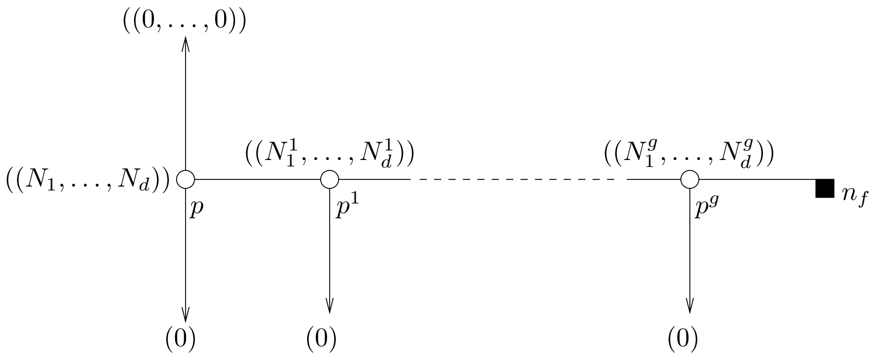

FIGURE 9.

Lemma 5.6. *Let f have only one end which is not a dead end. With the previous notations, then*

(1) *N_i is divisible by $p \cdot p^1 \cdot \ldots \cdot p^g$, for $1 \leq i \leq d$,*
(2) *N_i^j is divisible by $p^j \cdot \ldots \cdot p^g$, for $1 \leq i \leq d$ and for $1 \leq j \leq g$.*

Proof. The face of the first Newton polyhedron of f is supported by a line whose equation is: $\beta q_i + \alpha_i p = N_i$, for $i = 1, \dots, d$. Moreover, it has as vertex the point $(0, \dots, 0, p \cdot p^1 \cdot \dots \cdot p^g \cdot n_f)$. Since for all $i = 1, \dots, d$,

$$q_i \cdot p \cdot p^1 \cdot \dots \cdot p^g \cdot n_f = N_i, \tag{5.2}$$

then the first statement is proved.

Following the notations of (2.2), the corresponding Newton map σ is

$$\begin{array}{rccl} \sigma := \sigma_{\Gamma,1}: & \mathbb{K}[[\mathbf{x}]][z] & \longrightarrow & \mathbb{K}[[\mathbf{y}]][z_1] \\ & h(\mathbf{x}, z) & \mapsto & h(y_1^{p_1}, \dots, y_d^{p_d}, y_1^{q_1'} \cdot \dots \cdot y_d^{q_d'} \cdot (z_1 + 1)). \end{array}$$

Then $\sigma(f) = y_1^{\frac{N_1}{c_1}} \cdot \dots \cdot y_d^{\frac{N_d}{c_d}} \cdot g$, g is divided by no x_i. Now the compact face of the Newton polyhedron of g is supported by a line whose equation will be of the form $\beta q_i^1 + \alpha_i p^1 = N_i^1$, for $i = 1, \dots, d$, and goes through the point $(\frac{N_1}{c_1}, \dots, \frac{N_d}{c_d}, p^1 \cdot \dots \cdot p^g \cdot n_f)$. Then

$$p^1 \cdot \dots \cdot p^g \cdot n_f (q_i p_i p^1 + q_i^1) = N_i^1 \tag{5.3}$$

and N_i^1 is divisible by $p^1 \cdot \dots \cdot p^g$ for all i. The proof goes by induction. □

Lemma 5.7. *Let f have only one end which is not a dead end. With the previous notations, then the two numerical data associated to each given vertex v^j*

$$(((N_1^j, \dots, N_d^j)), p^j), \quad ((Q_1^j, \dots, Q_d^j), p^j)$$

satisfy

(1) $N_i = p \cdot p^1 \cdot \dots \cdot p^g \cdot Q_i$,
(2) $N_i^j = p^j \cdot \dots \cdot p^g \cdot Q_i^j$, $1 \leq i \leq d$,
(3) N_i *is divisible by* $p \cdot p^1 \cdot \dots \cdot p^g$, *for* $1 \leq i \leq d$ *and for* $1 \leq j \leq g$.

