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ON A CONJECTURE BY A. DURFEE

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Dedicated to María Ruas and to Terry Gaffney on the Occasion of their Sixteenth Birthday

Abstract. We show how *superisolated surface singularities* can be used to find a counterexample to the following conjecture by A. Durfee [8]: for a complex polynomial $f(x, y, z)$ in three variables vanishing at 0 with an isolated singularity there, "the local complex algebraic monodromy is of finite order if and only if a resolution of the germ $(\{f = 0\}, 0)$ has no cycles".

A pair (C_1, C_2) of reduced plane curves in $\mathbb{P}^2$ is called a *Zariski pair* if it satisfies the following conditions: (1) There exist tubular neighborhoods $T(C_i)$ (i = 1, 2) and a homeomorphism $h : T(C_1) \to T(C_2)$ such that $h(C_1) = C_2$. (2) There exists no homeomorphism $f : \mathbb{P}^2 \to \mathbb{P}^2$ such that $f(C_1) = C_2$. We show a Zariski pair whose corresponding superisolated surface singularities, one has complex algebraic monodromy of finite order and the other not (answering a question by J. Stevens).

Introduction

In this paper we show an example of a *superisolated surface singularities* $(V, 0) \subset (\mathbb{C}^3, 0)$ such that a resolution of the germ $(V, 0)$ has no cycles and the local complex algebraic monodromy of the germ $(V, 0)$ is not of finite order, contradicting a conjecture proposed by A. Durfee [8].

For completness in the first section we recall well known results about monodromy of the Milnor fibration, about normal surface singularities and state the question by Durfee.

In the second section we recall results on superisolated surface singularities and with them we study in detail the counterexample.

Definition 0.1. A pair (C_1, C_2) of reduced plane curves in $\mathbb{P}^2$ is called a *Zariski pair* if it satisfies the following conditions:

(1) There exist tubular neighborhoods $T(C_i)$ (i = 1, 2) and a homeomorphism $h : T(C_1) \to T(C_2)$ such that $h(C_1) = C_2$.

(2) There exists no homeomorphism $f : \mathbb{P}^2 \to \mathbb{P}^2$ such that $f(C_1) = C_2$.

In the last section we show a Zariski pair (C_1, C_2) of curves of degree d given by homogeneous polynomials $f_1(x, y, z)$ and $f_2(x, y, z)$ whose corresponding superisolated surface singularities $(V_1, 0) = (\{f_1(x, y, z) + l^{d+1} = 0\}, 0) \subset (\mathbb{C}^3, 0)$ and $(V_2, 0) = (\{f_2(x, y, z) + l^{d+1} = 0\}, 0) \subset (\mathbb{C}^3, 0)$ (l is a generic hyperplane) satisfy: 1) $(V_1, 0)$ has complex algebraic monodromy of finite order and 2) $(V_2, 0)$ has complex algebraic monodromy of infinite order (answering a cuestion by J. Stevens).

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1. Invariants of singularities

1.1. **Monodromy of the Milnor fibration.** Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be an analytic function defining a germ $(V, 0) := (f^{-1}\{0\}, 0) \subset (\mathbb{C}^{n+1}, 0)$ of a hypersurface singularity. The *Milnor fibration* of the holomorphic function f at 0 is the C^∞ locally trivial fibration $f| : B_\varepsilon(0) \cap f^{-1}(\mathbb{D}^*_\eta) \to \mathbb{D}^*_\eta$, where $B_\varepsilon(0)$ is the open ball of radius ε centered at 0, $\mathbb{D}_\eta = \{z \in \mathbb{C} : |z| < \eta\}$ and $\mathbb{D}^*_\eta$ is the open punctured disk ($0 < \eta \ll \varepsilon$ and ε small enough). Milnor's classical result also shows that the topology of the germ $(V, 0)$ in $(\mathbb{C}^{n+1}, 0)$ is determined by the pair $(S^{2n+1}_\varepsilon, L^{2n-1}_V)$, where $S^{2n+1} = \partial B_\varepsilon(0)$ and $L^{2n-1}_V := S^{2n+1}_\varepsilon \cap V$ is the *link* of the singularity.

Any fiber $F_{f,0}$ of the Milnor fibration is called the *Milnor fiber* of f at 0. The *monodromy transformation* $h : F_{f,0} \to F_{f,0}$ is the well-defined (up to isotopy) diffeomorphism of $F_{f,0}$ induced by a small loop around $0 \in \mathbb{D}_\eta$. The *complex algebraic monodromy of* f *at* 0 is the corresponding linear transformation $h_* : H_*(F_{f,0}, \mathbb{C}) \to H_*(F_{f,0}, \mathbb{C})$ on the homology groups.

If $(V, 0)$ defines a germ of isolated hypersurface singularity then $\tilde{H}_j(F_{f,0}, \mathbb{C}) = 0$ but for $j = 0, n$. In particular the non-trivial complex algebraic monodromy will be denoted by $h : H_n(F_{f,0}, \mathbb{C}) \to H_n(F_{f,0}, \mathbb{C})$ and $\Delta_V(t)$ will denote its characteristic polynomial.

1.2. **Monodromy Theorem and its supplements.** The following are the main properties of the monodromy operator, see e.g. [11]:

(a) $\Delta_V(t)$ is a product of cyclotomic polynomials.
(b) Let N be the maximal size of the Jordan blocks of h, then $N \leq n + 1$.
(c) Let N_1 be the maximal size of the Jordan blocks of h for the eigenvalue 1, then $N_1 \leq n$.
(d) The monodromy h is called of *finite order* if there exists $N > 0$ such that $h^N = Id$. Lê D.T. [12] proved that the monodromy of an irreducible plane curve singularity is of finite type.
(e) This result was extended by van Doorn and Steenbrink [7] who showed that if h has a Jordan block of maximal size $n+1$ then $N_1 = n$, i.e. there exist a Jordan block of h of maximal size n for the eigenvalue 1.

