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ON THE LOG-CANONICAL THRESHOLD FOR GERMS OF PLANE CURVES

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Dedicated to Lê Dũng Tráng on the Occasion of His Sixteenth Birthday

Abstract. In this article we show that for a given, reduced or non reduced, germ of a complex plane curve, there exists a local system of coordinates such that its log-canonical threshold at the singularity can be explicitly computed from the intersection of the boundary of its Newton polygon in such coordinates (degenerated or not) with the diagonal line.

1. Introduction

Let f be the germ of an analytic function at a point p on a complex d-dimensional manifold X such that $f(p) = 0$. Let $\pi : Y \to X$ be an embedded resolution of the hypersurface $f^{-1}\{0\}$ defined by the zero locus of f. Let $E_i, i \in I$, be the irreducible components of the divisor $\pi^{-1}(f^{-1}\{0\})$ and let $I_p := \{i \in I \mid p \in \pi(E_i)\}$. For each $j \in I$, we denote by N_j the multiplicity of E_j in the divisor of the function $f \circ \pi$ and we denote by $\nu_j - 1$ the multiplicity of E_j in the divisor of $\pi^*(\omega)$ where ω is a non-vanishing holomorphic d-form in a neighbourhood of $p \in X$. The pair (ν_i, N_i) is called the *numerical data* of the irreducible component E_i.

The *log-canonical threshold* of f at p is defined by

$$c_p(f) := \min_{i \in I_p} \left\{ \frac{\nu_i}{N_i} \right\}.$$

see [8, Proposition 8.5]. It does not depend on the resolution π since $-c_p(f)$ is the closest root to the origin of the *Bernstein-Sato polynomial* $b_{f,p}(s)$ of f at p, see [8, Theorem 10.6] or [9, 17]. Since $f(p) = 0$ then $b_{f,p}(s) = (s+1)\tilde{b}_{f,p}(s)$ where $\tilde{b}_{f,p}(s)$ is the reduced Bernstein-Sato polynomial of f at p introduced by M. Saito in [13]. Let $R_{f,p}$ be the set of roots of $\tilde{b}_{f,p}(-s)$ and $\alpha_{f,p} := \min R_{f,p}$.

The following result by M. Saito, Corollary 3.3 in [13] gives a bound for $c_p(f)$ in the non-degenerate case. We introduce a preliminary notation.

Let $\mathbf{x} := (x_1, \ldots, x_d)$ be a local system of coordinates at $p \in X$ such that the formal completion of $\mathcal{O} := \mathcal{O}_{X,x}$ is $\mathbb{C}[[\mathbf{x}]]$. Let $f = \sum_{\mathbf{n} \in \mathbb{N}^d} a_{\mathbf{n}} \mathbf{x}^{\mathbf{n}} \in \mathbb{C}[[\mathbf{x}]]$ be the formal power series defining the germ of f at p. Then:

- The *support* of f is the set $\operatorname{Supp}(f) = \{\mathbf{n} \in \mathbb{N}^d : a_{\mathbf{n}} \neq 0\}$.
- The *Newton polyhedron* $\Gamma(f)$ of f is the convex hull in $\mathbb{R}^d_+$ of the set $\bigcup_{\mathbf{n} \in \operatorname{Supp}(f)} (\mathbf{n} + (\mathbb{R}_+)^d)$.
- The *Newton polytope* or *Newton diagram* $ND(f)$ of f is the union of all compact faces of $\Gamma(f)$.
- The set of all compact faces is denoted by $CF(f)$.

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- The *principal part* of f is the polynomial $f|_{ND(f)} := \sum_{\mathbf{n}\in ND(f)} a_{\mathbf{n}}\mathbf{x}^{\mathbf{n}}$.
- For any $\tau \in CF(f)$ we denote by f_τ the polynomial $\sum_{\mathbf{n}\in\tau} a_{\mathbf{n}} x^{\mathbf{n}}$.

The principal part of f at p is called *non-degenerate* if for each closed proper face $\tau \in CF(f)$, the subscheme of the torus $(\mathbb{C}^*)^d$ defined by

$$\frac{\partial f_\tau}{\partial x_1} = \ldots = \frac{\partial f_\tau}{\partial x_d} = 0$$

is empty.

Theorem 1.1. [13] *If the principal part of f at p is non-degenerate then $\alpha_{f,p} \geq 1/t$ for $(t,\ldots,t) \in \partial\Gamma(f)$. In the isolated singularity case the equality holds.*

If f defines an isolated singularity the equality follows from results by [6, 12] (and [14] in the case $\alpha_{f,0} \leq 1$) combined with [10].

Remark 1.2. Note that the boundary $\partial\Gamma(f)$ of the Newton polyhedron $\Gamma(f)$ consists of compact and non compact faces of $\Gamma(f)$.

The main result of this paper deals with germs of plane curves, i.e. $d = 2$.

