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THE DENEF–LOESER ZETA FUNCTION IS NOT A TOPOLOGICAL INVARIANT

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Abstract

An example is given which shows that the Denef–Loeser zeta function (usually called the topological zeta function) associated to a germ of a complex hypersurface singularity is not a topological invariant of the singularity. The idea is the following. Consider two germs of plane curves singularities with the same integral Seifert form but with different topological type and which have different topological zeta functions. Make a double suspension of these singularities (consider them in a 4-dimensional complex space). A theorem of M. Kervaire and J. Levine states that the topological type of these new hypersurface singularities is characterized by their integral Seifert form. Moreover the Seifert form of a suspension is equal (up to sign) to the original Seifert form. Hence these new singularities have the same topological type. By means of a double suspension formula the Denef–Loeser zeta functions are computed for the two 3-dimensional singularities and it is verified that they are not equal.

Introduction

Let $f : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)$ be a germ of holomorphic function whose zero locus $(V, 0) = (f^{-1}(0), 0)$ has an isolated singularity at 0. For any positive integer d, J. Denef and F. Loeser defined the *local topological zeta functions* $Z^{(d)}_{\text{top},0}(f, s) \in \mathbb{Q}(s)$, $d \in \mathbb{Z}_{>0}$ of f. They are rational functions and were introduced as a kind of limit of the p-adic Igusa zeta function, see [**5**]. This first definition was in terms of an embedded resolution of the germ $(V, 0) = (f^{-1}(0), 0)$ (see formula (1.1)); such functions came from the p-adic Igusa zeta function, which allowed the proof that the given definitions were independent of the chosen particular resolution. In [**6**], an intrinsic definition of $Z^{(d)}_{\text{top},0}(f, s)$ was given using arc spaces and the motivic zeta function.

The fact that these functions depend only on topological properties of the resolution is the reason for the *topological* term in the original definition. In order to study the relationship with the topological type of the singularity it is necessary to consider invariants of this topological type. Let S^{2n+1} be the boundary of a *small* ball centred at the origin and let $K^{2n-1} := f^{-1}(0) \cap S^{2n+1}$ be the link of the hypersurface. The Milnor fibration theorem states that the topology of $(V, 0)$ is determined by the pair (S^{2n+1}, K^{2n-1}). There exists a fibration from $S^{2n+1} \setminus K^{2n-1}$ over the unit circle. A fibre F of this fibration is called a Milnor fibre and its non-vanishing homology groups are $H_k(F, \mathbb{Z}), k = 0, n$. An important invariant related with this pair is its integral Seifert form; it is a bilinear form over $\mathbb{Z}$ defined on $H_n(F, \mathbb{Z})$ in terms of the linking form on S^{2n+1}, see [**9**] or [**2**], which determines other invariants as the intersection form or the homological monodromy.

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A theorem of M. Kervaire and J. Levine states that the topological type of the pair (S^{2n+1}, K^{2n-1}), that is, the topological type of $(V, 0)$, is determined by its integral Seifert form, see A. Durfee's paper [**9**], whenever $n \geqslant 3$. It is also true if $n = 1$ and f is irreducible.

For an isolated singularity f, it is difficult to compute either the Seifert form or an embedded resolution, if n is large. We call the suspension of f a singularity in $(\mathbb{C}^{n+2}, 0)$ defined by $F := f + z^2$, where z is the new variable. There is a simple relationship between the Seifert forms of f and F.