Proof. Let us consider the first vertex; in this case $Q_i = q_i$ and the result follows from (5.2). After the first Newton map, we deduce from (5.1) that $Q_i^1 = p_i q_i p^1 + q_i^1$. We deduce the formula from (5.3). The general proof follows by induction. □

5.3. **Computation of resultants.** Let f and g be two Weierstraß polynomials which are comparable power series and such that their Newton trees have only one end which is not a dead end. The main result in this section is to show that the resultant $R_z(f, g)$ can be read from the coloured Newton tree of fg decorated with its local numerical data. This result can be seen as a generalization of Corollaire 5 in [6] where P. González-Pérez gave information about the Newton polyhedron of the resultant of two quasi-ordinary hypersurfaces satisfying an appropriate non-degeneracy condition.

We consider the coloured Newton tree of fg decorated with its local numerical data. As we have seen, to compute the resultant we can use the factorisation of the Newton transforms even if they do not come from factorisations of f. For the sake of simplicity, after the separation of f and g we summarise the Newton tree with a coloured box.

Proposition 5.8. *Let f and g be two Weierstraß polynomials whose Newton trees have only one end which is not a dead end. If they separate on the first Newton polyhedron then*

$$R_z(f, g) = x_1^{M_1} \cdot \dots \cdot x_d^{M_d} \cdot u$$

where u is a unit and $(M_1, \cdots, M_d)$ is obtained by multiplication of the numbers adjacent (i.e. only neighbours) to the path going from the blue end (representing f) to the red end (representing g) including the decoration of the ends.

Taking into account Remark 2.11 we define the following concept.

Definition 5.9. Let us assume that f is ν-quasi-ordinary and let Γ be the edge associated to f. We say that a ν-quasi-ordinary power series $g(\mathbf{x}, z) \in \mathbb{K}[[\mathbf{x}]][z]$ is Γ-*compatible* if its z-degree m coincides with the order of $g(\mathbf{0}, z)$ and for a Newton map σ associated to Γ we have that $\mathbf{y}^{\frac{mB^q}{c}}$ is the maximal monomial dividing $\sigma(g)$.

The following result is straightforward.

Lemma 5.10. *If f and g are* Γ-compatible *of degrees n and m, and σ is a Newton map associated to Γ, then*

$$\epsilon(R_z) = \mathbf{y}^{nm\frac{B^q}{c}} R_{z_1}(\mathbf{y}).$$

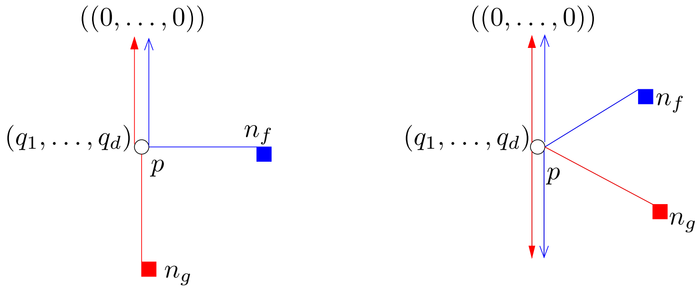

FIGURE 10. The right-hand case (resp. left-hand case) is denoted by R-case (resp. L-case)

Proof. The two possibilities appear in Figure 10. Let Γ be the edge associated to the first vertex and let ϵ be a Newton map associated to it. In both cases, f, g are Γ-compatibles, i.e., $\epsilon(f) = \mathbf{y}^{\deg f \frac{B^q}{c}} f_1$ and $\epsilon(g) = \mathbf{y}^{\deg g \frac{B^q}{c}} g_1$. Moreover, $R_{z_1}(f_1, g_1)$ is a unit since they separate at this edge.

In both cases $\deg f = pn_f$. For the same reasons, in the right hand case (R-case), $\deg g = pn_g$. In the left hand case (L-case) we have $\deg g = n_g$.

Applying Lemma 5.10, we obtain that there exists a unit $U(\mathbf{y}, z)$ such that:

$$\epsilon(R_z(f, g)) = \mathbf{y}^{\deg f \deg g \frac{B^q}{c}} U.$$

This equality implies that $R_z(f, g) = \mathbf{x}^M u(\mathbf{x}, z)$, where u is a unit, as in the statement. It remains to compute the value of M. Applying ϵ we deduce:

$$\mathbf{y}^{p\frac{M}{c}} = \mathbf{y}^{\deg f \deg g \frac{B^q}{c}}.$$

We obtain

$$M_i = \begin{cases} pn_f n_g q_i & \text{R-case,} \\ n_f n_g q_i & \text{L-case.} \end{cases}$$