Theorem 1.3. *Given a two-variable germ $f \in \mathcal{O}$ of an analytic function at p, there exists a system of coordinates (x,z) at p such that $c_p(f) = \frac{1}{t}$ for $(t,t) \in \partial\Gamma(f)$, where $\Gamma(f)$ is the Newton polyhedron of f in such coordinates (degenerated or not).*

Remark 1.4. This result has been independently obtained by M. Aprodu and D. Naie in [1]. Their result is about isolated singularities and their proof is based on the fact that the log-canonical threshold coincides with the first jumping number.

Our approach uses the relationship between the log-canonical threshold of f at p and the local topological zeta function $Z_{\mathrm{top},p}(f,s)$ of f at p introduced by Denef and Loeser [5]. We use the result that, in the two dimensional case, the log-canonical threshold $-c_p(f)$ is the pole of $Z_{\mathrm{top},p}(f,s)$ closest to the origin.

Let $\pi : Y \to X$ be a given embedded resolution of the germ of hypersurface $(f^{-1}\{0\},p)$; we use the notations introduced in the beginning of the section. For each subset $J \subset I$, we set

$$E_J := \bigcap_{j\in J} E_j \text{ and } \check{E}_J := E_J \setminus \bigcup_{j\notin J} E_{J\cup\{j\}}.$$

To f one associates the *local topological zeta function* of f at p

$$Z_{\mathrm{top},p}(f,s) := \sum_{J\subset I} \chi(\check{E}_J \cap \pi^{-1}\{p\}) \prod_{j\in J} \frac{1}{\nu_j + N_j s} \in \mathbb{Q}(s), \tag{1.1}$$

where χ denotes Euler-Poincaré characteristic.

In general many candidate poles $-\frac{\nu_j}{N_j}$ are not poles of $Z_{\mathrm{top},p}(f,s)$. In the case of dimension 2, W. Veys [15] determined all poles of $Z_{\mathrm{top},p}(f,s)$ by using the minimal embedded resolution of the germ f at p. This result was based on the structure of ordered tree of the resolution graph of f at p, weighting each exceptional curve E_i, $i \in I$, by $\frac{\nu_i}{N_i}$. Veys proved that the minimum $c_p(f)$ defines a connected part of the resolution graph and, moreover, one can deduce from his results that the minimum $c_p(f)$ is always a pole of $Z_{\mathrm{top},p}(f,s)$.

Theorem 1.5. [15] *Given a germ f of plane curve at p then $-c_p(f)$ is the closest pole to the origin of $Z_{\mathrm{top},p}(f,s)$.*

In [16], Veys gave a formula for $Z_{\text{top},p}(f,s)$ in terms of the log-canonical model of the pair $(\mathbb{C}^2, f^{-1}\{0\})$. In [2], the authors gave a formula for computing $Z_{\text{top},p}(f,s)$ by means of Newton maps and Eisenbud-Neumman splice diagrams. This is the description we are going to use here and from which we will prove our main result.

In §2 we give the properties of Eisenbud-Neumann diagrams we use in the proof. In §3 we give the proof of the main theorem.

Note that this result is generalized in [4] for the multivariable log-canonical threshold.

2. EN-diagrams

2.1. **Construction of Eisenbud and Neumann diagrams using Newton algorithm.**

This construction is explained in [3]. Let f be a germ of a plane curve, reduced or not, at $0 \in \mathbb{C}^2$. Let us choose some local coordinates (x,z) and write $f(x,z) = \sum_{\alpha,\beta\geq 0} A_{\alpha,\beta} x^\alpha z^\beta$. Let $\Gamma(f)$ be the Newton polyhedron of f and $ND(f)$ the Newton diagram of f in these coordinates. The boundary of $\partial\Gamma(f)$ consists of two half lines parallel to the axes and a polygonal line between them which coincides with the Newton polygon $ND(f)$ of f.

An Eisenbud-Neumann diagram (EN-diagram for short) for a germ of curve is a decorated tree with two types of vertices: standard ones and arrows. The decorations are integer numbers associated to pairs (e,v) where e is an edge of the tree, and v is a vertex which is an end of e; sometimes 1-decorations are not written. Near a vertex, there are at most 2 numbers different from 1 and they are coprime. The arrows represent the branches (the irreducible components) of the germ f (an arrow is the end of exactly one edge). The decoration at an arrow is the multiplicity of the corresponding branch in the germ. The data we encode in the EN-diagrams are related to the successive Newton polygons we get running the Newton algorithm.

