Алгебра и анализ
Том 11 (1999), вып. 5

St. Petersburg Math. J.
Vol. 11 (2000), No. 5

TOPOLOGY OF MEROMORPHIC GERMS AND ITS APPLICATIONS

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Dedicated to the memory of V. A. Rokhlin

It was not until recently that the germs of meromorphic functions became an object of study in the singularity theory. In [11] T. Suwa described the versal deformations of meromorphic germs. In [1] V.I. Arnol'd classified the meromorphic germs with respect to some equivalence relations. In [4] the present authors started the study of the topological properties of meromorphic germs. Some applications of the technique developed in [4] were described in [5] and [6].

The aim pursued by the authors in [4] was to elaborate notions and techniques for computing certain invariants of polynomials, such as the Euler characteristics of fibres, and the zeta-functions of the monodromy transformations associated with a polynomial (see [5]). For this reason, some important basic properties of the notions related to the topology of meromorphic germs were not discussed there, with a consequent lack of understanding of the general constructions. In the present paper we want to partially fill this gap. At the same time, we describe the relationship with some earlier results and their generalizations.

A polynomial P in $n+1$ complex variables gives rise to a map P from the affine complex space $\mathbb{C}^{n+1}$ to the complex line $\mathbb{C}$. It is well known that the map P is a C^∞-locally trivial fibration over the complement to a finite set in $\mathbb{C}$. The smallest of such sets is called the *bifurcation set* or the set of *atypical values* of the polynomial P. It is of interest to describe the topology of a fibre of that fibration, as well as the behavior of this topology under the monodromy transformations corresponding to loops around atypical values of the polynomial P. The monodromy transformation corresponding to a circle of large radius enclosing all atypical values (the monodromy transformation of the polynomial P at infinity) is particularly interesting.

The initial idea was to reduce the calculation of the zeta-function of the monodromy transformation at infinity (and, thus, the Euler characteristic of a generic fibre) for the polynomial P to local problems associated with various points at infinity, i.e., with points lying at the infinite hyperplane $\mathbb{CP}^n_\infty$ in the projective compactification $\mathbb{CP}^{n+1}$ of the affine space $\mathbb{C}^{n+1}$. For holomorphic germs, such a localization was used in [3]. This localization can be expressed in terms of an integral against the Euler characteristic, a notion that emerged in the works of V.A. Rokhlin's school (see [12]). However, the same technique does not apply to a polynomial function, because at a point of the infinite hyperplane $\mathbb{CP}^n_\infty$ the latter only determines a meromorphic (nonholomorphic) germ. Thus, the idea of reduction to the calculation of the local zeta-functions corresponding to various points of the hyperplane $\mathbb{CP}^n_\infty$ is impeded by the absence of certain notions

1991 *Mathematics Subject Classification.* Primary 32C18.
Key words and phrases. Germs of meromorphic functions, Milnor fibre, atypical values.
The first author was partially supported by Iberdrola, by INTAS (grant no. 96–0713), and by RFBR (grant no. 98–01–00612). The last two authors were partially supported by DGCYT (grant no. PB97-0284-C02-01).

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(such as the Milnor fibre, the monodromy transformation, etc.) for meromorphic germs. This necessitates defining the corresponding invariants and elaborating a technique for their calculation.

§1. Basic properties

A *meromorphic germ* at the origin in the complex space $\mathbb{C}^{n+1}$ is the ratio $f = \frac{P}{Q}$ of two holomorphic germs P and Q on $(\mathbb{C}^{n+1}, 0)$. For the goal describe above, the following equivalence relation is appropriate. Two meromorphic germs $f = \frac{P}{Q}$ and $f' = \frac{P'}{Q'}$ are *equal* if and only if $P' = P \cdot U$ and $Q' = Q \cdot U$ for some holomorphic germ U not equal to zero at the origin: $U(0) \neq 0$.

