PARABOLIC PROBLEMS WITH NONLINEAR DYNAMICAL BOUNDARY CONDITIONS AND SINGULAR INITIAL DATA

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1. INTRODUCTION

In this paper we consider the following parabolic problem with dynamic boundary conditions:

\[
\begin{align*}
  u_t + Au &= f(x, t, u, \nabla u), & x \in \Omega, & t > 0, \\
  (\gamma u)_t + Bu &= g(x, t, \gamma u), & x \in \Gamma, & t > 0, \\
  u(x, 0) &= u_0(x), & x \in \Omega, \\
  (\gamma u)(x, 0) &= v_0(x), & x \in \Gamma,
\end{align*}
\]

(1.1)

where \(\Omega\) is a bounded domain in \(\mathbb{R}^n\) of class \(C^2\), \(\Gamma = \partial \Omega\), \(\nu\) denotes the outer normal on \(\Gamma\), \(\gamma\) is the trace operator, and \(Au = -\Delta u + \omega u\), \(Bu = u_\nu + \omega u\). Although we will consider this particular case, the techniques we use can also be applied to the case of systems in which, as in [10], \(Au = -\partial_j (a_{jk} \partial_k u) + a_j \partial_j u + a_0, Bu = a_{jk} \nu^j \gamma \partial_k u + b_0 \gamma u\), with smooth-enough coefficients. On the nonlinear terms, \(f\) and \(g\), we assume that they are smooth functions with

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certain growth assumptions that will be specified later (see conditions (Hf) and (Hg) below).

Observe that the setting for (1.1) consists in writing an evolution equation for a two-component vector containing the function $u$ in $\Omega$ and the trace of $u$ on $\Gamma$. Therefore, (1.1) is considered as an evolution equation in a product of function spaces in $\Omega$ and $\Gamma$, and the condition that the latter component is the trace of the former has to be imposed.

The well-posedness of (1.1) was considered in [10] for the case of smooth and compatible initial data, that is, when $v_0$ is the trace of the smooth function $u_0$.

In this paper, we are interested in obtaining the largest possible growth on the nonlinear terms such that (1.1) can be solved with initial data in Lebesgue spaces, $L_{q_1}(\Omega) \times L_{q_2}(\Gamma)$, or even measures $\mathcal{M}(\Omega) \times \mathcal{M}(\Gamma)$, where $\mathcal{M}(\Omega)$ and $\mathcal{M}(\Gamma)$ denote the space of bounded Radon measures in $\Omega$ and on $\Gamma$, respectively. That is, we are interested in obtaining the critical exponents for $f$ and $g$ for these classes of initial data.

A general approach for this type of problem was introduced in [5], where it was used for the problem of critical exponents for the heat equation. This approach was then refined in [6], and it was used to determine the critical exponents for the heat equation with nonlinear boundary conditions.

The approach mentioned above can be summarized as follows. One first writes the equation in an abstract form

$$Z_t + AZ = \mathcal{F}(t, Z)$$

where $A$ represents the linear part of the equation and $\mathcal{F}$ includes all nonlinear terms. Then one shows that in a suitable Banach space $F$, $-A$ with domain $E = D(A)$ generates an analytic semigroup $e^{-At}$. One then constructs a suitable scale of interpolation spaces $X^\alpha$ for $\alpha \in [0, 2]$, such that $X^2 = E$, $X^1 = F$ and the restriction of $e^{-At}$ to $X^\alpha$ is also an analytic semigroup. Then the properties of the mapping $\mathcal{F}$ on different spaces of the scale plus the variation-of-constants formula

$$Z(t) = e^{-At}Z(0) + \int_0^t e^{-A(t-s)}\mathcal{F}(s, Z(s))\,ds$$

allow one to show that for $Z(0) \in F$ the equation has a unique solution, even when $\mathcal{F}$ grows critically.

Now we describe the contents of the paper. After Section 2, where we introduce preliminaries, we study the linear semigroup associated to equation (1.1) in Section 3. In this section we include an analysis of different families of scales of Banach spaces associated to the linear operator. These families
are related to different products of spaces of functions defined in Ω and on Γ that can be chosen to obtain a well-posed problem.

In Section 4 we refine again the abstract result given in [5, 6]. This is needed since the results developed in these papers do not apply directly to our situation. We do not include a detailed proof of this new abstract result since it can be obtained by minor modifications of the one given in [5].

The main results of the paper are contained in Section 5. Here we obtain that for initial data in certain spaces, \( \mathcal{M}(\Omega) \times \mathcal{M}(\Gamma) \) among others, there is a well-defined solution of (1.1). We need to analyze how the nonlinearity of the problem acts between the different scales of spaces defined in Section 3 in order to be able to apply the abstract results from Section 4. We describe this property of the nonlinearity by controlling the growth of \( f \) and \( g \) at infinity. As a matter of fact, the main hypotheses on \( f \) and \( g \) are expressed as the following:

**(Hf)** The function \( f \) is a \( C^1 \)-function, and there exist \( \rho_0, \tilde{\rho}_0, \rho_1, \tilde{\rho}_1 \geq 1 \) and a constant \( C > 0 \) such that for \( x \in \Omega, \ t \in \mathbb{R}^+, \ \xi \in \mathbb{R}, \ \eta \in \mathbb{R}^n, \)

\[
|\partial_\xi f(x, t, \xi, \eta)| \leq C(1 + |\xi|^{\rho_0 - 1} + |\eta|^{\tilde{\rho}_0 - 1^1}), \\
|\partial_\eta f(x, t, \xi, \eta)| \leq C(1 + |\xi|^{\tilde{\rho}_1 - 1} + |\eta|^{\rho_1 - 1}).
\]  

**(Hg)** The function \( g \) can be extended to a \( C^2 \)-function in \( \Omega \), which we denote also by \( g \), and there exists \( \rho_2 \geq 1 \) such that

\[
|\partial_\xi g(x, t, \xi)| \leq C(1 + |\xi|^{\rho_2 - 1}), \quad x \in \Omega, \ t \in \mathbb{R}^+, \ \xi \in \mathbb{R}. 
\]  