Milnor proved that the link L^{2n-1}_V is $(n-2)$-connected. Thus the link is an *integer (resp. rational) homology* $(2n-1)$*-sphere* if $H_{n-1}(L^{2n-1}_V, \mathbb{Z}) = 0$ (resp. $H_{n-1}(L^{2n-1}_V, \mathbb{Q}) = 0$). These can be characterized using the Wang's exact sequence which reads as (see e.g. [19, 21]):

$$0 \to H_n(L^{2n-1}_V, \mathbb{Z}) \to H_n(F_{f,0}, \mathbb{Z}) \xrightarrow{h-id} H_n(F_{f,0}, \mathbb{Z}) \to H_{n-1}(L^{2n-1}_V, \mathbb{Z}) \to 0.$$

Thus $\operatorname{rank} H_n(L^{2n-1}_V) = \operatorname{rank} H_{n-1}(L^{2n-1}_V) = \dim \ker(h - id)$ and:

- L^{2n-1}_V is a rational homology $(2n-1)$-sphere $\Longleftrightarrow \quad \Delta_V(1) \neq 0$,
- L^{2n-1}_V is an integer homology $(2n-1)$-sphere $\Longleftrightarrow \quad \Delta_V(1) = \pm 1$.

1.3. **Normal surface singularities.** Let $(V, 0) = (\{f_1 = \ldots = f_m = 0\}, 0) \subset (\mathbb{C}^N, 0)$ be a normal surface singularity with link $L_V := V \cap S^{2N-1}_\varepsilon$, L_V is a a connected compact oriented 3-manifold. Since $V \cap B_\varepsilon$ is a cone over the link L_V then L_V characterizes the topological type of $(V, 0)$. The link L_V is called a *rational homology sphere* ($\mathbb{Q}$HS) if $H_1(L_V, \mathbb{Q}) = 0$, and L_V is called an *integer homology sphere* ($\mathbb{Z}$HS) if $H_1(L_V, \mathbb{Z}) = 0$. One of the main problems in the stufy of normal surfaces is to determine which analytical properties of $(V, 0)$ can be read from the topology of the singularity, see the very nice survey paper by A. Nemethi [20].

The resolution graph $\Gamma(\pi)$ of a resolution $\pi : \tilde{V} \to V$ allows to relate analytical and topological properties of V. W. Neumann [22] proved that the information carried in any resolution graph is the same as the information carried by the link L_V. Let $\pi : \tilde{V} \to V$ be a *good* resolution of the singular point $0 \in V$. Good means that $E = \pi^{-1}\{0\}$ is a normal crossing divisor. Let $\Gamma(\pi)$ be the dual graph of the resolution (each vertex decorated with the genus $g(E_i)$ and the self-intersection E_i^2 of E_i in $\tilde{V}$). Mumford proved that the intersection matrix $I = \{E_i \cdot E_j\}$ is negative definite and Greuert proved the converse, i.e., any such graph comes from the resolution of a normal surface singularity.

Considering the exact sequence of the pair $(\tilde{V}, E)$ and using I is non-degenerated then

$$0 \longrightarrow \operatorname{coker} I \longrightarrow H_1(L_V, \mathbb{Z}) \longrightarrow H_1(E, \mathbb{Z}) \longrightarrow 0$$

and $\operatorname{rank} H_1(E) = \operatorname{rank} H_1(L_V)$. In fact L_V is a $\mathbb{Q}$HS if and only if $\Gamma(\pi)$ is a tree and every E_i is a rational curve. If additionally I has determinant ± 1 then L_V is an $\mathbb{Z}$HS.

1.4. **Number of cycles in the exceptional set E and Durfee's conjecture.** In general one gets

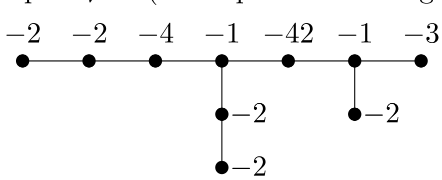
$$\operatorname{rank} H_1(L_V) = \operatorname{rank} H_1(\Gamma(\pi)) + 2\sum_i g(E_i),$$

where $\operatorname{rank} H_1(\Gamma(\pi))$ is the number of independent cycles of the graph $\Gamma(\pi)$. Let $n : \tilde{E} \to E$ be the normalization of E. A.H. Durfee showed in [8] that the *number of cycles* $c(E)$ *in* E, i.e. $c(E) = \operatorname{rank} H_1(E) - \operatorname{rank} H_1(\tilde{E})$, does not depend on the resolution and in fact it is equal to $c(E) = \operatorname{rank} H_1(\Gamma(\pi))$. Therefore, E contains cycles only when its irreducible components E_i intersects (including self-intersections) to form circular chains.

A. Durfee in [8] proposed the following conjecture: for a complex polynomial $f(x,y,z)$ in three variables vanishing at 0 with an isolated singularity there, "the local complex algebraic monodromy h is of finite order if and only if a resolution of the germ $(\{f = 0\}, 0)$ has no cycles". He showed that the conjecture is true in the following two cases:

(1) if f is weighted homogeneuos (the resolution graph is star-shaped and the monodromy is finite)
(2) if $f = g(x,y) + z^n$. Using Thom-Sebastiani [27], the monodromy of f is finite if and only if the monodromy of g is finite. Theorem 3 in [8] proves that monodromy of f is finite if and only a resolution of f has no cycles.