Any meromorphic germ $f = \frac{P}{Q}$ gives rise to a map of the complement to the indeterminacy locus $\{P = Q = 0\}$ into the complex projective line $\mathbb{CP}^1$. For any $c \in \mathbb{CP}^1$, there exists $\varepsilon_0 > 0$ such that for any ε with $0 < \varepsilon \leq \varepsilon_0$ the map

$$f : B_\varepsilon \setminus \{P = Q = 0\} \to \mathbb{CP}^1$$

is a C^∞-locally trivial fibration over a sufficiently small punctured neighborhood of c; here B_ε is the closed ball of radius ε centred at the origin in $\mathbb{C}^{n+1}$ (see [4]).

Definition. The fibre

$$\mathcal{M}_f^c = \left\{ z \in B_\varepsilon : f(z) = \frac{P(z)}{Q(z)} = c' \right\}$$

of that fibration (for c' sufficiently close to c) is called the *c-Milnor fibre* of the meromorphic germ f.

The Milnor fibre $\mathcal{M}_f^c$ is a (noncompact) n-dimensional complex manifold with boundary.

Definition. The monodromy transformation of the above fibration corresponding to a simple (small) loop around the value c is called the *c-monodromy transformation* of the meromorphic germ f.

Definition. A value $c \in \mathbb{CP}^1$ is said to be *typical* if the map $f : B_\varepsilon \setminus \{P = Q = 0\} \to \mathbb{CP}^1$ is a C^∞-locally trivial fibration over some neighbourhood of c (including the point c itself).

Observe that if c is a typical value, then the corresponding monodromy transformation is isotopic to the identity.

Theorem 1. *There exists a finite set $\Sigma \subset \mathbb{CP}^1$ such that for all $c \in \mathbb{CP}^1 \setminus \Sigma$ the c-Milnor fibres of f are diffeomorphic to one another, and the c-monodromy transformations are trivial (i.e., isotopic to the identity). In particular, the set of atypical values is finite.*

Proof. A *resolution* of the germ f is a modification of the space $(\mathbb{C}^{n+1}, 0)$ (i.e., a proper analytic map $\pi : \mathcal{X} \to \mathcal{U}$ of a smooth analytic manifold $\mathcal{X}$ onto a neighborhood $\mathcal{U}$ of the origin in $\mathbb{C}^{n+1}$ that is an isomorphism outside of a proper analytic subspace in $\mathcal{U}$) such that the preimage $\pi^{-1}(H)$ of the hypersurface $H = \{P = 0\} \cup \{Q = 0\}$ is a normal crossing divisor at every point of the manifold $\mathcal{X}$. We assume that the map π is an isomorphism outside of the hypersurface H.

The fact that the preimage $\pi^{-1}(H)$ is a divisor with normal crossings implies that in a neighborhood of any point of it there exists a local system of coordinates $y_0, y_1, \ldots, y_n$

such that the liftings $\widetilde{P} = P \circ \pi$ and $\widetilde{Q} = Q \circ \pi$ of the functions P and Q to the space $\mathcal{X}$ of modification are equal to $u \cdot y_0^{k_0} y_1^{k_1} \cdots y_n^{k_n}$ and $v \cdot y_0^{l_0} y_1^{l_1} \cdots y_n^{l_n}$, respectively; here $u(0) \neq 0$, $v(0) \neq 0$, and k_i and l_i are nonnegative.

Remark 1. The values 0 and ∞ in the projective line $\mathbb{CP}^1$ are used as distinguished points for convenience; then for the numerator and denominator we have the usual notion of resolution of a holomorphic germ.

Additional blow-ups can be made along the intersections of pairs of irreducible components of the divisor $\pi^{-1}(H)$; so, the lifting $\widetilde{f} = f \circ \pi = \frac{\widetilde{P}}{\widetilde{Q}}$ of the function f can be defined as a holomorphic mapping from the manifold $\mathcal{X}$ to the complex projective line $\mathbb{CP}^1$. This condition means that $\widetilde{P} = V \cdot P'$, $\widetilde{Q} = V \cdot Q'$, where V is a section of a line bundle (say, $\mathcal{L}$,) over $\mathcal{X}$, *and* P' and Q' are sections of the line bundle $\mathcal{L}^{-1}$ that have no common zeros on $\mathcal{X}$. We put $\widetilde{f}' = \frac{P'}{Q'}$.