The critical exponents for a given space of initial data are the least upper bounds for the numbers \( \rho_0, \rho_1, \rho_2, \tilde{\rho}_0, \tilde{\rho}_1 \) for which we have a well-posed problem in that space. Denote

\[
\rho_0^*(q) = 1 + \frac{2q}{n}, \quad \rho_1^*(q) = 1 + \frac{q}{n+q}, \quad \tilde{\rho}_0^*(q) = 1 + \frac{2q}{n+q}, \\
\tilde{\rho}_1^*(q) = 1 + \frac{q}{n}, \quad \rho_2^*(q) = 1 + \frac{q}{n-1}.
\]  

It is well known that the exponents \( \rho_0^*(q) \) or \( \rho_1^*(q) \) are critical for the well-posedness of the problem

\[
\begin{align*}
  u_t + Au & = f(x, t, u, \nabla u), & x \in \Omega, \ t > 0, \\
  \gamma u & = 0, & x \in \Gamma, \ t > 0, \\
  u(x, 0) & = u_0(x), & x \in \Omega
\end{align*}
\]  

in \( L_q(\Omega) \) if \( f(x, t, u, \nabla u) = |u|^\rho_0 \) or \( f(x, t, u, \nabla u) = |\nabla u|^{\rho_1} \), respectively (see [8] and [7], for example). Moreover, replacing the homogeneous Dirichlet boundary condition in (1.5) by the nonlinear boundary condition \( Bu = g(x, t, u) \) (where \( g \) satisfies (1.3)) one obtains well-posedness in \( L_q(\Omega) \) for \( \rho_2 < \tilde{\rho}_2^*(q) := 1 + \frac{2q}{n} \) (see [6] and notice that \( \tilde{\rho}_2^*(q) < \rho_2^*(q) \)). Our results in
Section 5 indicate that the exponents (1.4) are critical for the well-posedness of (1.1) in $L_q(\Omega) \times L_q(\Gamma)$. More precisely, we obtain the following assertions:

For initial conditions $(u_0, v_0) \in \mathcal{M}(\Omega) \times \mathcal{M}(\Gamma)$, the following conditions are sufficient for the well-posedness (see Subsection 5.2, Theorem 5.11):

\[ \rho_0 < \rho_0^c(1), \quad \rho_1 < \rho_1^c(1), \quad \tilde{\rho}_0 < \tilde{\rho}_0^c(1), \quad \tilde{\rho}_1 < \tilde{\rho}_1^c(1), \quad \rho_2 < \rho_2^c(1). \]

For initial conditions $(u_0, v_0) \in L_{q^#}(\Omega) \times L_q(\Gamma)$, with $1 < q < \infty$ and $q^# = nq/(n + q - 1)$, the following conditions are sufficient (see Subsection 5.3, Theorem 5.13):

\[ \rho_0 < \rho_0^c(q^#), \quad \rho_1 < \rho_1^c(q^#), \quad \tilde{\rho}_0 < \tilde{\rho}_0^c(q^#), \quad \tilde{\rho}_1 < \tilde{\rho}_1^c(q^#), \quad \rho_2 < \rho_2^c(q^#). \]

We also study well-posedness in the space $L_q(\Omega) \times B_{qq}^{-1/q}(\Gamma)$ where $1 < q < \infty$ and $B_{qq}^s$ denotes the Besov space. In this case, we have to assume some additional technical assumption on $q$ and/or $q$, and then we obtain the following sufficient conditions (see Subsection 5.1, Theorem 5.7):

\[ \rho_0 < \rho_0^c(q), \quad \rho_1 < \rho_1^c(q), \quad \tilde{\rho}_0 < \tilde{\rho}_0^c(q), \quad \tilde{\rho}_1 < \tilde{\rho}_1^c(q), \quad (n - q)\rho_2 \leq n, \]

where the last condition is empty if $q \geq n$. If $q < n$ then this condition can be written in the form $\rho_2 \leq \rho_2^c(q^*)$, where $q^* := (n - 1)/q/(n - q)$ is the biggest exponent such that $B_{qq}^{-1/q}(\Gamma)$ is embedded into $L_q(\Gamma)$.

Finally, in Section 6 we include an analysis of critical exponents for an elliptic equation in the presence of dynamical boundary conditions.

2. Preliminaries

In this section we introduce some notation, known results and preliminaries for (1.1). For $s \in \mathbb{R}$ and $1 < q < \infty$ we consider the Bessel potential spaces $H^s_q(\Omega)$ with norm $\| \cdot \|_{s,q}$ and Besov spaces $B^s_q(\Gamma) := B^{s-1/q}_q(\Gamma)$ with norm $\| \cdot \|_{s,q}$. The norm in the Lebesgue space $H^0_q(\Omega) = L_q(\Omega)$ will be denoted simply by $\| \cdot \|_q$, and the norm in $L_q(\Gamma)$ by $\| \cdot \|_{q,\Gamma}$.

By $q'$ we denote the dual exponent to $q$; i.e., $1/q' + 1/q = 1$. By $a^+$ (respectively $a^-$) we mean a number $b > a$ (respectively $b < a$) which is sufficiently close to $a$. We put also $a \vee b := \max\{a, b\}$ for any $a, b \in \mathbb{R}$.

By $D(A)$ we denote the domain of definition of a linear operator $A$. If $X$ and $Y$ are Banach spaces then we denote by $\mathcal{L}(X, Y)$ the space of continuous linear operators $A : X \to Y$ and we set $\text{Isom}(X, Y) := \{ A \in \mathcal{L}(X, Y) : A$ is an isomorphism$. We write $X \cong Y$ if $X, Y$ are isomorphic and $X \hookrightarrow Y$ if $X$ is continuously imbedded into $Y$. 