Proposition A (see [**13**]). *Let $f : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)$ be a germ of holomorphic function with an isolated singularity at the origin and A its Seifert form. If $F := f + z^2 : (\mathbb{C}^{n+2}, 0) \longrightarrow (\mathbb{C}, 0)$ is the suspension of f, then the Seifert form associated to F is $(-1)^{n+1}A$.*

There is no such simple relation between embedded resolutions of f and F. Nevertheless, we only need these resolutions in order to compute $Z^{(d)}_{\mathrm{top},0}(f, s)$ and $Z^{(d)}_{\mathrm{top},0}(F, s)$. There is a general Thom–Sebastiani theory for the motivic zeta function, see [**7**]; the motivic zeta function is a more general invariant, also defined by Denef and Loeser, which determines $Z^{(d)}_{\mathrm{top},0}$. In this paper, we give by elementary means formulas for $Z_{\mathrm{top},0}(F, s)$ and $Z^{(2)}_{\mathrm{top},0}(F, s)$ in terms of $Z_{\mathrm{top},0}(\mathrm{F}, s)$ and $Z^{(2)}_{\mathrm{top},0}(\mathrm{F}, s)$.

The Seifert form of the link is not a complete topological invariant for $n = 1$ (reducible case) and $n = 2$. In [**8**], P. du Bois and F. Michel found plane curve singularities with the same integral Seifert form but which have different topological type and the suspension method was used in [**3**] to deduce the case $n = 2$. Fix $a, b \in \mathbb{N}$ odd integers. Let $C_{a,b}$ be the germ at the origin of the plane curve singularity defined by

$$f_{a,b}(x, y) = ((y^2 - x^3)^2 - x^{b+6} - 4yx^{(b+9)/2})((x^2 - y^5)^2 - y^{a+10} - 4xy^{(a+15)/2}).$$

P. du Bois and F. Michel proved that if $b \geqslant 11$ and $a \neq r + 8$ then the integral Seifert forms corresponding to the singularities $C_{a,b}$ and $C_{b-8,a+8}$ are isomorphic but they are not topologically equivalent.

Let $g_{a,b} = f_{a,b} + z^2 + u^2$ be the double suspension of the germ $f_{a,b}$ and let $(V_{a,b}, 0) = (g^{-1}_{a,b}(0), 0) \subset (\mathbb{C}^4, 0)$ be its corresponding germ of isolated hypersurface singularity.

If $b \geqslant 11$ and $b \neq a + 8$ then Proposition A shows that the Seifert forms of the singularities $V_{a,b}$ and $V_{b-8,a+8}$ are isomorphic. It turns out that the singularities $V_{a,b}$ and $V_{b-8,a+8}$ have the same topological type. We can compute $Z^{(d)}_{\mathrm{top},0}(g_{a,b}, s)$ from the resolution graph of $C_{a,b}$ given in [**8**]. An iterated application of the suspension formula for $Z^{(d)}_{\mathrm{top},0}$, $d = 1, 2$, will give us $Z^{(1)}_{\mathrm{top},0}(g_{a,b}, s)$ without computing its embedded resolution. Then, we will verify that

$$Z^{(1)}_{\mathrm{top},0}(g_{a,b}, s) \neq Z^{(1)}_{\mathrm{top},0}(g_{b-8,a+8}, s), \quad b \geqslant 11 \text{ and } a \neq r + 8,$$

and we deduce that the Denef–Loeser zeta function is not a topological invariant.

The paper is organized as follows. In § 1 we compute the Denef–Loeser zeta function of a suspension singularity in terms of Denef–Loeser zeta functions of the original singularity. As an application in § 2 we compute the Denef–Loeser zeta function of the double suspensions of the previous plane curves and check that they are different. In § 3, we discuss this invariant and μ-constant deformations. Finally in § 4 we prove that if the monodromy conjecture for the Denef–Loeser zeta function is true for a germ of a hypersurface singularity then it is true for its suspension.

1. *The Denef–Loeser zeta function of the suspension* $f + z^2$

We recall the first Denef–Loeser definitions of the *local topological zeta functions* $Z^{(d)}_{\mathrm{top},0}(f,s) \in \mathbb{Q}(s)$, $d \in \mathbb{Z}_{>0}$. They are rational functions associated to any germ $f : (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$.