2.1.1. *Newton polygon part.* We start with the first Newton polygon. This first Newton polygon will be represented on the EN-diagram by a sequence of edges and vertices (and arrows eventually at top and bottom) drawn along a vertical line. Each compact face of $ND(f)$ is represented by a vertex. There is an edge between two consecutive vertices which corresponds to two consecutive compact faces. We go along the Newton diagram from the left to the right and along the EN-diagram from the top to the bottom. There is a vertex at the top (resp. bottom) if $ND(f)$ hits the z-axis (resp. x-axis) and an arrow otherwise. This arrow represents the branch $x=0$ (resp. $z=0$). These vertices or arrows at the top or the bottom correspond to the non compact faces of the boundary of $\partial\Gamma(f)$.

Consider a face γ of $ND(f)$ with slope $-\frac{q}{p}$ with q and p coprime, the equation of γ is $q\alpha + p\beta = N$; denote by v the associated vertex. For v, we write p at the extremity of the edge above the corresponding vertex and q at the extremity of the edge under the corresponding vertex.

2.1.2. *Newton maps.* Fix γ as above and let $f_\gamma(x,z) = \sum_{(\alpha,\beta)\in\gamma} A_{\alpha,\beta}x^\alpha z^\beta$. Let us denote by (α_1,β_1) the right extremity of γ. It is easily seen that there exists $P_\gamma(t) \in \mathbb{C}[t]$, $P_\gamma(0) \neq 0$, such that $f_\gamma(1,t) = t^{\beta_1}P_\gamma(t^q)$. From the corresponding vertex on the EN-diagram we will draw as many *horizontal* (non-vertical) edges as the number $v(\gamma)$ of distinct roots of $P_\gamma(t)$.

Now we go to the next step in the Newton algorithm. We choose a root a of $P_\gamma(t)$, i.e. a non vertical edge starting from the vertex corresponding to γ. If this root is a simple one, the corresponding branch is separated and we draw an arrow at the end of the corresponding edge. If a is a root of multiplicity k, we perform the algorithm. Let us choose a q^{th}-root $\tilde{a}$ of a (the algorithm does not depend on the specific choice of $\tilde{a}$). The rational map $\pi_{\gamma,a}$ given by $x = x_1^q,\ z = (z_1 + \tilde{a})x_1^p$ is called the *Newton map* associated with the face γ and the root a. From the Newton algorithm we get

$$f_1(x_1, z_1) := x_1^{-N} f(x_1^q, (z_1 + \tilde{a})x_1^p) \in \mathbb{C}\{x_1, z_1\},$$

since γ has as equation $q\alpha + p\beta = N$ and a is a root of $P_\gamma(t)$.

If f_1 is a k-th power of a polynomial with multiplicity 1, the corresponding branch has multiplicity k, then we draw an arrow at the end of the edge and write the multiplicity. If not, we consider $f_{\gamma,a} := f(x_1^q, (z_1 + \tilde{a})x_1^p) = x_1^N f_1(x_1, z_1)$.

We consider the diagram associated to the Newton polygon of $x_1^N f_1(x_1, z_1)$. It has an arrow at the top decorated by N. We draw new vertices corresponding to the faces of $ND(f_{\gamma,a})$, starting from the left on the Newton diagram and from the top on the EN-diagram. It ends by a vertex or an arrow. We glue this diagram to the corresponding edge, deleting the top arrow. We will change below the decorations after this process.

2.1.3. *New decorations.* To obtain the new decorations we consider every vertex $\tilde{v}$ corresponding to a face $\tilde{\gamma}$ of $ND(f_{\gamma,a})$ with slope $-\tilde{q}/\tilde{r}$, with $\tilde{r}$ and $\tilde{q}$, positive integers prime to each other. Then, we write $\tilde{q}$ on the extremity of the edge under $\tilde{v}$, near the vertex, and replace $\tilde{r}$ by $\tilde{p} = \tilde{r} + pq\tilde{q}$ on the extremity of the edge above $\tilde{v}$, near the vertex. The vertex v corresponding to γ will be called the *preceeding vertex* of $\tilde{v}$ for all vertices constructed from $ND(f_{\gamma,a})$.

We go on until all irreducible components of f are separated. We get as many arrows as the number of irreducible components of f. If the germ is not reduced, we have written the multiplicity of a branch, in front of the arrow, which represents this branch. Finally one gets the EN-diagram $D(f)$ of the germ f at the origin.

Example 2.1. Consider the germ given by

$$f(x, z) = (x^2 - z^3)^2(x^3 - z^2)^2 + x^6z^3 + x^5z^5 + x^4z^7$$

The graph of the first Newton polygon is given in Figure 1.

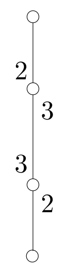


FIGURE 1

We apply the two Newton maps and we obtain:

$$f_1(x_1, z_1) = x_1^{20}(x_1^4 + 9z_1^2 + ...) \quad f_2(x_1, z_1) = x_1^{20}(x_1 + 4z_1^2 + ...).$$

The graphs of f_1 et f_2 are in Figure 2 and the EN-diagram of f is in Figure 3.