It is known (see [16]) that if \( s > 1/q \), then the trace operator \( \gamma \in \mathcal{L}(H^s_q(\Omega), B^s_q(\Gamma)) \). It is also known that there exists a continuous right inverse \( T \in \mathcal{L}(B^s_q(\Gamma), H^s(\Omega)) \) of \( \gamma \); i.e., \( \gamma(Tg) = g \) for \( g \in B^s_q(\Gamma) \). Moreover, \( Tg \) can (and will) be defined as the unique (generalized) solution of the problem

\[
\mathcal{A}u = 0 \quad \text{in} \quad \Omega,
\]

\[
u = g \quad \text{on} \quad \Gamma.
\]

Similarly, let \( Sf \) denote the solution of the problem

\[
\mathcal{A}u = f \quad \text{in} \quad \Omega, \\
u = 0 \quad \text{on} \quad \Gamma.
\]

Then \( S \in \mathcal{L}(H^s_q(\Omega), H^{s+2}_q(\Omega)) \) for any \( s \geq 0 \).

Following [12], we identify the dual space \( (L_q(\Omega))' \) of \( L_q(\Omega) \) with \( L_{q'}(\Omega) \), the dual of \( H^s_q(\Omega) \) with a subspace of \( H^{-s}_q(\Omega) \) for \( s > 0 \) and the dual space of \( B^s_q(\Gamma) \) with \( B^{-s}_q(\Gamma) \). If \( Z := (u, v) \in L_q(\Omega) \times L_{q'}(\Gamma) \) and \( \Upsilon := (\varphi, \psi) \in L_{q'}(\Omega) \times L_q(\Gamma) \), we set

\[
\langle Z, \Upsilon \rangle = \langle u, \varphi \rangle_\Omega + \langle v, \psi \rangle_\Gamma = \int_\Omega u \varphi \, dx + \int_\Gamma v \psi \, dS.
\]

Throughout the paper we shall use standard embedding, interpolation and duality properties for Bessel potential and Besov spaces (see [16], for example).

3. The linear semigroup

In this section we study the linear operators associated to problem (1.1). For this, following [10], we set

\[
E_1 := \{(u, v) \in H^2_q(\Omega) \times B^2_q(\Gamma) : \gamma u = v\}, \\
F_1 := L_q(\Omega) \times B^2_q(\Gamma), \\
A_1 : E_1 \rightarrow F_1 \\
(u, v) \mapsto (\mathcal{A}u, Bu).
\]

It is shown in [10] that choosing \( \omega \) sufficiently large, the triplet \( (A, E, F) := (A_1, E_1, F_1) \) has the following properties:

\[
(\lambda + A) \in Isom(E, F) \quad \text{for} \quad \Re \lambda \geq 0, \\
(3.1)
\]

\(-A\) generates a strongly continuous analytic semigroup \( e^{-tA} \) in \( F \).

This implies that one can solve the linear equation \( Z_t + A_1Z = 0 \) with \( Z = (u, v) \) and initial data \( Z_0 \in F_1 \). Since the solution lies in \( E_1 \), observe
that this implies that $v = \gamma(u)$ for $t > 0$ and that $Z$ solves
\begin{align*}
  u_t + Au &= 0 & &\text{in } \Omega, \\
  u(0) &= u_0 \\
  v_t + Bu &= 0 & &\text{on } \Gamma, \quad (3.2)
\end{align*}
with $(u_0, v_0) = Z_0 \in F_1$.

Now, following [1], a completely abstract argument can be carried out so one can solve (3.2) for a larger class of initial data. The sketch of such a procedure is as follows. If a triplet $(A, E, F)$ satisfies (3.1) then one can construct the extrapolated space $G$ defined as the completion of $F$ under the norm $\|A^{-1} \cdot \|$. The realization of $A$ in $G$, which we still denote by $A$, is an extension of $A$ and $A : F \to G$. Moreover the triplet $(A, F, G)$ satisfies that
\begin{align}
  (\lambda + A) &\in Isom(F, G) \text{ for } \Re \lambda \geq 0, \\
  (\lambda) &\text{ generates a strongly continuous analytic semigroup } e^{-tA} \text{ in } G. 
\end{align}

At this point we can construct now the scale of interpolation-extrapolation spaces as follows. Let $[\cdot, \cdot]_\eta, 0 < \eta < 1$, be the complex interpolation functor. If we denote $X^{1+\eta} = [E, F]_\eta, X^\eta = [F, G]_\eta, 0 \leq \eta \leq 1$, then we have that the realization of the operator $A$ in $X^\eta$, which we still denote by $A$, is such that the triplet $(A, X^{1+\eta}, X^\eta)$ satisfies (3.1) for every $0 \leq \eta \leq 1$. Moreover we have the estimates
\begin{align}
  \|e^{-tA} x\|_{X^\eta} \leq C t^{\eta}\|x\|_{X^\eta}, &\quad 0 \leq \alpha \leq \eta \leq 2. 
\end{align}
We can apply this abstract procedure starting with the triplet $(A_1, E_1, F_1)$ introduced above, and then we construct the extrapolated space $G_1$ and the scale of interpolation-extrapolation spaces $X^{1+\eta}_1 = [E_1, F_1]_\eta, X^\eta_1 = [F_1, G_1]_\eta, 0 \leq \eta \leq 1$, which satisfy all the above.

In particular, we can now solve (3.2) with initial data $(u_0, v_0) = Z_0 \in G_1$, and by the smoothing effect of the semigroup, the solution enters $E_1$ after positive time. Some description of $G_1$ and the interpolated spaces we just constructed will be given further below.

On the other hand, by [10], $A_1$ also admits a unique continuous extension to $A_0 \in Isom(E_0, F_0)$, where $E_0 = L^q_0(\Omega) \times B^0_q(\Gamma)$, and $F_0$ is the completion of $E_0$ with respect to an appropriate norm and the triplet $(A, E, F) := (A_0, E_0, F_0)$ satisfies again (3.1).

Using the abstract procedure above we can construct the space $G_0$ and the scale $X^\beta_0$ for $0 \leq \beta \leq 2$ as $X^{1+\eta}_0 = [E_0, F_0]_\eta, X^\eta_0 = [F_0, G_0]_\eta, 0 \leq \eta \leq 1,$ which satisfy all the above.

In particular, we can now solve the linear equation $Z_t + A_0 Z = 0$ with $Z = (u, v)$ and initial data $Z_0 \in G_0$, and the solution lies in $E_0$ for positive
times. After giving some description of the spaces obtained by the abstract procedure above, it will be shown below that in fact $Z$ solves also (3.2).