1.5. **Main result.** In this paper we show that the conjecture is not true in general, and for that we use superisolated surface singularities. Let $(V,0) \subset (\mathbb{C}^3, 0)$ be the germ of normal surface singularity defined by $f := (xz - y^2)^3 - ((y-x)x^2)^2 + z^7 = 0$. Then the minimal good resolution graph Γ_V of (the superisolated singularity) $(V,0)$ is

Γ_V :


where every dot denotes a rational non-singular curve with the corresponding self-intersection. Thus the link L_V is a rational homology sphere and in particular this graph is a tree, i.e. it has no cycles. But the complex algebraic monodromy of f at 0 has not finite order because there exists a Jordan block of size 2×2 for an eigenvalue $\neq 1$.

2. Superisolated surface singularities

Definition 2.1. A hypersurface surface singularity $(V,0)\subset(\mathbb{C}^3,0)$ defined as the zero locus of $f=f_d+f_{d+1}+\cdots\in\mathbb{C}\{x,y,z\}$ (where f_j is homogeneous of degree j) is *superisolated*, SIS for short, if the singular points of the complex projective plane curve $C:=\{f_d=0\}\subset\mathbb{P}^2$ are not situated on the projective curve $\{f_{d+1}=0\}$, that is $\operatorname{Sing}(C)\cap\{f_{d+1}=0\}=\emptyset$. Note that C must be reduced.

The SIS were introduced by I. Luengo in [17] to study the μ-constant stratum. The main idea is that for a SIS the embedded topological type (and the equisingular type) of $(V,0)$ does not depend on the choice of f_j's (for $j>d$, as long as f_{d+1} satisfies the above requirement), e.g. one can take $f_j=0$ for any $j>d+1$ and $f_{d+1}=l^{d+1}$ where l is a linear form not vanishing at the singular points [18].

2.1. **The minimal resolution of a SIS.** Let $\pi:\tilde{V}\to V$ be the monoidal transformation centered at the maximal ideal $\mathfrak{m}\subset\mathcal{O}_V$ of the local ring of V at 0. Then $(V,0)$ is a SIS if and only if $\tilde{V}$ is a non-singular surface. Thus π is the *minimal resolution* of $(V,0)$. To construct the resolution graph $\Gamma(\pi)$ consider $C=C_1+\ldots+C_r$ the decomposition in irreducible components of the reduced curve C in $\mathbb{P}^2$. Let d_i (resp. g_i) be the degree (resp. genus) of the curve C_i in $\mathbb{P}^2$. Then $\pi^{-1}\{0\}\cong C=C_1+\ldots+C_r$ and the self-intersection of C_i in $\tilde{V}$ is $C_i\cdot C_i=-d_i(d-d_i+1)$, [17, Lemma 3]. Since the link L_V can be identified with the boundary of a regular neighbourhood of $\pi^{-1}\{0\}$ in $\tilde{V}$ then the topology of the tangent cone determines the topology of the abstract link L_V [17].

2.2. **The minimal good resolution of a SIS.** The minimal good resolution of a SIS $(V,0)$ is obtained after π by doing the minimal embedded resolution of each plane curve singularity $(C,P)\subset(\mathbb{P}^2,P)$, $P\in\operatorname{Sing}(C)$. This means that the support of the minimal good resolution graph Γ_V is the same as the minimal embedded resolution graph Γ_C of the projective plane curve C in $\mathbb{P}^2$. The decorations of the minimal good resolution graph Γ_V are as follows:

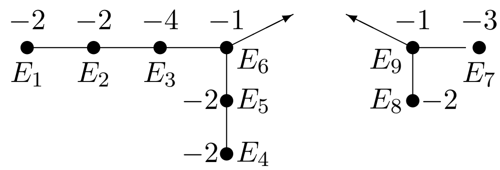
1) the genus of (the strict trasnform of) each irreducible component C_i of C is a birational invariant and then one can computed it as an embedded curve in $\mathbb{P}^2$. All the other curves are non-singular rational curves.

2) Let C_j be an irreducible component of C such that $P\in C_j$ and with multiplicity $n\geq 1$ at P. After blowing-up at P, the new self-intersection of the (strict transform of the) curve C_j in the (strict transform of the) surface $\tilde{V}$ is $C_j^2-n^2$. In this way one constructs the minimal good resolution graph Γ of $(V,0)$.

In particular the theory of hypersurface superisolated surface singularities "contains" in a canonical way the theory of complex projective plane curves.

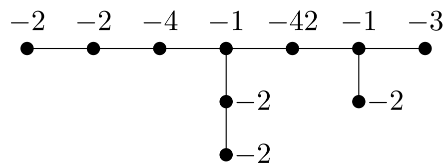
Example 2.2. If $(V,0)\subset(\mathbb{C}^3,0)$ is a SIS with an irreducible tangent cone $C\subset\mathbb{P}^2$ then L_V is a rational homology sphere if and only if C is a rational curve and each of its singularities (C,p) is locally irreducible, i.e a cusp.

Example 2.3. For instance, if $f=f_6+z^7$ is given by the equation $f_6=(xz-y^2)^3-((y-x)x^2)^2$. The plane projective curve C defined by $f_6=0$ is irreducible with two singular points: $P_1=[0:0:1]$ (with a singularity of local singularity type u^3-v^{10}) and $P_2=[1:1:1]$ (with a singularity of local singularity type $\mathbb{A}_2$) which are locally irreducible. Let $\pi:X\to\mathbb{P}^2$ be the minimal embedded resolution of C at its singular points P_1, P_2. Let $E_i, i\in I$, be the irreducible components of the divisor $\pi^{-1}(f^{-1}\{0\})$.