Let $\pi : (Y,\mathscr{D}) \longrightarrow (\mathbb{C}^{n+1},0)$ be an embedded resolution of the germ of hypersurface $(V,0)$ with exceptional divisor $\mathscr{D} := \pi^{-1}(0)$. Let $E_i, i \in I$, be the irreducible components of the divisor $\pi^{-1}(f^{-1}(0))$. For each subset $J \subset I$ we set

$$E_J := \bigcap_{j\in J} E_j \quad \text{and} \quad \check{E}_J := E_J \setminus \bigcup_{j\notin J} E_{J\cup\{j\}}.$$

For each $j \in I$, we denote by N_j the multiplicity of E_j in the divisor of the function $f \circ \pi$ and we denote by $v_j - 1$ the multiplicity of E_j in the divisor of $\pi^*(\omega)$ where ω is a non-vanishing holomorpic $(n+1)$-form in a neighbourhood of the origin in $\mathbb{C}^{n+1}$.

To f and $d \in \mathbb{N}\setminus\{0\}$ one associates the *local topological zeta function* of f and d

$$Z^{(d)}_{\mathrm{top},0}(f,s) := \sum_{\substack{J\subset I\\ \forall j\in J: d|N_j}} \chi(\check{E}_J \cap \mathscr{D}) \prod_{j\in J} \frac{1}{v_j + N_j s} \in \mathbb{Q}(s), \tag{1.1}$$

where χ denotes the Euler–Poincaré characteristic. Throughout the paper we only use $Z_{\mathrm{top},0}(f,s) = Z^{(1)}_{\mathrm{top},0}(f,s)$ and $Z^{(2)}_{\mathrm{top},0}(f,s)$. Let us rename them the *Denef–Loeser zeta functions* of f and d.

THEOREM 1.1. *Let* $f : (\mathbb{C}^{n+1},0) \longrightarrow (\mathbb{C},0)$ *be a germ of holomorphic function. If* $F = f + z^2 : (\mathbb{C}^{n+2},0) \longrightarrow (\mathbb{C},0)$ *then the following equalities hold:*

$$Z_{\mathrm{top},0}(F,s) = \frac{1}{2s+1} + \frac{s(2s+3)}{2(s+1)(2s+1)} Z_{\mathrm{top},0}\left(f, s+\frac{1}{2}\right) - \frac{3s}{2(s+1)} Z^{(2)}_{\mathrm{top},0}\left(f, s+\frac{1}{2}\right)$$

and

$$Z^{(2)}_{\mathrm{top},0}(F,s) = \frac{1}{2s+1} - \frac{2s+3}{2(2s+1)} Z_{\mathrm{top},0}\left(f, s+\frac{1}{2}\right) - \frac{1}{2} Z^{(2)}_{\mathrm{top},0}\left(f, s+\frac{1}{2}\right).$$

Proof. The key points in this proof are the proper birational morphism (PBM) principle, the stratum principle and the Fubini principle of [**4**], applied to the Denef–Loeser zeta functions $Z_{\mathrm{top},0}$ and $Z^{(2)}_{\mathrm{top},0}$.

We can suppose that in the resolution π that if $E_J \neq \emptyset$ then there is at most one $j \in J$ such that N_j is odd. Otherwise, one can do additional blow-ups along the intersection of two exceptional divisors having odd multiplicities arising in the above-mentioned situation. Let us denote by $\mathscr{P}_k(I)$ the subset of $\mathscr{P}(I)$ such that $J \in \mathscr{P}_k(I)$ if and only if there are exactly k elements in J with odd N-invariant. As in [**4**], we can define $Z_{\mathrm{top},0}(\eta,f,s)$ and $Z^{(2)}_{\mathrm{top},0}(\eta,f,s)$ for functions f and maximal forms η. Then, we have