Now we can interpolate the results above as follows. Observe first that $E_1 \subset E_0$, and by construction $F_1 \subset F_0$ which gives in turn $G_1 \subset G_0$. Denote as before by $[\cdot, \cdot]_\theta$, $0 < \theta < 1$, the complex interpolation functor. Set $E_\theta = [E_1, E_0]_\theta$, $F_\theta = [F_1, F_0]_\theta$, and let $A_\theta$ be the realization of the operator $A_0$ in $F_\theta$; i.e., $A_\theta Z = A_0 Z$ for $Z \in D(A_\theta) := \{W \in D(A_0) : A_0 W \in F_\theta\}$.

From the results in [10], it turns out that the triplet $(A_\theta, E_\theta, F_\theta)$ satisfies (3.1), and the abstract techniques of interpolation and extrapolation give us the space $G_\theta$ and the scale of spaces $X^{1+\eta}_\theta = [E_\theta, F_\theta]_\eta$, $X^{\eta}_\theta = [F_\theta, G_\theta]_\eta$, $0 \leq \eta \leq 1$, which satisfy all the above. When we need to stress the dependence of the scales on the parameter $q$ we will write $X^{\eta}_\theta(q)$, $X^{1+\eta}_\theta(q)$. It can also be shown that, as expected, $G_\theta = [G_1, G_0]_\theta$. In particular, we can now solve the linear equation $Z_t + A_0 Z = 0$ with $Z = (u, v)$ and initial data $Z_0 \in G_\theta$, and the solution lies in $E_\theta$ for positive times.

The structure of these spaces and operators is described in the following diagram:

$$
\begin{array}{cccccc}
E_1 & \xrightarrow{A_1} & F_1 & \xrightarrow{A_1} & G_1 & \xrightarrow{A_1} & X^{1+\eta}_1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E_\theta & \xrightarrow{A_\theta} & F_\theta & \xrightarrow{A_\theta} & G_\theta & & \text{and} & X^{1+\eta}_\theta & \xrightarrow{A_\theta} & X^\eta_\theta & 0 \leq \eta \leq 1, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
E_0 & \xrightarrow{A_0} & F_0 & \xrightarrow{A_0} & G_0 & & X^{1+\eta}_0 & \xrightarrow{A_0} & X^\eta_0 \\
\end{array}
$$

where the arrows $\downarrow$ represent the natural embeddings.

It is clear that to make the techniques above useful for practical purposes, some description and properties of the spaces we just constructed must be given. One of the technical difficulties of (3.2) is that only an incomplete description of the scales $X^\eta_\theta$ is available. This will translate into some results for (1.1) which may probably be improved with a better knowledge of the scales.

We start by observing that it is a straightforward generalization of some results in [10] that

$$
\begin{align*}
E_\theta &= \{(u, v) \in H^{2\theta}_q(\Omega) \times B^{2\theta}_q(\Gamma) : \gamma u = v\} \quad \text{if } 1/q < 2\theta \leq 2, \\
E_\theta &\hookrightarrow H^{2\theta}_q(\Omega) \times B^{2\theta}_q(\Gamma) \quad \text{if } 2\theta = 1/q, \\
E_\theta &\doteq H^{2\theta}_q(\Omega) \times B^{2\theta}_q(\Gamma) \quad \text{if } 0 \leq 2\theta < 1/q, \\
F_\theta &= \left\{(\varphi, \psi) \in H^{2-2\theta}_q(\Omega) \times B^{2-2\theta}_q(\Gamma) : \gamma \varphi = \psi\right\}' \\
&\quad \text{if } 0 \leq 2\theta < 1 + 1/q,
\end{align*}
$$

(3.5)
\[ F_\theta \leftarrow \left[ H^{2-2\theta}_q(\Omega) \times B^{2-2\theta}_q(\Gamma) \right]' \quad \text{if} \quad 2\theta = 1 + 1/q, \]
\[ F_\theta \equiv \left[ H^{2-2\theta}_q(\Omega) \times B^{2-2\theta}_q(\Gamma) \right]' = H^{2\theta - 2}_q(\Omega) \times B^{2\theta - 1}_q(\Gamma) \]
\[ \quad \text{if} \quad 1 + 1/q < 2\theta \leq 2. \]

The following lemmata give us information about some of the spaces of the scales that we have just constructed. They give us a description of \( X^\eta_\theta \) for some values of \( \eta \) and \( \theta \).

**Lemma 3.1.** Let \( 1 + 1/q < 2\theta \leq 2 \). Then
\[ X^{1+\eta}_{\theta} \equiv [E_\theta, F_\theta]_\eta \leftarrow H^{2\theta - 2(1-\eta)}_q(\Omega) \times B^{2\theta + \eta - 1}_q(\Gamma), \quad \text{for} \quad \eta \in [0, 1]. \quad (3.6) \]

**Proof.** We have
\[ E_\theta \leftarrow H^{2\theta}_q(\Omega) \times B^{2\theta}_q(\Gamma), \quad F_\theta \equiv H^{2\theta - 2}_q(\Omega) \times B^{2\theta - 1}_q(\Gamma), \]
and the result follows by interpolation. \( \square \)

**Lemma 3.2.** For \( \theta \geq 1/2 \) with \( 2\theta \neq 1 + 1/q \) and \( \eta \in [0, 1] \), we have
\[ F_{\theta-(1-\eta)/2} \equiv [F_\theta, F_{\theta-1/2}]_\eta \leftarrow [F_\theta, G_\theta]_\eta \equiv X^\eta_\theta. \]

**Proof.** A direct application of (3.5) implies that \( E_{\theta-1/2} \leftarrow F_\theta \) for any \( \theta \in [1/2, 1] \). Moreover since \( A_{\theta-1/2} : E_{\theta-1/2} \rightarrow F_{\theta-1/2} \) and \( A_\theta : F_\theta \rightarrow G_\theta \) are isomorphisms, and both the operators \( A_\theta \) and \( A_{\theta-1/2} \) are realizations of the same operator, we will also have that \( F_{\theta-1/2} \leftarrow G_\theta \), for any \( \theta \in [1/2, 1] \). This last embedding and the properties of the interpolation give us the result. \( \square \)

Observe that the arguments in the proof of Lemma 3.2 guarantee that \( E_0 \leftarrow F_{1/2} \leftarrow G_1 \). This implies, in particular, that the solution of the linear equation \( Z_t + A_0 Z = 0 \) with \( Z = (u, v) \) and initial data \( Z_0 \in G_0 \) lies in \( E_1 \) for positive times and solves (3.2).