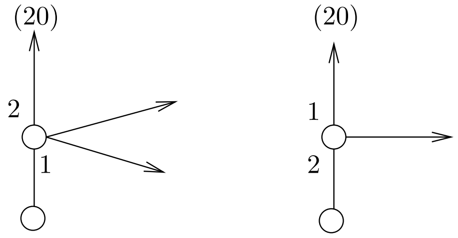


FIGURE 2

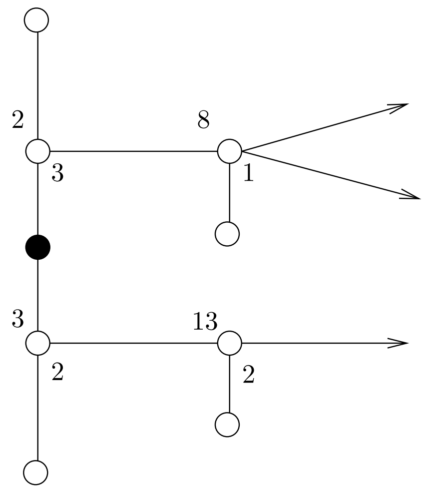


FIGURE 3

We call *vertical* the edges between two vertices corresponding to a Newton polygon and *horizontal* the edges corresponding to Newton maps. The diagrams depend on the system of coordinates.

Remark 2.2. For each edge $e := [v_1, v_2]$ we denote by Δ_e the edge determinant, which is the product of the numbers appearing on the edge, minus the product of the numbers adjacent to the edge. By construction all the edge determinants are positive integers.

2.2. **Minimal diagrams.**

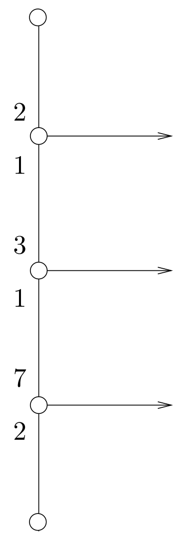
For a vertex v of $D(f)$, its *valency* δ_v is the number of edges and arrows meeting v. From the EN-diagram $D(f)$ one gets the minimal EN-diagram $D_m(f)$ using the following process:

(1) We delete all the edges bearing only 1, with an extremity attached to a vertex of valency 1, and the vertex of valency 1 as well.
(2) We delete all vertices of valency two, replacing the two edges which end at this vertex by one.

Theorem 2.3. [7] *The minimal EN-diagram of a germ determines and is determined by the topological type of the germ.*

Remark 2.4. The minimal EN-diagram does not depend on the choice of coordinates if we forget about vertical and horizontal edges. Otherwise it does depend on the system of coordinates.

Example 2.5. The two diagrams of Figure 4 give a presentation of the same minimal diagram in two different systems of coordinates. On the left f is non degenerate.



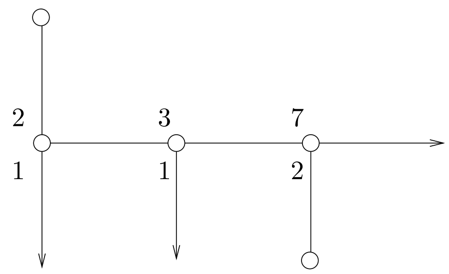


FIGURE 4

2.3. **Computation of the local topological Zeta function.**

Let $f:(\mathbb{C}^2,0)\to(\mathbb{C},0)$ be an analytic function germ which is singular at the origin. Let $f=\prod_i f_i^{m_i}$ be its decomposition into irreducible germs. Let $D(f)$ be the EN-diagram of f and let v be a vertex of $D(f)$. Each vertex v has associated a composition of Newton maps, which allows to associate then to v the pair $(\nu_v,N_v)\in\mathbb{N}^2$ as defined in the Introduction; this pair can be computed from the decorations using the following propositions.

Proposition 2.6. *The N_v can be computed by the following rules.*

- *If v corresponds to a face γ of the Newton polygon $ND(f)$ of f contained in a line $q\alpha+p\beta=N$, p,q coprime, one has $N_v=N$.*
- *In general, the number N_v can be computed on the diagram as the sum over all arrows of the product of the numbers adjacent to the path from the vertex v to each arrow.*

Proposition 2.7 ([3])**.** *The ν_v can be computed by the following rules.*

- *If v corresponds to a segment γ of the Newton polygon $ND(f)$ of f contained in a line $q\alpha+p\beta=N$, p,q coprime, one has $\nu_v=p+q$.*
- *If v corresponds to a segment γ (contained in a line $q\alpha+p\beta=N$, p,q coprime) of $ND(f_{\gamma_0,a})$ and v_0 is its preceeding vertex then $\nu_v=q\nu_{v_0}+p$.*

Proof. Let $x^{\nu-1}dx\wedge dz$ be an holomorphic form. We perform the change

$$x=x_1^q,\ z=(z_1+\tilde{a})x_1^p$$

and we get $x_1^{(\nu-1)q+q+p-1}dx_1\wedge dz_1$ which proves the claim. □

Remark 2.8. Each vertex of an EN-diagram is associated to an exceptional divisor in a resolution and (ν,N) for the vertex of the diagram coincides with (ν,N) for the corresponding divisor.

