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EULER CHARACTERISTIC OF THE MILNOR FIBRE OF PLANE SINGULARITIES

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Abstract. We give a formula for the Euler characteristic of the Milnor fibre of any analytic function f of two variables. This formula depends on the intersection multiplicities, the Milnor numbers and the powers of the branches of the germ of the curve defined by f. The goal of the formula is that it use neither the resolution nor the deformations of f. Moreover, it can be use for giving an algorithm to compute it.

1. Introduction

In this note we deal with germs of analytic functions f of two complex variables with $f(0) = 0$ and its factorization $f = f_1^{q_1} \cdots f_r^{q_r}$ into irreducible factors, such that $f_i/f_j, 1 \leq i, j \leq r$, are as power series not units. Let $(C, 0)$ be the germ of the plane curve defined by the local equation $f = 0$ and let $(C_i, 0)$, $i = 1, \dots , r$, be its reduced branches defined by $f_i = 0$.

The local curve C defines a *link with multiplicities* $L := C \cap S^3_\varepsilon$, in the sphere of radius $\varepsilon > 0$ around $0 \in \mathbb{C}^2$, which does not depend on ε provided ε is small enough. The link L consists of the components $C_i \cap S^3_\varepsilon$, with multiplicities q_i and determines the topological type of the germ C. Moreover, Milnor proved that the map $\frac{f}{|f|} : S^3_\varepsilon \setminus L \to S^1$ is a C^∞-locally trivial fibration, the *Milnor fibration.* Any fibre F of this fibration is called the *Milnor fibre of* f (see [M, Theorem 4.8]). A'Campo [A] and Eisenbud-Neumann [EN], using different methods, calculated many topological invariants of the fibration $\frac{f}{|f|}$ from the resolution graph or the splicing diagrams. We are only interested in the Euler characteristic $\chi(F)$ of the surface F. If f is reduced, i.e. every power q_i is equal to one, the Euler characteristic of F is $1 - \mu(C, 0)$, where $\mu(C, 0)$ is the Milnor number of the isolated singularity of C. Moreover the Euler characteristic of F is related to topological and geometric invariants of its branches by the well-known formula:

$$\chi(F) = -2 \sum_{1\leq i<j\leq r} (C_i, C_j)_0 + \sum_{i=1}^{r} (1 - \mu(C_i)\,),$$

where $(C_i, C_j)_0$ is the intersection multiplicity of C_i and C_j at the origin and $\mu(C_i)$ is the Milnor number of C_i at the origin (e.g. see [BK]).

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On the other hand, when f is non-reduced Schrauwen [S] expressed the Euler characteristic of F in terms of special points of suitable deformations of f. For calculating $\chi(F)$ in this case one can use the methods of A'Campo or Eisenbud-Neumann and construct the resolution graph or the splicing diagram.

The aim of this note is to give a closed formula for the Euler characteristic of F without the construction of these graphs.

For every $q \in \mathbb{N}^r$ set

$$F^q := \{z \in S_\epsilon : \prod_{1 \le i \le r, q_i \ne 0} (\frac{f_i}{|f_i|})^{q_i}(z) = 1 \text{ and } f_i(z) \ne 0 \quad \forall i = 1, \dots, r\}.$$

Note that, for ϵ small, the surface F^q is the Milnor fibre of the local curve $C^q := \{f_1^{q_1} \cdots f_r^{q_r} = 0\}$ if and only if all $q_i \ne 0$. If some q_i are zero, but $q \ne 0$, then F^q is the Milnor fibre of $\prod_{1 \le i \le r, q_i \ne 0} f_i^{q_i}$ with punctures, where the number of punctures equals $\sum_{1 \le i,j \le r, q_i \ne 0, q_j = 0} (C_i, C_j)_0\ (q_i)$. For $q = 0$ the space F^q is just the complement of the link of the curve C.

Our generalized and closed formula is:

$$\chi(F^q) = -\sum_{1 \le i < j \le r} (C_i, C_j)_0\ (q_i + q_j) + \sum_{i=1}^r q_i\,(1 - \mu(C_i)\,).$$

I am indebted to the referee for suggesting how to improve the presentation of the proof of the formula.

2. Proof of the formula

The formula follows from the two following lemmas.

Lemma 1. *The function $q \in \mathbb{N}^r \to \chi(F^q)$ is additive.*

Proof. Let $\pi : X \to \mathbb{C}^2$ be a proper modification of $\mathbb{C}^2$ above the origin such that, for every point on the divisor $E := \pi^{-1}(0)$, the total transform of the $\bigcup_{1 \le i \le r} C_i$ has normal crossing singularities. Let $\widetilde{C_i}$ be the strict transform of C_i by π and E_α, $\alpha \in A$, the components of E.

First assume $q \ne 0$. Put $f^q = \prod_{1 \le i \le r, q_i \ne 0} f_i^{q_i}$. Observe that F^q retracts on $E \setminus \left(\bigcup_{1 \le j \le r, q_j = 0} \widetilde{C_j}\right)$. With the formula of A'Campo we get:

$$\chi(F^q) = \sum_{\alpha \in A} m(f^q, E_\alpha)\,\chi(\check{E}_\alpha),$$

where $\check{E}_\alpha := E_\alpha \setminus \left(\bigcup_{\beta \ne \alpha} E_\beta \cup \bigcup_{1 \le i \le r} \widetilde{C_i}\right)$. Then

$$\chi(F^q) = \sum_{\alpha \in A} \sum_{i=1}^r q_i\, m(f_i, E_\alpha)\chi(\check{E}_\alpha),$$

since $m(f^q, E_\alpha) = \sum_{1 \le i \le r} q_i m(f_i, E_\alpha)$.

To prove the additivity it remains to observe that $\chi(F^0) = 0$. □

Put $e_i = (0, \dots, 1, \dots, 0)$. From the additivity we get:

$$\chi(F^q) = \sum_{i=1}^r q_i \chi(F^{e_i}).$$

Lemma 2.

$$\chi(F^{e_i}) = -\sum_{\substack{j=1,\dots,r\\ i\neq j}} (C_i, C_j)_0 + (1-\mu(C_i)).$$

Proof. Remember that F^{e_i} is the Milnor fibre F_i with $\sum_{1\leq j\leq r, j\neq i} (C_i, C_j)_0$ punctures. □

Remark that Lemma 1 holds for the case where the germs of the curves C_i are reduced and have no branch in common. Thus, if we assume

1. each f_i has no multiple components (i.e. f_i is squarefree) and
2. for $i, j \in \{1, \dots, r\}$, $i \neq j$, the germ $f_i f_j$ has no multiple components,

then we finally get for the Euler characteristic of the Milnor fibre of F of $f = f_1^{q_1} \cdots f_s^{q_s}, q_i > 0$, the formula:

$$\chi(F) = -\sum_{1\leq i<j\leq s} (C_i, C_j)_0\,(q_i + q_j) + \sum_{i=1}^{s} q_i\,(1-\mu(C_i, 0)).$$

To have this formula for squarefree factorization is particularly useful for inductive calculations. If R is a computable ring with $char(R) = 0$ and f is a polynomial in $R[x, y]$, then there exists an algorithm that computes a squarefree decomposition of f in $R[x, y]$ (see [BWK, Proposition 2.86, Corollary 2.92]). This is also a squarefree decomposition in $R\{x, y\}$ and one may then compute the intersection multiplicities and the Milnor numbers. I would like to thank Bernd Martin for showing me the implementation of this algorithm using the computer algebra system SINGULAR, [GPS].

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