4. **Abstract results**

In this section we give an abstract result on existence and uniqueness of solutions for problems of the form
\[ \dot{x} + Ax = F(t, x), \quad t > 0, \]
\[ x(0) = x_0. \quad (4.1) \]

The results presented here are extensions of the abstract results presented in [5, 6], and they are needed to deal with parabolic problems with dynamic boundary conditions. With this in mind we will indicate only the differences in the proofs and refer to [5, 6] for details.

We assume that we are given an interval \( I \subset [0, 2] \), a scale of Banach spaces \( X^\alpha, \alpha \in I \), and sectorial operators \( A_\alpha : D(A_\alpha) \subset X^\alpha \rightarrow X^\alpha, \alpha \in I \), with
Re \( \sigma(A_{\alpha}) > \omega > 0 \), where \( \sigma(A_{\alpha}) \) denotes the spectrum of \( A_{\alpha} \). We assume that \( X^\beta \hookrightarrow X^\alpha \) and that \( A_{\beta} \) is the realization of \( A_{\alpha} \) in \( X_{\beta} \) if \( \alpha, \beta \in I, \alpha < \beta \). Moreover, we assume that the linear semigroup generated by the operator \(-A_{\alpha}\) satisfies the inequality
\[
t^{\beta-\alpha} \| e^{-A_{\alpha} t} x \|_{\beta} \leq M \| x \|_{\alpha}, \quad \alpha \leq \beta, \quad \alpha, \beta \in I, \quad t > 0,
\]
where \( \| \cdot \|_{\beta} \) denotes the norm in \( X^\beta \). If no confusion seems likely, we write \( A \) instead of \( A_{\alpha} \).

The main differences in comparison with the results from \([5, 6]\) rely on the hypotheses for the nonlinearities. In our case we will assume the following hypothesis:

\((H)\) There exist numbers \( 0 < \varepsilon_i < 1, \rho_{ij} \geq 1 \) and \( \gamma_i < 1, i, j = 0, 1, \ldots, k \), such that \( 1 + \varepsilon_i, \gamma_i \in I \),

\[
1 > \gamma_i \geq \max_j [\varepsilon_i + (\rho_{ij} - 1) \varepsilon_j] =: \gamma, \\
\min_i \gamma_i =: \gamma > \varepsilon := \max_i \varepsilon_i,
\]

(4.2)

and the nonlinearity \( \mathcal{F} : [0, \infty) \times X^{1+\varepsilon} \to X^\alpha \) is such that \( \mathcal{F}(t, 0) \in X^\alpha \), \( \| \mathcal{F}(t, 0) \|_{\gamma_i} \leq C, i = 1, \ldots, k \), and \( \mathcal{F}(t, x) - \mathcal{F}(t, y) \) can be written as \( \mathcal{F}(t, x) - \mathcal{F}(t, y) = \sum_{i=0}^k f_i(t, x, y) \) with

\[
\| f_i(t, x, y) \|_{\gamma_i} \leq C \| x - y \|_{1+\varepsilon_i} \left( 1 + \sum_{j=0}^k (\| x \|_{1+\varepsilon_j} + \| y \|_{1+\varepsilon_j}^{-1}) \right),
\]

(4.3)

\( i = 1, \ldots, k \). Then, we look for functions that satisfy (4.1) in the sense that

\[
x(t) = e^{-At} x_0 + \int_0^t e^{-A(t-s)} \mathcal{F}(s, x(s)) \, ds
\]

(4.4)

with \( x_0 \in X^1 \). As in \([5, 6]\) we look for \( \varepsilon \)-regular solutions of (4.4) in the sense of \([6, \text{Definition 1.1}]\), i.e., a function \( x \in C([0, \tau], X^1) \cap C((0, \tau], X^{1+\varepsilon}) \) which satisfies (4.4), and we have the following result.

**Theorem 4.1.** Assume that the nonlinearity \( \mathcal{F} \) satisfies \((H)\). Let \( y_0 \in X^1 \) and denote \( B_r(y_0) = \{ x \in X^1 : \| x - y_0 \| < r \} \). Then there exist \( r = r(y_0) > 0 \) and \( \tau_0 = \tau_0(y_0) > 0 \) with the following properties: for any \( x_0 \in B_r(y_0) \) there exists a continuous function \( x : [0, \tau_0] \to X^1 \), with \( x(0) = x_0 \), which is the unique \( \varepsilon \)-regular mild solution to (4.1) starting at \( x_0 \). In addition, this solution satisfies

\[
x \in C((0, \tau_0], X^{1+\theta}), \quad t^\theta \| x(t, x_0) \|_{1+\theta} \xrightarrow{t \to 0^+} 0, \quad 0 < \theta < \gamma \land \sup I.
\]

Moreover, if \( x_0, z_0 \in B_r(y_0) \), then

\[
t^\theta \| x(t, x_0) - x(t, z_0) \|_{1+\theta} \leq C(\theta_0, \tau_0) \| x_0 - z_0 \|_1,
\]

(4.5)
$t \in [0, \tau_0]$, $0 \leq \theta \leq \theta_0 < \gamma \wedge \sup I$. If $\gamma, 1 + \gamma \in I$ and $t \to \mathcal{F}(t, x)$, as a map from $(0, \tau)$ to $X^2$, is locally Hölder continuous, uniformly on bounded sets of $x \in X^{1+2}$, then $x \in C^1((0, \tau_0], X^2) \cap C((0, \tau_0], X^{1+2})$, and $x(\cdot, x_0)$ satisfies the equation $\dot{x} + Ax = \mathcal{F}(t, x)$ in $X^2$ for $t \in (0, \tau_0)$.