Corollary 2.9. *Let v_0 be the preceeding vertex of the vertex v and $e := [v_0, v]$. Let p be the integer associated to (e, v) and let q be the other integer which is eventually not equal to 1 adjacent to e and near v. Then*

$$\nu_v = q\nu_0 + \Delta_e. \tag{2.1}$$

Proof. It is enough to consider the formula for the new decorations. □

Notation 2.10. Let v be a vertex of $D(f)$ and define $P_v(s) := \nu_v + N_v s$.

Theorem 2.11. [2] *Let f be a germ of complex analytic function in two variables with $f(0) = 0$. Let $D(f)$ be an EN-diagram of f, then*

$$Z_{top,0}(f,s) = \sum_{[v,v']\in\mathcal{E}} \frac{\Delta_{v,v'}}{P_v(s)P_{v'}(s)} + \sum_v \frac{2-\delta_v}{P_v(s)} + \sum_v \sum_{i\in\mathcal{F}_v} \frac{1}{(1+m_i s)P_v(s)},$$

where

- *δ_v is the valence of the vertex v,*
- *$\mathcal{E}$ is the set of edges of $D(f)$,*
- *$\mathcal{F}_v$ is the set of arrows which are connected to the vertex v by an edge and*
- *m_i is the multiplicity of the corresponding branch.*

A proof of Theorem 2.11 is obtained from [2, Corollary 5.2, Theorem 5.3 and Theorem 6.1]. We give an idea of the proof which uses induction.

Let h be a germ of curve singularity and assume $h = x^{N_0} z^M g(x, z)$, and $\omega = x^{\nu_0 - 1} dx \wedge dz$, where neither x nor z divide g and (h, ω) satisfies the *support condition*: "if $N_0 = 0$ then $\nu_0 = 1$". We can define $Z_{\mathrm{top},0}(h, \omega, s)$ following (1.1) in order to apply induction.

Let $\gamma_1, \ldots, \gamma_r$ be the compact edges of the Newton diagram of h. For each $1 \le d \le r$, the equation of γ_d will be $q_d\alpha + p_d\beta = N_d$ with $\gcd(q_d, p_d) = 1$. Let $v(\gamma_d)$ be the number of non-zero distinct roots of P_{γ_d}. Each one of these roots defines a Newton map and let $h_{d,j}$ be the pull-back of h under these Newton maps, for $1 \le d \le r$. Recall that γ_1 is the compact face with z-highest vertex, (this is a different convention than the one used in [2]). Applying [2, Theorem 5.3] for curves one can compute $Z_{\mathrm{top},0}(h, \omega, s)$ inductively as follows:

$$\begin{aligned} Z_{\mathrm{top},0}(h,w,s) =& \frac{q_r}{(N_r s + \nu_0 q_r + p_r)(Ms+1)^{\varepsilon}} + \frac{p_1}{(N_1 s + \nu_0 q_1 + p_1)(N_0 s + \nu_0)} \\ &+ \sum_{d=1}^{r-1} \frac{|p_d q_{d+1} - p_{d+1} q_d|}{(N_d s + \nu_0 q_d + p_d)(N_{d+1} s + \nu_0 q_{d+1} + p_{d+1})} \\ &- \sum_{d=1}^{r} \frac{v(\gamma_d)}{N_d s + \nu_0 q_d + p_d} + \sum_{d=1}^{r} \sum_{j=1}^{v(\gamma_d)} Z_{\mathrm{top},0}(h \circ \pi_{d,j}, w \circ \pi_{d,j}, s). \end{aligned}$$

where ε is zero if and only if z does not divides h.

Remark 2.12. Essentially this gives a first better set of candidate poles of the rational function $Z_{\mathrm{top},0}(h, w, s)$ defined by $N_0 s + \nu_0$, $Ms + 1$, and $N_d s + \nu_d$ where $\nu_d := \nu_0 q_d + p_d$, with $1 \le d \le r$.

Carine Reydy in her Ph.D. [11] showed a similar formula for the multivariable topological zeta function, one variable s_i for each irreducible component f_i of f. In fact, doing every $s_i = s$ we get Theorem 2.11.

3. The log-canonical threshold for germs of plane curves

Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$ be an analytic function germ which is singular at the origin. Let $f = \prod_i f_i^{m_i}$ be its decomposition into irreducible germs.