The minimal good resolution graph Γ_V of the superisolated singularity $(V,0)$ is

Γ_V :



2.3. **The embedded resolution of a SIS.** In [2], the first author has studied, for SIS, the Mixed Hodge Structure of the cohomology of the Milnor fibre introduced by Steenbrink and Varchenko, see e.g. [28]. For that he constructed in an effective way an embedded resolution of a SIS and described the MHS in geometric terms depending on invariants of the pair $(\mathbb{P}^2, C)$.

The first author determined the Jordan form of the complex monodromy on $H_2(F_{f,0},\mathbb{C})$ of a SIS. Let $\Delta_V(t)$ be the corresponding characteristic polynomial of the complex monodromy on $H_2(F_{f,0},\mathbb{C})$. Denote by $\mu(V,0) = \deg(\Delta_V(t))$ the Milnor number of $(V,0) \subset (\mathbb{C}^3,0)$.

Let $\Delta^P(t)$ be the characteristic polynomial (or Alexander polynomial) of the action of the complex monodromy of the germ (C,P) on $H_1(F_{g^P},\mathbb{C})$, (where g^P is a local equation of C at P and F_{g^P} denotes the corresponding Milnor fiber). Let μ^P be the Milnor number of C at P. Recall that if $n^P : \tilde{C}^P \to (C,P)$ is the normalization map then $\mu^P = 2\delta^p - (r^p - 1)$, where $\delta^P := \dim_{\mathbb{C}} n^P_*(\mathcal{O}_{\tilde{C}^P})/\mathcal{O}_{C,P}$ and r^p is the number of local irreducible components of C at P.

Let H be a $\mathbb{C}$-vector space and $\varphi : H \to H$ an endomorphism of H. The i-th Jordan polynomial of φ, denoted by $\Delta_i(t)$, is the monic polynomial such that for each $\zeta \in \mathbb{C}$, the multiplicity of ζ as a root of $\Delta_i(t)$ is equal to the number of Jordan blocks of size $i+1$ with eigenvalue equal to ζ.

Let Δ_1 and Δ_2 be the first and the second Jordan polynomials of the complex monodromy on $H_2(F_{f,0},\mathbb{C})$ of V and let Δ_1^P be the first Jordan polynomial of the complex monodromy of the local plane singularity (C,P). After the Monodromy Theorem these polynomials joint with $\Delta_V(t)$ and Δ^P, $P \in \mathrm{Sing}(C)$, determine de corresponding Jordan form of the complex monodromy. Let us denote the Alexander polynomial of the plane curve C in $\mathbb{P}^2$ by $\Delta_C(t)$, it was introduced by A. Libgober [13, 14] and F. Loeser and Vaquié [16].

Theorem 2.4 ([2]). *Let $(V,0)$ be a SIS whose tangent cone $C = C_1 \cup \ldots \cup C_r$ has r irreducible components and degree d. Then the Jordan form of the complex monodromy on $H_2(F_{f,0},\mathbb{C})$ is determined by the following polynomials*

(i) *The characteristic polynomial $\Delta_V(t)$ is equal to*

$$\Delta_V(t) = \frac{(t^d-1)^{\chi(\mathbb{P}^2\setminus C)}}{(t-1)} \prod_{P\in\mathrm{Sing}(C)} \Delta^P(t^{d+1}).$$

(ii) *The first Jordan polynomial is equal to*

$$\Delta_1(t) = \frac{1}{\Delta_C(t)(t-1)^{r-1}} \prod_{P\in\mathrm{Sing}(C)} \frac{\Delta_1^P(t^{d+1})\Delta_{(d)}^P(t)}{\Delta_{1,(d)}^P(t)^3},$$

where $\Delta_{(d)}^P(t) := \gcd(\Delta^P(t),(t^d-1)^{\mu^P})$ *and* $\Delta_{1,(d)}^P(t) := \gcd(\Delta_1^P(t),(t^d-1)^{\mu^P})$.

(iii) *The second Jordan polynomial is equal to*

$$\Delta_2(t) = \prod_{P\in\mathrm{Sing}(C)} \Delta_{1,(d)}^P(t).$$

Corollary 2.5 ([2, Corollaire 5.5.4])**.** *The number of Jordan blocks of size 2 for the eigenvalue* 1 *of the complex monodromy* h *is equal to*

$$\sum_{P\in\mathrm{Sing}(C)} (r^P-1)-(r-1). \tag{2.1}$$

Let $\tilde{D}_i$ be the normalization of D_i and $\tilde{C}$ the disjoint union the $\tilde{D}_i$ and $n:\tilde{C}\to C$ be the projection map. Thus the first Betti number of $\tilde{C}$ is $2g := 2\sum_i g(D_i)$ and the first Betti number of C is $2g+\sum_{P\in\mathrm{Sing}(C)}(r^P-1)-r+1$. Then $\sum_{P\in\mathrm{Sing}(C)}(r^P-1)-(r-1)$ is exactly the difference the first Betti numbers of C and $\tilde{C}$. In fact this non-negative integer is equal to the first Betti number of the minimal embedded resolution graph Γ_C of the projective plane curve C in $\mathbb{P}^2$, which is nothing but $\operatorname{rank} H_1(\Gamma_V)$.