On each component of the divisor $\pi^{-1}(H)$ and on all intersections of several of them, $\widetilde{f}'$ determines a map to the projective line $\mathbb{CP}^1$. These maps have finitely many critical values, say a_1, a_2, ..., a_s.

Remark 2. If the function $\widetilde{f}'$ is constant on a component of the divisor $\pi^{-1}(H)$ or on the intersection of some components, then this constant is a critical value. The value of the function $\widetilde{f}'$ on the intersection of $n+1$ components (this intersection is zero-dimensional) should also be viewed as a critical value.

Let $c \in \mathbb{CP}^1$ be different from a_1, a_2, ..., a_s. We show that for all c' in some neighborhood of c (including the point c itself) the c'-Milnor fibres of the meromorphic function f are diffeomorphic to one another, and that the c'-monodromy transformations are trivial.

Let $r^2(z)$ denote the square of the distance from z to the origin in the space $\mathbb{C}^{n+1}$, and let $\widetilde{r}^2(x) = r^2(\pi(x))$ be the lifting of this function to the space $\mathcal{X}$ of modification. In order to define the c'-Milnor fibre, we must choose $\varepsilon_0 > 0$ (the Milnor radius) so small that the level manifold $\{\widetilde{r}^2(x) = \varepsilon^2\}$ is transversal to $\{\widetilde{f}'(x) = c'\}$ for all ε with $0 < \varepsilon \leq \varepsilon_0$. Let $\varepsilon_0 = \varepsilon_0(c)$ be the Milnor radius for the value c. Since at the same time $\{\widetilde{f}'(x) = c\}$ is transversal to the components of the divisor $\pi^{-1}(H)$ and to all their intersections, ε_0 is also the Milnor radius for all c' belonying to a neighborhood of the point $c \in \mathbb{CP}^1$ (and the level manifold $\{\widetilde{f}'(x) = c'\}$ is transversal to the components of the divisor $\pi^{-1}(H)$ and to their intersections). This implies our claim. $\square$

Remark 3. The c-Milnor fibre for a generic value $c \in \mathbb{CP}^1$ can be called a *generic Milnor fibre* of the meromorphic germ f. It is easily seen that a generic Milnor fibre of a meromorphic germ can be viewed as embedded in the c-Milnor fibre for any value $c \in \mathbb{CP}^1$. Moreover, the Euler characteristic of a generic Milnor fibre of a meromorphic germ is equal to zero, and the zeta-function of the corresponding monodromy transformation (see [4]) is equal to $(1-t)^0 = 1$.

§2. Isolated singularities and the Euler characteristic of the 0-Milnor fibre

Let P be a polynomial in $n+1$ complex variables. Suppose that the closure $\overline{V}_{t_0} \subset \mathbb{CP}^{n+1}$ of the level set $V_{t_0} = \{P = t_0\} \subset \mathbb{C}^{n+1}$ in the complex projective space $\mathbb{CP}^{n+1} \supset \mathbb{C}^{n+1}$ has isolated singular points only. Let $A_1, \ldots, A_r$ be those of them that lie in the affine space $\mathbb{C}^{n+1}$, and let $B_1, \ldots, B_s$ be those lying in the infinite hyperplane $\mathbb{CP}^n_\infty$. For t sufficiently close to t_0 (thus, generic), the closure $\overline{V}_t \subset \mathbb{CP}^{n+1}$ of the level set $V_t = \{P = t\} \subset \mathbb{C}^{n+1}$ is nonsingular inside the space $\mathbb{C}^{n+1}$ and may have isolated singularities only at the points $B_1, \ldots, B_s$. It is known that

$$\chi(V_t) - \chi(V_{t_0}) = (-1)^{n+1}\left(\sum_{i=1}^{r} \mu_{A_i}(V_{t_0}) + \sum_{j=1}^{s}(\mu_{B_j}(\overline{V}_{t_0}) - \mu_{B_j}(\overline{V}_t))\right). \tag{1}$$

We shall present a somewhat more general statement about meromorphic germs. Combined with a formula in [6], this statement yields (1). (That formula expresses the difference between the Euler characteristics of two level sets of a polynomial P in terms of the meromorphic germs determined by P; of these level sets, one is taken generic, the other one is special.)