We use the result of W. Veys that the log-canonical threshold $-c_0(f)$ of the germ f at 0 is the pole closest to the origin of $Z_{\text{top},0}(f, s)$, see Theorem 1.5. To compute the (candidate) poles of $Z_{\text{top},0}(f, s)$ we will use Theorem 2.11. We start with a fixed system of coordinates and in order to apply induction, we will consider $Z_{\text{top},0}(h, \omega, s)$.

The strategy is to study the behaviour of $\frac{\nu}{N}$ as follows.

- First we study what happens along the vertical edges corresponding to the Newton polygon. We see that along these edges, the minimum of $\frac{\nu}{N}$ is obtained at the vertices corresponding to the segments cut by a fixed line passing through the origin (one or two segments if the line cut at a 0-face of the Newton polygon).
- Starting from the corresponding vertex, the function $\frac{\nu}{N}$ increases along the vertical line.
- If we start from a vertex on the first Newton polygon which does not realize the minimum then it increases along horizontal edges.
- It remains to study what happens starting from one vertex where the minimum is realized.

First we study what happens along vertical edges. Assume that we are in an intermediate step of the above procedure where h is a germ of curve singularity given by $h = x^{N_0} z^M g(x, z)$, and $\omega = x^{\nu_0 - 1} dx \wedge dz$. We keep the notations as in Remark 2.12.

Proposition 3.1. *Consider the Newton polyhedron $\Gamma(h)$ of h. The minimum of the set of quotients $\{\frac{\nu_0}{N_0}, \frac{1}{M}, \frac{\nu_d}{N_d}\}$ is attained on the (compact or not) faces of the boundary $\partial\Gamma(h)$ cut by the line $\{\alpha = \nu_0 t,\ \beta = t\}$. From this minimum the quotients will strictly increase along $\partial\Gamma(h)$.*

Proof. It is enough to show that the intersection point of the line $\{\alpha = \nu_0 t,\ \beta = t\}$ and a line $N = q\alpha + p\beta$ has coordinates $\alpha = \frac{\nu_0 N}{\nu_0 q + p}$ and $\beta = \frac{N}{\nu_0 q + p}$. ☐

As before let $\gamma_1, \dots, \gamma_r$ be the compact edges of the Newton diagram of h. For each $1 \le d \le r$, the equation of γ_d will be $q_d\alpha + p_d\beta = N_d$ with $\gcd(q_d, p_d) = 1$. Let $v(\gamma_d)$ be the number of non-zero distinct roots of h_{γ_d}. There are positive integers $m_{d,j}$, for $1 \le j \le v(\gamma_d)$, such that

$$h_{\gamma_d} = z^k x^a \prod_{j=1}^{v(\gamma_d)} (z^{q_d} - \tilde{a}_j^d x^{p_d})^{m_{d,j}}, \quad m_d := \sum_j m_{d,j}.$$

We perform the Newton map π_{γ_d, a_j^d} given by $x = x_1^{q_d},\ z = (z_1 + \tilde{a}_j^d) x_1^{p_d}$ and we will follow the induction process with

$$h_{d,j}(x_1, z_1) := h \circ \pi_{\gamma_d, a_j^d}(x, z) = x_1^{N_d} \tilde{h}(x_1, z_1) \text{ and } \pi^*_{\gamma_d, a_j^d}(w) = x_1^{\nu_0 q_d + p_d - 1} dx_1 \wedge dz_1.$$

We begin the study along horizontal edges.

Lemma 3.2. *The line $\{\alpha = (\nu_0 q_d + p_d)t,\ \beta = t\}$ hits the vertical line $\{\alpha = N_d\}$ of the polygon $\Gamma(h_{d,j})$ if and only if $\nu_d m_{d,j} < N_d$.*

Proof. Recall that $\nu_d := (\nu_0 q_d + p_d)$. The line $\{\alpha = (\nu_0 q_d + p_d)t,\ \beta = t\}$ hits the vertical line of the polygon $\Gamma(h_{d,j})$ if and only if for all compact faces of $\Gamma(h_{d,j})$ the corresponding $\frac{\nu}{N}$ is bigger than $\frac{\nu_0 q_d + p_d}{N_d}$. In fact this is the case if and only if the inequality $(\nu_0 q_d + p_d) m_{d,j} < N_d$ holds because in the new Newton diagram the highest z_1-height is $m_{d,j}$ and its corresponding compact 0-dim face has coordinates $(\alpha, \beta) = (N_d, m_{d,j})$. □

The following result is a consequence of the above Lemma.

Proposition 3.3. *If $\nu_d m_{d,j} < N_d$, then for all vertices of $D(f)$ having v_d as preceeding vertex we have $\frac{\nu}{N} > \frac{\nu_d}{N_d}$.*

Lemma 3.4. *Assume we are on the first Newton polygon of f and that $\nu_0 = 1$. Assume we have two vertices v_1 and v_2 and fix notation as in the Figure 5. Then*

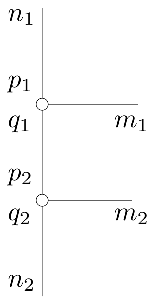


FIGURE 5

$\frac{\nu_1}{N_1} > \frac{\nu_2}{N_2}$ if and only if $m_2 q_2 + n_2 > m_1 p_1 + n_1$.