Corollary 2.6. *Let* $(V,0)$ *be a SIS whose tangent cone* $C = C_1\cup\ldots\cup C_r$ *has* r *irreducible components. Then the number of independent cycles* $c(E) = \operatorname{rank} H_1(\Gamma_V) = \sum_{P\in\mathrm{Sing}(C)}(r^P-1)-(r-1)$.

In particular E *has no cycles if and only if* $\sum_{P\in\mathrm{Sing}(C)}(r^P-1) = (r-1)$ *if and only if the complex monodromy* h *has no Jordan blocks of size 2 for the eigenvalue* 1.

Corollary 2.7 ([2, Corollaire 4.3.2])**.** *If for every* $P\in\mathrm{Sing}(C)$, *the local monodromy of the local plane curve equation* g^p *at* P *acting on the homology* $H_1(F_{g^P},\mathbb{C})$ *of the Milnor fibre* F_{g^P} *has no Jordan blocks of maximal size* 2 *then the corresponding SIS has no Jordan locks of size 3.*

Corollary 2.8. *Let* $(V,0)\subset(\mathbb{C}^3,0)$ *be a SIS with a rational irreducible tangent cone* $C\subset\mathbb{P}^2$ *of degree* d *whose singularities are locally irreducible. Then:*

(1) the link L_V *is a* $\mathbb{Q}HS$ *link and* E *has no cycles,*
(2) the complex monodromy on $H_2(F_{f,0},\mathbb{C})$ *has no Jordan blocks of size 2 for the eigenvalue* 1,
(3) the complex monodromy on $H_2(F_{f,0},\mathbb{C})$ *has no Jordan blocks of size 3.*
(4) The first Jordan polynomial is equal to

$$\Delta_1(t) = \frac{1}{\Delta_C(t)} \prod_{P\in\mathrm{Sing}(C)} \gcd(\Delta^P(t),(t^d-1)^{\mu^P}).$$

The proof follows from the previous description and the fact that if every $P\in\mathrm{Sing}(C)$ is locally irreducible then by Lê D.T. result (see 1.2) the plane curve singularity has finite order and $\Delta_1^P(t)=1$.

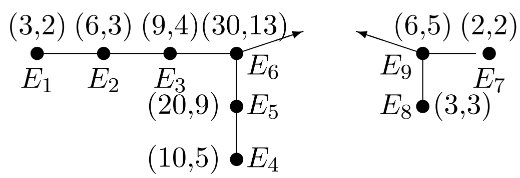
Corollary 2.9. *Let* $(V,0)\subset(\mathbb{C}^3,0)$ *be a SIS whose tangent cone* $C=C_1\cup\ldots\cup C_r$ *has* r *irreducible components. Assume that* $\sum_{P\in\mathrm{Sing}(C)}(r^P-1)=(r-1)$, *then:*

(1) E has no cycles,
(2) the complex monodromy on $H_2(F_{f,0},\mathbb{C})$ has no Jordan blocks of size 2 for the eigenvalue 1*,*
(3) the complex monodromy on $H_2(F_{f,0},\mathbb{C})$ has no Jordan blocks of size 3.
(4) The first Jordan polynomial is equal to

$$\Delta_1(t) = \frac{1}{\Delta_C(t)(t-1)^{r-1}} \prod_{P\in \mathrm{Sing}(C)} \gcd(\Delta^P(t), (t^d-1)^{\mu^P}).$$

The proof follows from Corollary 2.6 and the part (e) Monodromy Theorem 1.2.

2.4. **The first Jordan polynomial in Example 2.3.** As we described above, the plane projective curve C defined by $f_6 = (xz-y^2)^3 - ((y-x)x^2)^3 = 0$ is irreducible, rational and with two singular points: $P_1 = [0:0:1]$ (with a singularity of local singularity type $u^3 - v^{10}$) and $P_2 = [1:1:1]$ (with a singularity of local singularity type $\mathbb{A}_2$) which are unibranched. Let $\pi : X \to \mathbb{P}^2$ be the minimal embedded resolution of C at its singular points P_1, P_2. Let $E_i, i\in I$, be the irreducible components of the divisor $\pi^{-1}(f^{-1}\{0\})$. For each $j \in I$, we denote by N_j the multiplicity of E_j in the divisor of the function $f\circ\pi$ and we denote by $\nu_j - 1$ the multiplicity of E_j in the divisor of $\pi^*(\omega)$ where ω is the non-vanishing holomorpic 2-form $dx\wedge dy$ in $\mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty$. Then the divisor $\pi^*(C)$ is a normal crossing divisor. We attach to each exceptional divisor E_i its numerical data (N_i, ν_i).



Thus $\Delta^{P_1}(t) = \frac{(t-1)(t^{30}-1)}{(t^3-1)(t^{10}-1)} = \phi_{30}\phi_{15}\phi_6$ and $\Delta^{P_2}(t) = \frac{(t-1)(t^6-1)}{(t^3-1)(t^2-1)} = \phi_6$, where ϕ_k is the k-th cyclotomic polynomial. Thus, by Corollary 2.8, the only possible eigenvalues of with Jordan blocks of zize 2 are the roots of the polynomial $\Delta_1(t) = \frac{\phi_6^2}{\Delta_C(t)}$.

The proof of our main result will be finished if we show that the Alexander polynomial $\Delta_C(t) = \phi_6$. The Alexander polynomial, in particular of sextics, has been longly investigated by Artal-Bartolo [1], Artal-Bartolo and Carmona-Rúber [3], Degtyarev [6], Oka [24], Pho [25], Zariski [29] among others. In [23] Corollary 18, I.2, it is proved that $\Delta_C(t) = \phi_6$.