$$Z_{\mathrm{top},0}(F,s)=\sum_{J\in\mathscr{P}(I)}\chi(\check{E}_J\cap\pi^{-1}(0))Z_{\mathrm{top},0}\left(\left(\prod_{j\in J}x_j^{\nu_j-1}\right)\omega, z^2-\prod_{j\in J}x_j^{N_j},s\right)$$

and

$$Z^{(2)}_{\mathrm{top},0}(F,s)=\sum_{J\in\mathscr{P}(I)}\chi(\check{E}_J\cap\pi^{-1}(0))Z^{(2)}_{\mathrm{top},0}\left(\left(\prod_{j\in J}x_j^{\nu_j-1}\right)\omega, z^2-\prod_{j\in J}x_j^{N_j},s\right).$$

For the sake of simplicity let us assume that $J=\{0,1,\ldots,r\}$ and let $m_j:=[N_j/2]$ be the integer part of $N_j/2$. Moreover, let us assume that N_j is even for $j=1,\ldots,r$. By induction on m_r and r, we have:

$$\begin{aligned}Z_{\mathrm{top},0}&\left(\left(\prod_{j\in J}x_j^{\nu_j-1}\right)\omega, z^2-\prod_{j\in J}x_j^{N_j},s\right)\\&=\frac{1}{\nu_0}\sum_{k=1}^{r}m_k\left(\prod_{j=1}^{k}\frac{1}{\nu_j}\right)\left(\prod_{j=k}^{r}\frac{1}{\nu_j+m_j(2s+1)}\right)\\&\quad+\left(\prod_{j=1}^{r}\frac{1}{\nu_j+m_j(2s+1)}\right)Z_{\mathrm{top},0}(x_0^{\nu_0-1}\omega, z^2-x_0^{N_0},s).\end{aligned}$$

The same formula is true if we replace $Z_{\mathrm{top},0}$ by $Z^{(2)}_{\mathrm{top},0}$ everywhere. We need the following lemmas to prove Theorem 1.1.

LEMMA 1.2. *With the same notations as above,*
(i) *if N_0 is even then*

$$Z_{\mathrm{top},0}(x_0^{\nu_0-1}\omega, z^2-x_0^{N_0},s)=\frac{1}{\nu_0+m_0(2s+1)}\left(\frac{m_0}{\nu_0}+\frac{2}{1+s}-1\right)$$

and

$$Z^{(2)}_{\mathrm{top},0}(x_0^{\nu_0-1}\omega, z^2-x_0^{N_0},s)=\frac{1}{\nu_0+m_0(2s+1)}\left(\frac{m_0}{\nu_0}-1\right);$$

(ii) *if N_0 is odd then*

$$Z_{\mathrm{top},0}(x_0^{\nu_0-1}\omega, z^2-x_0^{N_0},s)=\frac{1}{2\nu_0+N_0(2s+1)}\left(\frac{N_0}{\nu_0}+\frac{1}{1+s}+1\right)$$

and

$$Z^{(2)}_{\mathrm{top},0}(x_0^{\nu_0-1}\omega, z^2-x_0^{N_0},s)=\frac{1}{2\nu_0+N_0(2s+1)}\left(\frac{2m_0+1}{\nu_0}-1\right).$$

The proof of this lemma is straightforward. Using induction one deduces the following two lemmas.

LEMMA 1.3. *If N_0 is even then*

$$\begin{aligned}Z_{\mathrm{top},0}&\left(\prod_{j=0}^{r}x_j^{\nu_j-1}\omega, z^2-\prod_{j=0}^{r}x_j^{N_j},s\right)\\&=\frac{1}{2s+1}\prod_{j=0}^{r}\frac{1}{\nu_j}-\frac{2s^2}{(s+1)(2s+1)}\prod_{j=0}^{r}\frac{1}{\nu_j+N_j(s+\frac{1}{2})}\end{aligned}$$

and

$$Z^{(2)}_{\text{top},0}\left(\prod_{j=0}^{r} x_j^{v_j-1}\omega, z^2-\prod_{j=0}^{r} x_j^{N_j}, s\right) = \frac{1}{2s+1}\prod_{j=0}^{r}\frac{1}{v_j} - \frac{2(s+1)}{2s+1}\prod_{j=0}^{r}\frac{1}{v_j+N_j(s+\frac{1}{2})}.$$