Since in the L-case p is in the path and in the R-case p is adjacent to the path, we have proved the formula. □

Proposition 5.11. *Let $f, g \in \mathbb{K}[[\mathbf{x}]][z]$ be two comparable power series such that their Newton trees have only one end which is not a dead end. Then, $R_z(f, g)$ can be read from the coloured Newton tree of fg decorated with its local numerical data.*

More precisely, the multiplicity of x_i in $R_z(f, g)$ is computed as follows. Consider the path from the blue end representing f to the red end representing g. First, we take the product of all the decorations adjacent to the path (the i^{th} coordinate in the case of d-uples), including the decorations of these ends. Finally, we multiply it by the gcd *of the vertices before the vertex where they separate.*

Example 5.12. Let $a, b, c \in \mathbb{N}$, a odd. Let $\alpha, \beta, \gamma \in \mathbb{K}$. Let

$$\begin{aligned} f =& ((z^{2a} - x^{2b}y^{2c+1})^2 + x^{4b+1}y^{4c+3})^2 + x^{2b+2}y^{2c+2}(z^{2a} - x^{2b}y^{2c+1})^3 + \\ &+ x^{4b+1}y^{2(2c+1)+2}(z^{2a} - x^{2b}y^{2c+1})^2 \\ g =& (z^{2a} - \alpha x^{2b}y^{2c+1})^2 + z^{2a}(x^{2b+2}y^{2c+2} + x^{2b+1}y^{2c+3}). \end{aligned}$$

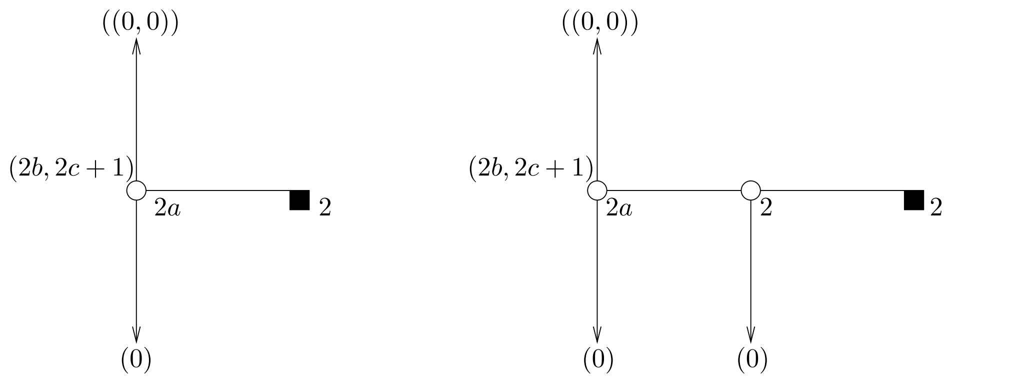

Figure 11. Tree of g on the left and tree of f on the right

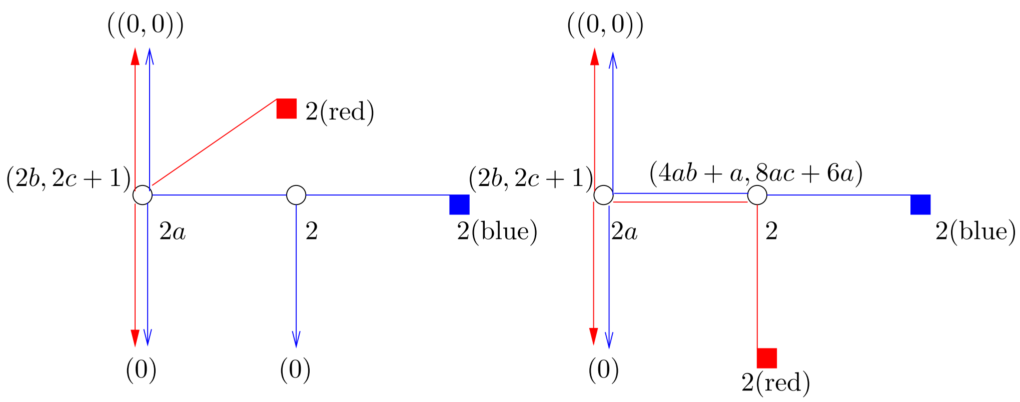

Figure 12. Tree of fg for $\alpha \neq 1$ (left) and $\alpha = 1$ (right)

Figure 11 shows the trees of f and g and Figure 12 shows the bi-coloured tree of fg. In both cases f and g are comparable. If $\alpha \neq 1$, $R_z(f, g) = x^{32ab}y^{16a(2c+1)}$ since the adjacent weights are the degrees of f and g at the black boxes and $2, 2, 2a$ and $(2b, 2c + 1)$. If $\alpha = 1$, $R_z(f, g) = x^{8(4ab+a)}y^{4(8ac+6a)}$.