Proof. By Propositions 2.6 and 2.7 one has $\nu_1 = p_1 + q_1, \nu_2 = p_2 + q_2$ and

$$N_1 = p_1 q_1 m_1 + p_1(q_2 m_2 + n_2) + q_1 n_1,$$
$$N_2 = p_2 q_2 m_2 + q_2(p_1 m_1 + n_1) + p_2 n_2.$$

This can be used as definition of n_1 and n_2. Then

$$\nu_1 N_2 - \nu_2 N_1 = (q_1 p_2 - q_2 p_1)(m_2 q_2 + n_2 - m_1 p_1 + n_1),$$

and we are done since the edge determinant $q_1 p_2 - q_2 p_1$ is always positive (see Remark 2.2). □

Proposition 3.5. *If γ_d is a face of the first Newton polygon, and if the minimum of $\frac{\nu}{N}$ on the first Newton polygon is not attained on γ_d then $m_d \nu_d - N_d < 0$.*

Proof. Let v_0 be a vertex where the minimum of $\frac{\nu}{N}$ is attained. Let v be a nearby vertex where the minimum is not attained. We use the previous lemma with v_0 as v_2 and v as v_1. We have $q_0 m_0 + n_0 > p_v m_v + n_v$. We prove that $m_v \nu_v - N_v < 0$ which will imply that, for all j, $m_{v,j} \nu_v - N_v < 0$. Thus

$$\begin{aligned} m_v(p_v + q_v) - N_v &= m_v(p_v + q_v) - p_v q_v m_v - p_v(q_0 m_0 + n_0) - q_v n_v \\ &< m_v(p_v + q_v) - p_v q_v m_v - p_v(p_v m_v + n_v) - q_v n_v. \end{aligned}$$

Since $p_v \geq 1$ then $m_v(p_v + q_v) - N_v < (p_v + q_v)(m_v(1 - p_v) - n_v) \leq 0$ □

Proposition 3.6. *If $m_{v,j} \nu_v - N_v < 0$ then after the corresponding Newton map, $m_{v'} \nu_{v'} - N_{v'} < 0$ for all vertices v' which admit v as preceeding vertex.*

Proof. We can write $N = N^0 + pqN^1$, where $N^1 = q_1 m_1 + \sum q_i m_i + N^2$ where the sum is taken on the vertices between v and v_1.

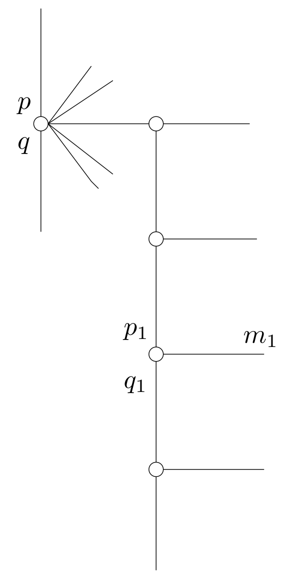


FIGURE 6

Thus N_1 can be also written as $N_1 = p_1 q_1 m_1 + q_1(\sum p_i m_i) + p_1 N^2 + q_1 N^0$. We will use $p_i > pq q_i$. Since

$$\begin{aligned} N_1 &> p_1 q_1 m_1 + q_1 pq(\sum q_i m_i) + p_1 N^2 + q_1 N^0 \\ &= p_1 q_1 m_1 + q_1 pq(\sum q_i m_i) + p_1 N^2 + q_1(N - pqN^1) \end{aligned}$$

then

$$N_1 > q_1 m_1(p_1 - qpq_1) + (p_1 - pqq_1)N^2 + q_1 N.$$

Finally,

$$m_1\nu_1 - N_1 = m_1(\nu q_1 + (p_1 - pqq_1) - N_1 < (m\nu - N)q_1 + m_1(p_1 - pqq_1) + Nq_1 - N_1.$$

By hypothesis $m\nu - N < 0$ then

$$m_1\nu_1 - N_1 < (m\nu - N)q_1 + m_1(p_1 - pqq_1)(1 - q_1) < 0. \qquad \square$$

Suppose now that we are in the first Newton polygon, that is the Newton polygon of f. Suppose there exists a face γ_d is $q_d\alpha + p_d\beta = N_d$, with $\gcd(q_d, p_d) = 1$ and let v_d be the corresponding vertex. Assume that on γ_d there are $v(\gamma_d)$ different roots and and for each root a_j^d its multiplicity is $m_{d,j}$.