LEMMA 1.4. *If N_0 is odd then*

$$Z_{\text{top},0}\left(\prod_{j=0}^{r} x_j^{v_j-1}\omega, z^2-\prod_{j=0}^{r} x_j^{N_j}, s\right)$$
$$= \frac{1}{2s+1}\prod_{j=0}^{r}\frac{1}{v_j} + \frac{s(2s+3)}{2(s+1)(2s+1)}\prod_{j=0}^{r}\frac{1}{v_j+N_j(s+\frac{1}{2})}$$

and

$$Z^{(2)}_{\text{top},0}\left(\prod_{j=0}^{r} x_j^{v_j-1}\omega, z^2-\prod_{j=0}^{r} x_j^{N_j}, s\right) = \frac{1}{2s+1}\prod_{j=0}^{r}\frac{1}{v_j} - \frac{2s+3}{2(2s+1)}\prod_{j=0}^{r}\frac{1}{v_j+N_j(s+\frac{1}{2})}.$$

In [**5**] J. Denef and F. Loeser showed the following equality:

$$1 = Z_{\text{top},0}(f,0) = \sum_{J\in\mathscr{P}(I)}\chi(\check{E}_J\cap\pi^{-1}(0))\prod_{j\in J}\frac{1}{v_j}.$$

Note also that the formulæ for the Denef–Loeser zeta function can be rewriten as

$$Z_{\text{top},0}(f,s) := \sum_{J\in\mathscr{P}_0(I)\cup\mathscr{P}_1(I)}\chi(\check{E}_J\cap\pi^{-1}(0))\prod_{j\in J}\frac{1}{v_j+N_js}$$

and

$$Z^{(2)}_{\text{top},0}(f,s) := \sum_{J\in\mathscr{P}_0(I)}\chi(\check{E}_J\cap\pi^{-1}(0))\prod_{j\in J}\frac{1}{v_j+N_js}.$$

These formulæ give the proof of Theorem 1.1. □

2. *Computing examples*

We compute now the Denef–Loeser zeta function for $C_{a,b}$. Since a and b are odd, we denote $a = 2a_1 + 1$ and $b = 2b_1 + 1$. Figure 1 contains the dual graph of the embedded resolution of the plane curve singularity $C_{a,b}$.

In Table 1 we list the invariants of the resolution of $C_{a,b}$. Multiplicities can be found in [**8**]. There is an easy recursive formula to compute the v-invariant. If E_i is an exceptional component of the resolution obtained by the blowing-up of a point P_i, let $\mathscr{P}(i)$ be the set of indices j such that $P_i \in E_j$ before the blow-up. It is well known that $\#\mathscr{P}(i) \leqslant 2$. Then

$$v_i = \sum_{j\in\mathscr{P}(i)}(v_j-1)+2.$$

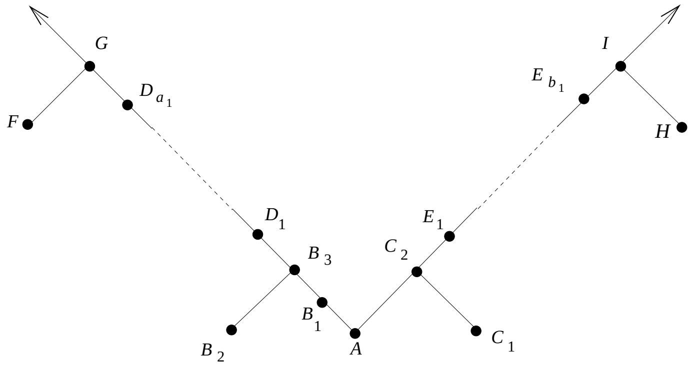

FIGURE 1.