Finally, assume that all nonlinearities are subcritical; that is, $\gamma_i > \gamma$ for all $1 \leq i \leq k$. Then $r$ can be chosen arbitrarily large. That is, the time of existence can be chosen uniformly on bounded sets of $X^1$.

**Proof.** Under the assumptions above, one can repeat the considerations in [5] or [6] in order to prove the unique solvability of (4.4) in the ball $B_\mu := \{ x \in K : \| x \|_K \leq \mu \}$ of the Banach space

$$K := \{ x \in C((0, \tau], X^{1+\varepsilon}) : \| x \|_K := \sup_{0 < t \leq \tau} \sup_i t^{\varepsilon_i} \| x(t) \|_{1+\varepsilon_i} < \infty \},$$

(4.6)

where $\mu$ and $\tau$ are sufficiently small. More precisely, one obtains an $\varepsilon$-regular solution. Moreover, this solution is unique in the class of $\varepsilon$-regular solutions.

The only significant change concerns the proofs of Lemmata 2 and 3 in [5] (see also [6]), where one has to use estimates of the following type:

$$t^\theta \left\| \int_0^t e^{-(t-s)A} \left( \mathcal{F}(s, x(s)) - \mathcal{F}(s, y(s)) \right) \, ds \right\|_{1+\theta}$$

$$= t^\theta \left\| \int_0^t e^{-(t-s)A} \sum_{i=0}^k f_i(s, x(s), y(s)) \, ds \right\|_{1+\theta}$$

$$\leq C t^\theta \left\| \int_0^t (t-s)^{\gamma_i-1-\theta} \| f_i(s, x(s), y(s)) \|_{\gamma_i} \, ds \right\|_{1+\varepsilon_i}$$

$$\leq \tilde{C} t^\theta \left\| \int_0^t \sum_{i=0}^k (t-s)^{\gamma_i-1-\theta} \| x(s) - y(s) \|_{1+\varepsilon_i} \right. \left. \times \sum_{j=0}^k \left[ (1 + \| x(s) \|_{1+\varepsilon_j} + \| y(s) \|_{1+\varepsilon_j}) \right] \, ds \right\|_{1+\varepsilon_i}$$

$$\leq \tilde{C} t^\theta \left\| \int_0^t \sum_{i=0}^k (t-s)^{\gamma_i-1-\theta} \| x(s) - y(s) \|_{1+\varepsilon_i} \right. \left. \times \left( s^{\varepsilon_j(\rho_j-1)} + (s^{\varepsilon_j} \| x(s) \|_{1+\varepsilon_j})^{\rho_j-1} + (s^{\varepsilon_j} \| y(s) \|_{1+\varepsilon_j})^{\rho_j-1} \right) \, ds \right\|_{1+\varepsilon_i}$$

$$\leq c(\mu) \| x - y \|_K.$$
5. Problems with Dynamical Boundary Conditions

In this section, we apply the abstract results of Section 4 to problem (1.1) and obtain results on existence and uniqueness of solutions when the initial data lie in different spaces.

If we denote by $A$ the linear operator (and its different realizations) introduced in Section 3, $Z = (u, v)$ and

$$
\mathcal{F}(t, Z) = \mathcal{F}(t, u, v) = \left( f(\cdot, t, u(\cdot), \nabla u(\cdot)), g(\cdot, t, v(\cdot)) \right),
$$

then problem (1.1) can be written as

$$
\begin{align*}
\dot{Z} + AZ &= \mathcal{F}(t, Z), \ t > 0, \\
Z(0) &= Z_0.
\end{align*}
$$

(5.1)

First, for the case of smooth and compatible initial data, assume $1 + 1/q > 2\eta > 2\Theta > 1/q$, $u_0 \in H^2_q(\Omega)$ and $v_0 = \gamma u_0$; i.e., $Z_0 := (u_0, v_0) \in E_\Theta$. It follows from [12] and the properties of the Nemytskii mapping $\mathcal{F}$ that under the corresponding assumptions on $\rho_0$, $\rho_1$, $\rho_2$, $\tilde{\rho}_0$ and $\tilde{\rho}_1$ in (1.2) and (1.3), the problem (1.1) has a unique maximal weak solution $u$ defined on its maximal existence interval $[0, T)$. Moreover, $u$ is the first component of $Z$, where $Z \in C([0, T), E_\Theta)$ satisfies the variation-of-constants formula

$$
Z(t) = e^{-tA_\Theta}Z_0 + \int_0^t e^{-(t-s)A_\Theta} \mathcal{F}(s, Z(s)) \, ds
$$

and $v = \gamma(u)$.

Below we shall prove the existence of a unique solution of (5.1) in an appropriate Banach space for nonsmooth and noncompatible initial data. We will make use of the scales constructed in Section 3 and we will take $Z_0 \in X^\eta_\xi$ for different values of $\eta$ and $\xi$. Depending on $\eta$ and $\xi$ we will derive different conditions on the growth of the nonlinearities in terms of the exponents $\rho_0$, $\rho_1$, $\rho_2$, $\tilde{\rho}_0$ and $\tilde{\rho}_1$.

Since the corresponding solution $Z(t)$ will belong to $E_\Theta$ for some $2\Theta > 1/q$ and any $t > 0$, one can easily check that this solution will be a solution in the sense of [10] for any $t > 0$. Moreover, one can use the methods from [14] to get further regularity properties of $Z$.

In view of the abstract results of Section 4 we need to study how the nonlinearity $\mathcal{F}$ acts between the different spaces of the scale of Banach spaces constructed for $A$.

Our goal is to prove that for the scale of Banach spaces $X^\eta_\xi$, for some fixed $\xi$, the nonlinearity $\mathcal{F}$ fulfils hypothesis (H). Once this is done, Theorem 4.1 will guarantee that problem (5.1) is well posed. Notice that condition (H) will impose restrictions on the possible growth of the nonlinearities $f$ and $g$. 
that will be translated into restrictions on the exponents \( \rho_0, \rho_1, \rho_2, \tilde{\rho}_0 \) and \( \tilde{\rho}_1 \).