Proof of Proposition 5.11. We proceed by induction on the number of Newton steps, after which the two series separate. Assume that f and g separate on the first Newton polyhedron. Proposition 5.8 proves this case.

Now we assume that f and g separate after i Newton maps. We will consider the L-case or the R-case to distinguish how they separate, see Figure 10. Let $p, p^1, \ldots, p^{i-1}$ be the common decorations of f and g. As in Proposition 5.8, we denote by n_f and n_g the degrees of the transforms of f and g after the i common Newton maps. If Γ is the first segment, both f and g are Γ-compatible and then we consider a Newton map ϵ associated to this segment.

For the sake of simplicity, when necessary *equality* means *equality up to a unit*. From Lemma 5.10, we have

$$\mathbf{y}^{p\frac{N}{c}} = \epsilon(R_z(f, g)) = \mathbf{y}^{mn\frac{B^q}{c}} R_{z_1}(f_1, g_1),$$

where $R_z(f,g)=\mathbf{x}^N$, $m:=\deg f=n_f p\prod_{j=1}^i p^j$ and

$$n:=\deg g=\begin{cases} n_g p\prod_{j=1}^i p^j & R\text{-case},\\ n_g p\prod_{j=1}^{i-1} p^j & L\text{-case}.\end{cases}$$

Since they do not separate at the first vertex, we have

$$f_1(\mathbf{0},z_1)=(z_1^p-\mu)^m,\quad g_1(\mathbf{0},z_1)=(z_1^p-\mu)^n,\quad \mu\in\mathbb{K}^*.$$

We may suppose that $\mu=1$. Let us denote $\hat{f}_1(\mathbf{y},z_1):=f_1(\mathbf{y},z_1+1)$; we construct $\hat{g}_1$ in the same way. Since the translation by any other p-root of unity acts in the same way, then $R_{z_1}(f_1,g_1)=(R_{z_1}(\hat{f}_1,\hat{g}_1))^p$. Let us assume that $R_{z_1}(\hat{f}_1,\hat{g}_1)=\mathbf{y}^{\hat{N}}$. We deduce that

$$N=\frac{mn}{p}B^q+c\cdot\hat{N}=c\cdot\hat{N}+\begin{cases} n_f n_g p\left(\prod_{j=1}^i p^j\right)^2 B^q & R\text{-case},\\ n_f n_g p\left(\prod_{j=1}^{i-1} p^j\right)^2 B^q & L\text{-case}.\end{cases}$$

By induction hypothesis, since $\hat{f}_1$ and $\hat{g}_1$ separate after $i-1$ Newton steps, we know that

$$\hat{N}=\begin{cases} n_f n_g p^i\left(\prod_{j=1}^{i-1} c^j\right)\cdot Q^{i,1} & R\text{-case},\\ n_f n_g\left(\prod_{j=1}^{i-1} c^j\right)\cdot Q^{i,1}, & L\text{-case}.\end{cases}$$

We denote by c^j the gcd-tuple of the j^{th} Newton step. The d-tuple Q^i corresponds to the local numerical data of the $(i-1)^{\text{th}}$ vertex for $\hat{f}_1,\hat{g}_1$, which is the i^{th} vertex for f,g.

It is easily seen that

$$Q^i=Q^{i,1}+p\prod_{j=1}^{i-1}\frac{B^q}{c\cdot\prod_{j=1}^{i-1}c^j}.$$

Putting all these data together, we obtain

$$N=\begin{cases} n_f n_g p^i Q^i\cdot c\cdot\prod_{j=1}^{i-1} c^j & R\text{-case},\\ n_f n_g Q^i\cdot c\cdot\prod_{j=1}^{i-1} c^j, & L\text{-case}.\end{cases}\qquad\square$$

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