Theorem 2. *Let $f = \frac{P}{Q}$ be a meromorphic germ on the space $(\mathbb{C}^{n+1}, 0)$ such that the numerator P has an isolated critical point at the origin and, for $n = 1$, the germs of the curves $\{P = 0\}$ and $\{Q = 0\}$ have no common irreducible components. Then for a generic $t \in \mathbb{C}$ we have*

$$\chi(\mathcal{M}_f^0) = (-1)^n(\mu(P, 0) - \mu(P + tQ, 0)).$$

Here $\mu(g, 0)$ stands for the usual Milnor number of the holomorphic germ g at the origin.

Proof. The Milnor fibre $\mathcal{M}_f^0$ of the meromorphic germ f admits the following description. Let ε be sufficiently small (thus, ε is the Milnor radius for the holomorphic germ P). Then

$$\mathcal{M}_f^0 = B_\varepsilon(0) \cap (\{P + tQ = 0\} \setminus \{P = Q = 0\})$$

for $t \neq 0$ with $|t|$ sufficiently small (thus, t is generic). We note that the zero-level set $\{P + tQ = 0\}$ is nonsingular off the origin for $|t|$ sufficiently small. Since the space $B_\varepsilon(0) \cap \{P = Q = 0\}$ is homeomorphic to a cone, its Euler characteristic is equal to 1. Therefore,

$$\chi(\mathcal{M}_f^0) = \chi(B_\varepsilon(0) \cap (\{P + tQ = 0\}) - 1.$$

Now Theorem 2 is a consequence of the following well-known fact (see, e.g., [2]).

Proposition. Let $P : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function with an isolated critical point at the origin, and let P_t be its deformation ($P_0 = P$). If ε is sufficiently small, then, for $|t|$ small,

$$(-1)^n(\chi(B_\varepsilon(0) \cap \{P_t = 0\}) - 1)$$

is equal to the number of the critical points of P (counted with multiplicities) that split out of the zero level set, i.e., to

$$\mu(P, 0) - \sum_{Q \in \{P_t = 0\} \cap B_\varepsilon} \mu(P_t, Q). \qquad \square$$

Example 1. The example of $P = xy$ and $Q = x$ shows that the absence of common components of the curves $\{P = 0\}$ and $\{Q = 0\}$ is necessary for $n = 2$.

Example 2. In Theorem 2 the difference of Milnor numbers (up to a sign) can be replaced by the (equal) difference of the Euler characteristics of the corresponding Milnor fibres (of the germs P and $P + tQ$). However, the formula obtained this way fails if the germ P has a nonisolated critical point at the origin. This is shown by the example of $f = \frac{x^2+z^2y}{z^2}$.

Formula (1) is a direct consequence of Theorem 2 and of [6, Theorem 2, formula (2)].

§3. Topological triviality of the family $\{P + tQ\}$ and the typical values of meromorphic germs

As before, let $f = \frac{P}{Q}$ be a meromorphic germ on $(\mathbb{C}^{n+1}, 0)$ such that the holomorphic germ P has an isolated critical point at the origin.

Theorem 3. *The value 0 is typical for the meromorphic germ f if and only if $\chi(\mathcal{M}_f^0) = 0$.*

Proof. The "only if" part follows from the definition and Theorem 2.