TABLE 1.

	N	v
A	8	2
B_1	12	3
B_2	14	4
B_3	28	7
C_1	10	3
C_2	20	5
D_i, $1 \leqslant i \leqslant a_1$	$28 + 2i$	$7 + i$
E_i, $1 \leqslant i \leqslant b_1$	$20 + 2i$	$5 + i$
F	$28 + a$	$8 + a_1$
G	$56 + 2a$	$14 + a$
H	$20 + b$	$6 + b_1$
I	$40 + 2b$	$10 + b$

Applying Veys' formula [**15**] one has

$$\begin{aligned} Z_{\mathrm{top},0}(C_{a,b}, s) = {} & \frac{1}{14 + a + (56 + 2a)s}\left(1 + \frac{1}{1+s}\right) \\ & + \frac{1}{7 + 28s}\left(\frac{a}{14 + 2a + (56 + 2a)s} + 1\right) \\ & + 11\frac{1}{7 + 28s} \cdot \frac{1}{5 + 20s} + \frac{1}{5 + 20s}\left(\frac{b}{10 + b + (40 + 2b)s} + 1\right) \\ & + \frac{1}{10 + b + (40 + 2b)s}\left(1 + \frac{1}{1+s}\right) \end{aligned}$$

and

$$\begin{aligned} Z^{(2)}_{\mathrm{top},0}(C_{a,b}, s) = {} & -\frac{1}{14 + a + (56 + 2a)s} + \frac{1}{7 + 28s}\left(\frac{a}{14 + 2a + (56 + 2a)s} + 1\right) \\ & + 11\frac{1}{7 + 28s} \cdot \frac{1}{5 + 20s} + \frac{1}{5 + 20s}\left(\frac{b}{10 + b + (40 + 2b)s} + 1\right) \\ & - \frac{1}{10 + b + (40 + 2b)s}. \end{aligned}$$

Take the easiest case, that is, $b = 11$ and $a = 1$. Then the Denef–Loeser zeta functions of the curves are

$$Z_{\mathrm{top},0}(C_{1,11}, s) = \frac{1}{35}\frac{11\,025 + 112\,665s + 401\,056s^2 + 565\,684s^3 + 239\,808s^4}{(15+58s)(1+s)(1+4s)^2(21+62s)},$$

$$Z^{(2)}_{\mathrm{top},0}(C_{1,11}, s) = \frac{1}{35}\frac{7245 + 60\,060s + 151\,156s^2 + 105\,408s^3}{(15+58s)(1+4s)^2(21+62s)},$$

and

$$Z_{\mathrm{top},0}(C_{3,9}, s) = \frac{1}{35}\frac{11\,305 + 114\,081s + 402\,960s^2 + 566\,452s^3 + 239\,808s^4}{(17+62s)(1+s)(1+4s)^2(19+58s)},$$

$$Z^{(2)}_{\mathrm{top},0}(C_{3,9}, s) = \frac{1}{35}\frac{7525 + 61\,196s + 151\,924s^2 + 105\,408s^3}{(17+62s)(1+4s)^2(19+58s)}.$$

Their first suspensions $C_{1,11} + z^2$ and $C_{3,9} + z^2$ have the following Denef–Loeser zeta functions:

$$Z_{\mathrm{top},0}(C_{1,11} + z^2, s) = \frac{1}{35}\frac{242\,168s^4 + 905\,219s^3 + 1\,266\,245s^2 + 782\,880s + 180\,180}{(22+29s)(3+4s)^2(26+31s)(s+1)},$$

$$Z_{\mathrm{top},0}(C_{3,9} + z^2, s) = \frac{1}{35}\frac{242\,168s^4 + 906\,147s^3 + 1\,269\,225s^2 + 786\,240s + 181\,440}{(24+31s)(3+4s)^2(24+29s)(s+1)},$$

and

$$Z^{(2)}_{\mathrm{top},0}(C_{1,11} + z^2, s) = Z^{(2)}_{\mathrm{top},0}(C_{3,9} + z^2, s) = \frac{1}{35}\frac{232s + 185}{(3+4s)^2}.$$