Consider a generic line L_∞ in $\mathbb{P}^2$, in our example the line $z=0$ is generic, and define $f(x,y) = f_6(x,y,1)$. Consider the (global) Milnor fibration given by the homogeneous polynomial $f_6 : \mathbb{C}^3 \to \mathbb{C}$ with Milnor fibre F. Randell [26] proved that $\Delta_C(t)(t-1)^{r-1}$ is the characteristic polynomial of the algebraic monodromy acting on $F : T_1 : H_1(F,\mathbb{C}) \to H_1(F,\mathbb{C})$.

Lemma 2.10 (Divisibility properties,[13])**.** *The Alexander polinomial $\Delta_C(t)(t-1)^{r-1}$ divides $\prod_{P\in\mathrm{Sing}(C)} \Delta^P(t)$ and the Alexander polynomial at infinity $(t^d-1)^{d-2}(t-1)$. In particular the roots of the Alexander polynomial are d-roots of unity.*

To compute the Alexander polynomial $\Delta_C(t)$ we combined the method described in [1] with the methods given in [13], [16] and [9].

Consider for $k = 1,\dots,d-1$ the ideal sheaf $\mathcal{I}^k$ on $\mathbb{P}^2$ defined as follows:

- If $Q \in \mathbb{P}^2 \setminus \mathrm{Sing}(C)$ then $\mathcal{I}^k_Q = \mathcal{O}_{\mathbb{P}^2,Q}$.

- If $P \in \mathrm{Sing}(C)$ then $\mathcal{I}_P^k$ is the following ideal of $\mathcal{O}_{\mathbb{P}^2,P}$: if $h \in \mathcal{O}_{\mathbb{P}^2,P}$ then $h \in \mathcal{I}_P^k$ if and only if the vanishing order of $\pi^*(h)$ along each E_i is, at least, $-(\nu_i - 1) + [\frac{kN_i}{d}]$ (where [.] stands for the integer part of a real number).

For $k \geq 0$ the following map

$$\sigma_k : H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k-3)) \to \bigoplus_{P \in \mathrm{Sing}(C)} \mathcal{O}_{\mathbb{P}^2,P}/\mathcal{I}_P^k : h \mapsto (h_P + \mathcal{I}_P^k)_{P \in \mathrm{Sing}(C)}$$

is well defined (up to scalars) and the result of [13] and [16] reinterpreted in this language as [1] and [9] reads as follows:

Theorem 2.11 (Libgober, Loeser-Vaquié)**.**

$$\Delta_C(t) = \prod_{k=1}^{d-1} (\Delta^k(t))^{l_k}, \tag{2.2}$$

where $\Delta^k(t) := (t - \exp(\frac{2k\pi i}{d})(t - \exp(\frac{-2k\pi i}{d})$ *and* $l_k = \dim \operatorname{coker} \sigma_k$

In our case, by the Divisibility properties (Lemma 2.10), $\Delta_C(t)$ divides $\Delta^{P_1}(t)\Delta^{P_2}(t) = \phi_{30}\phi_{15}\phi_6^2$. Thus, by Theorem 2.11, we are only interested in the case $k = 1$ and 5, $\Delta^1(t) = \Delta^5(t) = \phi_6 = (t^2 - t - 1)$. In case $k = 1$, we have $l_1 = 0$.

In case $k = 5$, the ideal $\mathcal{I}_{P_1}^5$ is the following ideal of $\mathcal{O}_{\mathbb{P}^2,P_1}$:

$$\mathcal{I}_{P_1}^5 = \{h \in \mathcal{O}_{\mathbb{P}^2,P_1} : (\pi^* h) \geq E_1 + 3E_2 + 4E_3 + 4E_4 + 8E_5 + 13E_6\}$$

and with the local change of coordinates $u = x - y^2, w = y$, the generators of the ideal are $\mathcal{I}_{P_1}^5 =< uw, u^2, w^5 >$ and the dimension of the quotient vector space $\mathcal{O}_{\mathbb{P}^2,P_1}/\mathcal{I}_{P_1}^5$ is 6. A basis is given by $1, u, w, w^2, w^3, w^4$. The ideal

$$\mathcal{I}_{P_2}^5 = \{h \in \mathcal{O}_{\mathbb{P}^2,P_2} : (\pi^* h) \geq 0E_7 + 0E_8 + E_9\} = \mathfrak{m}_{\mathbb{P}^2,P_2}$$

and the dimension of the quotient vector space $\mathcal{O}_{\mathbb{P}^2,P_2}/\mathcal{I}_{P_2}^5$ is 1. A basis is given by 1.

If we take as a basis for the space of conics $1, x, y, x^2, y^2, xy$, the map σ_5

$$\sigma_5 : H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) \to \mathcal{O}_{\mathbb{P}^2,P_1}/\mathcal{I}_{P_1}^5 \times \mathcal{O}_{\mathbb{P}^2,P_2}/\mathcal{I}_{P_2}^5 = \mathbb{C}^6 \times \mathbb{C} : h \mapsto (h + \mathcal{I}_{P_1}^5, h + \mathcal{I}_{P_2}^5)$$

is given in such coordinates by (using $u = x - y^2$)

$$\begin{array}{llll}
\sigma_5(1) & = (1,0,0,0,0,0,1) & \sigma_5(x) & = (0,1,0,1,0,0,1) \\
\sigma_5(y) & = (0,0,1,0,0,0,1) & \sigma_5(x^2) & = (0,0,0,0,0,1,1) \\
\sigma_5(y^2) & = (0,0,0,1,0,0,1) & \sigma_5(xy) & = (0,0,0,0,1,0,1),
\end{array}$$

and then σ_5 is injective and $\dim \operatorname{coker} \sigma_5 = 7 - 6 + 0 = 1$. The key point is that $u \notin \mathcal{I}_{P_1}^5$.