The "if" part is a consequence of the result proved by A. Parusiński in [8] (or, rather, a concequence of its proof). A Parusiński proved that if $\mu(P) = \mu(P+tQ)$ for $|t|$ sufficiently small, then the family of maps $P_t = P + tQ$ is topologically trivial. In particular, the family of germs of the hypersurfaces $\{P_t = 0\}$ is topologically trivial. For $n \neq 2$ this was proved by D.T.Lê and C.P. Ramanujam [7]. However, in order to apply this result to the situation in question, we need to have a topological trivialization of the family $\{P_t = 0\}$ that keeps the subset $\{P = Q = 0\}$ and is smooth off the origin. For the family $P_t = P + tQ$, such a trivialization was explicitly constructed in [8] without any restriction on the dimension. □

Example 3. If the germ of the function P has a nonisolated critical point at the origin, then this characterization is no longer true. As an example we can take $P(x, y) = x^2y^2$ and $Q(x, y) = x^4 + y^4$.

§4. A generalization of the Parusiński–Pragacz formula for the Euler characteristic of a singular hypersurface

Let X be a compact complex manifold, and let $\mathcal{L}$ be a holomorphic line bundle on X. For a section s of $\mathcal{L}$ not identically equal to zero, let $Z := \{s = 0\}$ be its zero locus (a hypersurface in the manifold X). Let s' be another section of the bundle $\mathcal{L}$ such that its zero locus Z' is nonsingular and transversal to a Whitney stratification of the hypersurface Z. In [9, Proposition 7] A. Parusiński and P. Pragacz proved a statement, which, in terms of [6], can be written as follows:

$$\chi(Z') - \chi(Z) = \int_{Z \setminus Z'} (\chi_x(Z) - 1)\, d\chi, \tag{2}$$

where $\chi_x(Z)$ is the Euler characteristic of the Milnor fibre of the germ of s at the point x (the definition of the integral against the Euler characteristic can be found in [12] or [6]).

We present a more general formula, which includes this one as a particular case.

Theorem 4. *Let s be as above, and let s' be a section of the bundle $\mathcal{L}$ such that its zero locus Z' is nonsingular. If f is the meromorphic function s/s' on the manifold X, then*

$$\chi(Z') - \chi(Z) = \int_{Z \setminus Z'} (\chi_x(Z) - 1)\, d\chi + \int_{Z \cap Z'} \chi^0_{f,x} d\chi, \tag{3}$$

where $\chi^0_{f,x}$ is the Euler characteristic of the 0-Milnor fibre of the meromorphic germ f at the point x.

Proof. Let F_t be the level set $\{f = t\}$ of the (global) meromorphic function f on the manifold X (with the indeterminacy set $\{s = s' = 0\}$), i.e., $F_t = \{s - ts' = 0\} \setminus \{s =$

$s' = 0\}$. By [6], for a generic value t we can write

$$\chi(F_{\rm gen}) - \chi(F_0) = \int_{F_0} (\chi^0_{f,x}(Z) - 1)\, d\chi + \int_{\{s=s'=0\}} \chi^0_{f,x} d\chi,$$

where $\chi^0_{f,x}$ is the Euler characteristic of the 0-Milnor fibre of the meromorphic germ f at x. We have $F_0 = Z \setminus (Z \cap Z')$ and $F_\infty = Z' \setminus (Z \cap Z')$, and in this case F_∞ is a generic level set of the meromorphic function f (since its closure is nonsingular). Therefore, $\chi(F_0) = \chi(Z) - \chi(Z \cap Z')$, $\chi(F_{\rm gen}) = \chi(Z') - \chi(Z \cap Z')$. Finally, for $x \in F_0$ the germ of the function f at x is holomorphic, whence $\chi^0_{f,x} = \chi_x(Z)$. $\square$

If the hypersurface Z' is transversal to all strata of a Whitney stratification of the hypersurface Z, then for $x \in Z \cap Z'$ the Euler characteristic $\chi^0_{f,x}$ is equal to 0 (see [10, Proposition 5.1]) and formula (2) turns into (3).

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Received 13/APR/1999

Translated by THE AUTHORS