We will need to distinguish different cases according to where we choose our initial data. Although it would be possible to perform the forthcoming analysis for initial data in \( X^\eta_\xi \) for a large variety of \( \eta \) and \( \xi \), we will restrict ourselves to some specific choices of \( \eta \) and \( \xi \), as we now explain.

One of our motivations for the study of (1.1) with nonsmooth and noncompatible initial data was to consider initial values \( Z_0 \) in \( L_{q_1}(\Omega) \times L_{q_2}(\Gamma) \), for \( 1 < q_1, q_2 < \infty \), or in \( \mathcal{M}(\Omega) \times \mathcal{M}(\Gamma) \) (which is close to the case \( q_1 = q_2 = 1 \), in some sense). However, these spaces are not included in the scales of spaces described in Section 3. Therefore, we have to restrict ourselves to some spaces from these scales which are as “close” as possible to \( L_{q_1}(\Omega) \times L_{q_2}(\Gamma) \) and to \( \mathcal{M}(\Omega) \times \mathcal{M}(\Gamma) \). This will lead us to the following three particular cases:

i) **Initial data in** \( L_q(\Omega) \times B_q^1(\Gamma), 1 < q < \infty. \)**

Observe that in this case \( F_1 = L_q(\Omega) \times B^1_q(\Gamma) \), and therefore we will study the equation in the scale of spaces given by \( X^\eta = X_1^\eta, \eta \in [0, 1] \). This case is treated in Subsection 5.1.

ii) **Initial data in** \( \mathcal{M}(\Omega) \times \mathcal{M}(\Gamma) \).

If we choose \( q \) close to 1 then there exists a small \( \delta > 0 \) such that \( \mathcal{M}(\Omega) \times \mathcal{M}(\Gamma) \hookrightarrow F_{1-\delta/2} \hookrightarrow X_1^{1-\delta} \) (it is enough to choose \( \delta > n/q^* \); then these embeddings follow from (3.5) and Lemma 3.2). Therefore, we will use the scale \( X^\eta = X_1^{\eta-\delta} \). This case is treated in Subsection 5.2.

iii) **Initial data in** \( L_{q#}(\Omega) \times L_q(\Gamma), \) \( q# = nq/(n + q - 1), 1 < q < \infty. \)**

If \( \delta > 0 \) is small enough and \( 2\xi = 1 + 1/q + \delta \), then (3.5) and Lemma 3.2 imply \( L_{q#}(\Omega) \times L_q(\Gamma) \hookrightarrow H^{1/q-1}_q(\Omega) \times B^1_q(\Gamma) \hookrightarrow F_{\xi-\delta+1/2} \hookrightarrow X_\xi^{1-\delta+}. \) Therefore we will use the scale \( X^\eta = X_\xi^{\eta-\delta+} \). This case is treated in Subsection 5.3.

Recall from the Introduction that in all the cases mentioned above we plan to prove well-posedness under the conditions

\[
\rho_i < \rho^{\circ}_i(q), \quad \tilde{\rho}_i < \tilde{\rho}^{\circ}_i(q), \quad i = 0, 1, \tag{5.2}
\]

where \( q \) should be replaced by \( q, 1 \) or \( q# \) if \( u_0 \) belongs to \( L_q(\Omega), \mathcal{M}(\Omega) \) or \( L_{q#}(\Omega), \) respectively. The critical exponents \( \rho^{\circ}_i(q) \) and \( \tilde{\rho}^{\circ}_i(q) \), \( i = 0, 1, \) fulfill

\[
\rho^{\circ}_i(q) = \rho^{\circ}_0(q) \frac{n}{n+q}, \quad \tilde{\rho}^{\circ}_i(q) - 1 = (\rho^{\circ}_0(q) - 1) \frac{n}{n+q}, \quad \tilde{\rho}^{\circ}_i(q) - 1 = (\rho^{\circ}_i(q) - 1) \frac{n^*}{n}.
\]

Hence, if \( \rho_0, \rho_1, \tilde{\rho}_0, \) and \( \tilde{\rho}_1 \) satisfy (5.2), then we can always increase their values in such a way that the new values will still satisfy (5.2) and, moreover,
\[
\begin{align*}
\rho_1 &= \rho_0 \frac{n}{n+q}, \quad \bar{\rho}_0 - 1 = (\rho_0 - 1) \frac{n}{n+q}, \quad \bar{\rho}_1 - 1 = (\rho_1 - 1) \frac{n+q}{n} \quad (5.3)
\end{align*}
\]

(it is sufficient to choose \(\rho_0\) sufficiently close to \(\rho^*_0(q)\) and define \(\rho_1, \bar{\rho}_0,\) and \(\bar{\rho}_1\) by (5.3). By doing this we greatly simplify some proofs below).

We decompose \(\mathcal{F}\) as \(\mathcal{F}(t, \mathcal{Z}) = \mathcal{F}_1(t, \mathcal{Z}) + \mathcal{F}_2(t, \mathcal{Z}) = (\bar{\mathcal{F}}_1(t, u), 0) + (0, \bar{\mathcal{F}}_2(t, v))\), where \(\bar{\mathcal{F}}_1(t, u) = f(\cdot, t, u(\cdot), \nabla u(\cdot))\) and \(\bar{\mathcal{F}}_2(t, v) = g(\cdot, t, v(\cdot))\).

If no confusion seems likely, we shall identify the operator \(\mathcal{F}_i\) with \(\bar{\mathcal{F}}_i\) for \(i = 1, 2\) and, for example, we shall write \(\mathcal{F}_1(t, u)\) instead of \(\bar{\mathcal{F}}_1(t, u)\). We shall prove that \(\mathcal{F}\) fulfills the assumption (4.3) in the following form:

\[
\begin{align*}
\mathcal{F}_1(t, u_1) - \mathcal{F}_1(t, u_2) &= f_0(t, u_1, u_2) + f_1(t, u_1, u_2), \\
\mathcal{F}_2(t, v_1) - \mathcal{F}_2(t, v_2) &= f_2(t, v_1, v_2),
\end{align*}
\]

where

\[
\begin{align*}
f_0(t, u_1, u_2) &= f(\cdot, t, u_1(\cdot), \nabla u_1(\cdot)) - f(\cdot, t, u_2(\cdot), \nabla u_1(\cdot)), \\
f_1(t, u_1, u_2) &= f(\cdot, t, u_2(\cdot), \nabla u_1(\cdot)) - f(\cdot, t, u_2(\cdot), \nabla u_2(\cdot)), \\
f_2(t, v_1, v_2) &= g(\cdot, t, v_2(\cdot)) - g(\cdot, t, v_1(\cdot)).
\end{align*}
\]

The following results describe how the Nemytskii operators \(f_0, f_1\) and \(f_2\) act on Besov potential spaces on \(\Omega\) and on Besov spaces on \(\Gamma\), respectively.