3. Zariski pairs

Let us consider $C \subset \mathbb{P}^2$ a reduced projective curve of degree d defined by an equation $f_d(x,y,z) = 0$. If $(V,0) \subset (\mathbb{C}^3, 0)$ is a SIS with tangent cone C, then the link L_V of the singularity is completely determined by C. Let us recall, that L_V is a Waldhausen manifold and its plumbing graph is the dual graph of the good minimal resolution. In order to determine L_V we do not need the embedding of C in $\mathbb{P}^2$, but only its embedding in a regular neighborhood. The needed data can be encoded in a combinatorial way.

Definition 3.1. Let $\mathrm{Irr}(C)$ be the set of irreducible components of C. For $P \in \mathrm{Sing}(C)$, let $B(P)$ be the set of local irreducible components of C. The *combinatorial type* of C is given by:

- A mapping $\deg : \mathrm{Irr}(C) \to \mathbb{Z}$, given by the degrees of the irreducible components of C.
- A mapping top : $\mathrm{Sing}(C) \to \mathrm{Top}$, where Top is the set of topological types of singular points. The image of a singular point is its topological type.
- For each $P \in \mathrm{Sing}(C)$, a mapping $\beta_P : T(P) \to \mathrm{Irr}(C)$ such that if γ is a branch of C at P, then $\beta_)(\gamma)$ is the global irreducible component containing γ.

Remark 3.2. There is a natural notion of isomorphism of combinatorial types. It is easily seen that combinatorial type determines and is determined by any of the following graphs (with vertices decorated with self-intersections):

- The dual graph of the preimage of C by the minimal resolution of $\mathrm{Sing}'(C)$. The set $\mathrm{Sing}'(C)$ is obtained from $\mathrm{Sing}(C)$ by forgetting ordinary double points whose branches belong to distinct global irreducible components. We need to mark in the graph the r vertices corresponding to $\mathrm{Irr}(C)$.
- The dual graph of the minimal good minimal of V. Since minimal intersection is unique, it is not necessary to mark vertices.

Note also that the combinatorial type determine the characteristic polynomial $\Delta_V(t)$ of V (see Theorem 2.4).

Definition 3.3. A *Zariski pair* is a set of two curves $C_1, C_2 \subset \mathbb{P}^2$ with the same combinatorial type but such that $(\mathbb{P}^2, C_1)$ is not homeomorphic to $(\mathbb{P}^2, C_2)$. An *Alexander-Zariski pair* $\{C_1, C_2\}$ is a Zariski pair such that the Alexander polynomials of C_1 and C_2 do not coincide.

In [2], (see here Theorem 2.4) it is shown that Jordan form of complex monodromy of a SIS is determined by the combinatorial type and the Alexander polynomial of its tangent cone. The first example of Zariski pair was given by Zariski, [29, 30]; there exist sextic curves with six ordinary cusps. If these cusps are (resp. not) in a conic then the Alexander polynomial equals $t^2 - t + 1$ (resp. 1). Then, it gives an Alexander-Zariski pair. Many other examples of Alexander-Zariski pairs have been constructed (Artal,[1], Degtyarev [6]).

We state the main result in [2].

Theorem 3.4. *Let V_1, V_2 be two SIS such that their tangent cones form an Alexander-Zariski pair. Then V_1 and V_2 have the same abstract topology and characteristic polynomial of the monodromy but not the same embedded topology.*

Recall that the Jordan form of the monodromy is an invariant of the embedded topology of a SIS (see Theorem 2.4); since it depends on the Alexander polynomial $\Delta_C(t)$ of the tangent cone.

3.1. **Zariski pair of reduced sextics with only one singular point of type $\mathbb{A}_{17}$.** Our next Zariski-pair example (C_1, C_2) can be found in [1, Théorème 4.4]. The curves $C_i, i = 1, 2$ are reduced sextics with only one singular point P of type $\mathbb{A}_{17}$, locally given by $u^2 - v^{18}$.

(I) the irreducible componentes of C_1 are two non-singular cubics. These cubics meet at only one point P which moreover is an inflection point of each of the cubics, i.e. the tangent line to the singular point P goes through the infinitely near points P, P_1 and P_2 of C_1. The equations of C_1 are given for instance by $\{f_1(x, y, z) := (zx^2 - y^3 - ayz^2 - bz^3)(zx^2 - y^3 - ayz^2 - cz^3) = 0\}$, with $a, b, c \in \mathbb{C}$ generic.

(II) the irreducible componentes of C_2 are two non-singular cubics. These cubics meet at only one point P which is not an inflection point of any of the cubics, i.e. the tangent line to the singular point P goes through the infinitely near points P, P_1 of C_1 but it is not going through P_2. The equations of C_1 are given for instance by $\{f_2(x,y,z) := (zx^2 - y^2x - yz^2 - a_1(z^3 - y(xz - y^2)))(zx^2 - y^2x - yz^2 - a_2(z^3 - y(xz - y^2))) = 0\}$ with $a_1, a_2 \in \mathbb{C}$ generic.