The Denef–Loeser zeta functions of $V_{1,11}$ and $V_{3,9}$ are different because

$$Z_{\mathrm{top},0}(V_{1,11}, s) = \frac{1}{35}\frac{239\,808s^4 + 3\,298\,868s^3 + 10\,502\,888s^2 + 12\,683\,713s + 5\,301\,625}{(73+58s)(5+4s)^2(83+62s)(s+1)}$$

and

$$Z_{\mathrm{top},0}(V_{3,9}, s) = \frac{1}{35}\frac{239\,808s^4 + 3\,299\,636s^3 + 10\,515\,288s^2 + 12\,714\,681s + 5\,322\,625}{(79+62s)(5+4s)^2(77+58s)(s+1)}.$$

We note that since the Denef–Loeser zeta function is not a topological invariant then the motivic zeta function, first introduced by J. Denef and F. Loeser, is not a topological invariant, see [**6**].

3. *Questions on μ-constant families*

Though the examples in the last section are topologically equivalent, we do not know if they are in a μ-constant family (and we conjecture that it is not the case). The following interesting question is still open.

QUESTION 3.1. Is the Denef–Loeser zeta function constant in a μ-constant family of isolated hypersurface singularities?

The answer is 'yes' in many other examples worked out by the authors. It is also true in families with simultaneous embedded resolution, by the very definition of the Denef–Loeser zeta functions. Since in the case of curves it is well known that the condition μ-constant is equivalent to the same embedded resolution, therefore any μ-constant family of plane curve singularities has a constant Denef–Loeser zeta function. Note that this is not true for the Bernstein polynomial (for a conjecture relating poles of the Denef–Loeser zeta function with roots of the Bernstein polynomial of the singularity see [**5**, **10**, **16**]).

Another kind of interesting example comes from families of singularities which are non-degenerate with respect to the Newton polyhedra, since in this case an explicit formula for the Denef–Loeser zeta function can be found in [**5**]. It may be interesting to consider μ-constant families such that any element is non-degenerate with respect to its Newton polyhedron, but this one is non-constant.

Examples of such families were worked out by Briançon and Speder, for example, consider the μ-constant family of weighted homogeneous polynomials

$$f_t(x,y,z) = x^6 + z^{12} + y^{11}z + txy^{10}.$$

A priori, since the Newton polyhedron is not the same for $t = 0$, one could expect to obtain different Denef–Loeser zeta functions $Z^{(d)}_{\mathrm{top},0}(f_t, s)$ by computing them from the Newton polyhedron formaulæ of [**5**]. Nevertheless, the results

$$Z^{(1)}_{\mathrm{top},0}(f_t,s) = \frac{52s+4}{(s+1)(12s+4)}, \quad Z^{(2)}_{\mathrm{top},0}(f_t,s) = \frac{52}{(12s+4)}$$

do not depend on t. This is not surprising since in this case the family $\{f_t = 0\}$ has a strong simultaneous resolution in the sense of B. Teissier. The resolution only needs two blow-ups. The first one blows-up the origin and the second one blows-up along the projective line obtained as the intersection of the first exceptional divisor and the strict transform of the zero locus of f_t.

Another interesting example is the following family, considered by K. Altmann in [**1**]:

$$f_t(x,y,z) = x^5 + z^5 + y(yz + tx^2)^2 + y^6.$$

It gives an example of equisingular deformation below the Newton boundary. In fact for $t \neq 0$ the Newton polyhedron is degenerated. In this case for any t the isolated surface singularity defined by f_t is *superisolated*. In [**4**], the authors gave a formula for the Denef–Loeser zeta function for such singularities. This formula is the same for all t.