Lemma 3.7. *If $m_{d,j}\nu_d > N_d$ then either $p_d = 1$ or $q_d = 1$.*

Proof. Since we are in the first Newton polygon then $\nu_d = p_d + q_d$ and $N_d = p_d q_d m_{d,j} + N'_d$, with $N'_d \geq 0$. We can also deduce this fact from the following argument: If γ_d has as boundary points (a_{d+1}, b_{d+1}) and (a_d, b_d), from the left to the right on the Newton polygon, then $N_d = a_{d+1}q_d + b_d p_d + q_d p_d \sum m_{d,j}$.

Then $m_{d,j}\nu_d > N_d$ if and only if $m_{d,j}(p_d + q_d - p_d q_d) > N'_d (\geq 0)$. Thus $p_d + q_d - p_d q_d \geq 1$ which implies that either $p_d = 1$ or $q_d = 1$ and $p_d + q_d - p_d q_d = 1$. $\square$

Lemma 3.8. *Assume that $m_{d,j}\nu_d > N_d$ and consider a vertex $\tilde{v}$ which has v_d as a preceeding vertex with decorations $\tilde{q}$ and $\tilde{p}$. Then $\tilde{p} \geq 2$ and $\tilde{\nu} = \tilde{p} + \tilde{q}$.*

Proof. Let $\tilde{N} = \tilde{q}\alpha + \tilde{r}\beta$ be equation of the corresponding compact face of $ND(h_{d,j})$ with numerical data $(\tilde{N}, \tilde{\nu})$. We know that the decorations of the vertex $\tilde{v}$ in the EN-diagram are $\tilde{q}$ and $\tilde{p} = \tilde{r} + p_d q_d \tilde{q} \geq 2$ since $\tilde{r}, p_d, q_d, \tilde{q} \geq 1$.

Finally since $p_d + q_d - p_d q_d = 1$ then

$$\tilde{\nu} = \nu_d \tilde{q} + \tilde{r} = (p_d + q_d)\tilde{q} + \tilde{p} - p_d q_d \tilde{q} = (p_d + q_d - p_d q_d)\tilde{q} + \tilde{p}. \qquad \square$$

Proposition 3.9. *If $m_{d,j}\nu_d > N_d$, then for a vertex $\tilde{v}$ which has v_d as a preceeding vertex with decorations $\tilde{q}$ and $\tilde{p}$ one has $\tilde{\nu} = \tilde{p} + \tilde{q}$ and $\tilde{q} = 1$.*

Proof. If $\tilde{\mu}\tilde{\nu} > \tilde{N}$ then either $\tilde{p} = 1$ or $\tilde{q} = 1$. But in Lemma 3.8 it is shown that $\tilde{p} \geq 2$ then $\tilde{q} = 1$. $\square$

Corollary 3.10. *If the minimum $\frac{\nu}{N}$ is not attained on the first Newton polygon of f, then in the EN-diagram, there is a vertex v_d with either $p_d = 1$ or $q_d = 1$ and after that $q_1 = 1, \ldots q_{k-1} = 1$ until the minimum is reached at the k-th vertex. At that point $\nu_k = p_k + q_k$.*

Proof. It is an immediate consequence of Proposition 3.9. $\square$

Proof of Theorem 1.3. Since we have in the EN-diagram either $p_d = 1$ or $q_d = 1$ and after a sequence of $1's$, i.e. $q_1 = 1, \ldots q_{k-1} = 1$, then we can change coordinates until being in the case where the minimum is attained on the first Newton polygon.

We have to prove that when the minimum $\frac{\nu}{N}$ is attained on the first Newton polygon, $-\frac{\nu}{N}$ is actually a pole of the topological zeta function. If the minimum is attained on two consecutive vertices, then we have a double pole, which is actually a pole. Now we have to consider the case where the minimum is attained once on the vertex v. We have, for all vertex v' and all arrow $i \in \mathcal{F}_v$:

$$\frac{1}{(\nu_v + N_v s)(\nu_{v'} + N_{v'} s)} = \frac{A_{v,v'}}{(\nu_v + N_v s)} + \frac{B}{(\nu_{v'} + N_{v'} s)}$$
$$\frac{1}{(\nu_v + N_v s)(1 + m_i s)} = \frac{A_{v,i}}{(\nu_v + N_v s)} + \frac{B}{(1 + m_i s)}$$

where $A_{v,v'} = \frac{N_v}{N_v \nu_{v'} - N_{v'} \nu_v}$ and $A_{v,i} = \frac{N_v}{N_v - m_i \nu_v}$. Since $\frac{\nu_v}{N_v} < \frac{\nu_{v'}}{N_{v'}}$ then $A_{v,v'} > 0$ and we have also $A_{v,i} > 0$. Similar computations as before show that $A_{v,v'} \geq \frac{1}{\Delta_{v,v'}}$ and $A_{v,i} > 1$. Then the residue at $-\frac{\nu_v}{N_v}$ does not vanish. $\square$

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