Consider the superisolated surface singularities $(V_1, 0) = (\{f_1(x,y,z) + l^7 = 0\}, 0) \subset (\mathbb{C}^3, 0)$ and $(V_2, 0) = (\{f_2(x,y,z) + l^7 = 0\}, 0) \subset (\mathbb{C}^3, 0)$ (l is a generic hyperplane). In both cases the tangent cone has two irreducible components and it has only one singular point P of local type $u^2 - v^{18}$ and therefore $\Delta^P(t) = (t^{18} - 1)(t - 1)/(t^2 - 1) = \phi_{18}\phi_9\phi_6\phi_3\phi_1$, where ϕ_k is the k-th cyclotomic polynomial. Thus the number of local branches is 2 and $\sum_{P \in \mathrm{Sing}(C)}(r^P - 1) = (r - 1)$. By Corollary 2.9, for $(V_i, 0)$, $i = 1, 2$, the complex monodromy has no Jordan blocks of size 2 for the eigenvalue 1, and it has no Jordan blocks of size 3. Moreover the first Jordan polynomial is equal to

$$\Delta_1(t) = \frac{\gcd(\Delta^P(t), (t^6 - 1)^{\mu^P})}{\Delta_{C_i}(t)(t - 1)} = \frac{\phi_6\phi_3}{\Delta_{C_i}(t)}. \tag{3.3}$$

To compute $\Delta_{C_i}(t)$ we use the same ideas as in Theorem 2.11.

Lemma 3.5. *For the point P at the curve C_1 the ideals $\mathcal{I}_P^k = \mathcal{O}_{\mathbb{P}^2,P}$ if $k \leq 3$, $\mathcal{I}_P^4 =< y^3, z > \mathcal{O}_{\mathbb{P}^2,P}$ and $\mathcal{I}_P^5 =< y^6, z - y^3 - ay^4 - by^5 > \mathcal{O}_{\mathbb{P}^2,P}$.*

Lemma 3.6. *For the point P at the curve C_2 the ideals $\mathcal{I}_P^k = \mathcal{O}_{\mathbb{P}^2,P}$ if $k \leq 3$, $\mathcal{I}_P^4 =< y^3, z - y^2 > \mathcal{O}_{\mathbb{P}^2,P}$ and $\mathcal{I}_P^5 =< y^6, z - y^2 - y^5 > \mathcal{O}_{\mathbb{P}^2,P}$.*

Thus $\Delta_{C_i}(t) = \phi_6^{\dim \operatorname{coker} \sigma_5} \phi_3^{\dim \operatorname{coker} \sigma_4}$.

Thefore the map σ_4 is

$$\sigma_4 : H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \simeq \mathbb{C}^3 \to \mathcal{O}_{\mathbb{P}^2,P}/\mathcal{I}_P^4 \simeq \mathbb{C}^3\mathbb{C}^3,$$

and if we chosse as basis of the first space $\{1, y, z\}$ and of the second $\{1, y, y^2\}$ then

(1) by Lemma 3.5, for C_1 the dimension $\dim \operatorname{coker} \sigma_4 = \dim \ker \sigma_4 = 1$.
(2) by Lemma 3.6, for C_2 the dimension $\dim \operatorname{coker} \sigma_4 = \dim \ker \sigma_4 = 0$.

On the other hand for the map σ_5

$$\sigma_5 : H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)) \simeq \mathbb{C}^6 \to \mathcal{O}_{\mathbb{P}^2,P}/\mathcal{I}_P^5 \simeq \mathbb{C}^6,$$

if we choose as basis of the first space $\{1, y, z, y^2, yz, z^2\}$ and of the second $\{1, y, y^2, y^3, y^4, y^5\}$ then we can compute

(3) by Lemma 3.5, for C_1 the dimension $\dim \operatorname{coker} \sigma_5 = \dim \ker \sigma_5 = 1$.
(4) by Lemma 3.6, for C_2 the dimension $\dim \operatorname{coker} \sigma_5 = \dim \ker \sigma_5 = 0$.

Therefore, $\Delta_{C_1}(t) = \phi_6\phi_3$ and $\Delta_{C_2}(t) = 1$ and by (3.3) we have proved that the pair (C_1, C_2) is a Alexander-Zariski pair.

Example 3.7. Consider the superisolated surface singularities $(V_1, 0) = (\{f_1(x,y,z) + l^7 = 0\}, 0) \subset (\mathbb{C}^3, 0)$ and $(V_2, 0) = (\{f_2(x,y,z) + l^7 = 0\}, 0) \subset (\mathbb{C}^3, 0)$ (l is a generic hyperplane). Then the complex algebraic monodromy of $(V_1, 0) \subset (\mathbb{C}^3, 0)$ has finite order and the complex algebraic monodromy of $(V_2, 0) \subset (\mathbb{C}^3, 0)$ has not finite order

This answer a question proposed to us by J. Stevens: find a Zariski pair C_1, C_2 such that for the corresponding SIS surface singularities $(V_1, 0) \subset (\mathbb{C}^3, 0)$ and $(V_2, 0) \subset (\mathbb{C}^3, 0)$ one has a finite order monodromy and the other it does not.

There are also examples of Zariski pairs which are not Alexander-Zariski pairs (see [23], [3], [4]). Some of them are distinguished by the so-called characteristic varieties introduced by Libgober [15]. These are subtori of $(\mathbb{C}^*)^r$, $r := \#\operatorname{Irr}(C)$, which measure the excess of Betti numbers of finite Abelian coverings of the plane ramified on the curve (as Alexander polynomial does it for cyclic coverings).

Problem 3.8. How can one translate characteristic varieties of a projective curve in terms of invariants of the SIS associated to it?

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