4. *Monodromy conjecture*

It is clear from the definition that each exceptional divisor of an embedded resolution $\pi : (Y, \mathscr{D}) \longrightarrow (\mathbb{C}^{n+1}, 0)$ of the germ $(V, 0)$ gives a candidate pole of the rational function $Z_{\mathrm{top},0}(f,s)$. Nevertheless only a few of them give a pole of $Z_{\mathrm{top},0}(f,s)$. This fact is related with the *monodromy conjecture* for Denef–Loeser zeta functions, see [**5**].

The *local monodromy conjecture* states that if s_0 is a pole for the Denef–Loeser zeta function $Z^{(d)}_{\mathrm{top},0}(f,s)$ of the local isolated singularity defined by f, then $\exp(2i\pi s_0)$ is an eigenvalue of the complex algebraic monodromy of the germ $(f^{-1}(0), 0) \subset (\mathbb{C}^{n+1}, 0)$.

The monodromy conjecture has been proved for curves by F. Loeser [**10**], see also [**14**, **15**], for some families of surfaces, see [**12**] and [**4**]; and for all singularities of hypersurfaces defined by analytic functions which are non-degenerated with respect to its Newton polyhedra (the polyhedra verifying another extra condition), see [**11**].

COROLLARY 4.1. *Let* $f : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)$ *be an analytic function whose zero locus defines a germ of isolated hypersurface singularity. Suppose that* $Z_{\mathrm{top},0}(f,s)$ *and* $Z^{(2)}_{\mathrm{top},0}(f,s)$ *verify the monodromy conjecture. Then the Denef–Loeser zeta functions* $Z_{\mathrm{top},0}(F,s)$ *and* $Z^{(2)}_{\mathrm{top},0}(F,s)$ *of its suspension* $F = f + z^2 : (\mathbb{C}^{n+2}, 0) \longrightarrow (\mathbb{C}, 0)$ *also verify the monodromy conjecture.*

Proof. It is well known that if $\exp(2i\pi s_0)$ is an eigenvalue of the algebraic monodromy of f then $\exp(2i\pi(s_0+1/2))$ is an eigenvalue of the algebraic monodromy of its suspension F.

Moreover since f and F are reduced germs then the poles $s_0 = -1/2$ and $s_0 = -1$ of $Z_{\mathrm{top},0}(F,s)$ and $Z^{(2)}_{\mathrm{top},0}(F,s)$ verify the monodromy conjecture.

One deduces from Theorem 1.1 that the remaining poles of $Z_{\mathrm{top},0}(F,s)$ and $Z^{(2)}_{\mathrm{top},0}(F,s)$ are obtained by adding 1/2 to the poles of $Z_{\mathrm{top},0}(f,s)$ and $Z^{(2)}_{\mathrm{top},0}(f,s)$. Then the result is proved. □

Note added in proof, November 2001. We would like to thank F. Loeser, who after reading this paper has pointed out to us that the following formula for any suspension $f + z^n$ can be deduced from results in [**7**]:

$$Z_{\mathrm{top},0}(f + z^n, s) = \left(\frac{n-1}{n}\right)\left(\frac{s}{s+1}\right)\left(\frac{s+1+1/n}{s+1/n}\right) Z_{\mathrm{top},0}\left(f, s+\frac{1}{n}\right) - \frac{s}{s+1}\sum_{e|n, e\neq 1} \frac{(e+1)\phi(e)}{n} Z^{(e)}_{\mathrm{top},0}\left(f, s+\frac{1}{n}\right) + \frac{1}{ns+1},$$

where $\phi(e)$ denotes the *Euler ϕ-function* (and similar formulas for the twisted Denef–Loeser zeta functions $Z^{(d)}_{\mathrm{top},0}$). In particular, Corollary 4.1 remains true for the suspension $F = f + z^n$.

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