**Lemma 5.1.** Let \(f\) fulfill (Hf) with \(\rho_0, \rho_1, \bar{\rho}_0,\) and \(\bar{\rho}_1\) satisfying (5.3). Let \(\alpha_0\) and \(\theta\) fulfill

\[
\begin{align*}
\frac{n}{n+q} &\leq \alpha_0 < \frac{n}{q}, \\
-\frac{n}{q} &< \theta \leq 0 \quad \text{and} \quad \rho_0 = \frac{n - \theta q}{n - \alpha_0 q}.
\end{align*}
\]

Put \(\alpha_1 := \alpha_0(n + q)/n\). Then

\[
\begin{align*}
|f_0(t, u_1, u_2)|_{\theta, q} &\leq C|u_1 - u_2|_{\alpha_0, q} \left(1 + \sum_{i=1}^2 (|u_i|_{\alpha_i, q}^{\rho_0} + |u_i|_{\alpha_i, q}^{\bar{\rho}_0 - 1})\right), \\
|f_1(t, u_1, u_2)|_{\theta, q} &\leq C|u_1 - u_2|_{\alpha_1, q} \left(1 + \sum_{i=1}^2 (|u_i|_{\alpha_i, q}^{\rho_1} + |u_i|_{\alpha_i, q}^{\bar{\rho}_1 - 1})\right).
\end{align*}
\]

**Proof.** Put \(r_i = \frac{nq}{n - \theta q} \rho_i, \ i = 0, 1,\) and notice that our assumptions guarantee \(r_i > \rho_i, \ i = 0, 1,\) The growth assumption (Hf), (5.3) and Hölder’s inequality imply

\[
\begin{align*}
|f_0(t, u_1, u_2)|_{r_0/\rho_0} &\leq C|u_1 - u_2|_{r_0} \left(1 + \sum_{i=1}^2 (|u_i|_{r_0}^{\rho_0 - 1} + |\nabla u_i|_{r_1}^{\bar{\rho}_0 - 1})\right), \\
|f_1(t, u_1, u_2)|_{r_1/\rho_1} &\leq C|\nabla (u_1 - u_2)|_{r_1} \left(1 + \sum_{i=1}^2 (|\nabla u_i|_{r_1}^{\rho_1 - 1} + |u_i|_{r_0}^{\bar{\rho}_1 - 1})\right).
\end{align*}
\]
Since our assumptions and the choice of \( r_i \) imply
\[
H^q_{\alpha_i}(\Omega) \hookrightarrow L_{r_0}(\Omega), \ H^{\alpha_i-1}_q(\Omega) \hookrightarrow L_{r_1}(\Omega), \text{ and } L_{r_i/r_1}(\Omega) \hookrightarrow H^q_{\theta}(\Omega), \ i = 0, 1,
\]
the estimates (5.5) guarantee (5.4).

In some situations we will need to impose some extra restrictions on the nonlinearity \( g \). We express it as the following:

**(Hg+)** The function \( g \) satisfies hypothesis (Hg) and

\[
|\partial_\xi \partial_\zeta g(x, t, \xi)| \leq C(1 + |\xi|^{\rho_2-1}) \quad |\partial_\zeta \partial_\xi g(x, t, \xi)| \leq C(1 + |\xi|^{\rho_2-2}) \quad x \in \Omega, \ t \in \mathbb{R}^+, \ \xi \in \mathbb{R} \quad (5.6)
\]

(if \( \rho_2 \leq 2 \), then the last inequality should be read as \( |\partial_\xi \partial_\zeta g(x, t, \xi)| \leq C \)).

**Lemma 5.2.** Let \( g \) satisfy (Hg) for some \( \rho_2 \geq 1 \). Let \( 1 < q < n \) and let \( \rho_2 = (n-q\theta)/(n-q\alpha) \), where one of the following conditions holds:

i) \( 0 \leq (1-n+n/q) \leq \theta < 1/q \) and \( 1/q < \alpha < n/q \),

ii) \( 1/q < \theta \leq 1, \ 1 \leq \alpha < n/q \), \( \rho_2 \geq 2 \) and \( g \) satisfies (Hg+).

Then the Nemyskii operator \( G = G(t): B^q_{\alpha}(\Gamma) \to B^q_{\theta}(\Gamma): v \mapsto g(\cdot, t, v(\cdot)) \), satisfies

\[
\|G(v_1) - G(v_2)\|_{\theta,q} \leq C\|v_1 - v_2\|_{\alpha,q} (1 + \|v_1\|_{\alpha,q}^{\rho_2-1} + \|v_2\|_{\alpha,q}^{\rho_2-1}). \quad (5.7)
\]

**Proof.** i) Put \( p := q(n-1)/(n-\theta q) \). Then \( 1 \leq p < q \) and \( L_p(\Gamma) \hookrightarrow B^\theta_q(\Gamma) \), so that (1.3) and Hölder’s inequality imply

\[
\|G(v_1) - G(v_2)\|_{\theta,q} \leq C\|G(v_1) - G(v_2)\|_{p,\Gamma} \\
\leq C\|v_1 - v_2\|_{ps,\Gamma} (1 + \|v_1\|_{ps\rho_2-1,\Gamma}^{\rho_2-1} + \|v_2\|_{ps\rho_2-1,\Gamma}^{\rho_2-1}) \quad (5.8)
\]

for any \( s > 1 \). The choice \( s = \rho_2 = (n-q\theta)/(n-q\alpha) \) guarantees

\[
\frac{\alpha - n}{q} = -\frac{n-1}{ps} = -\frac{n-1}{ps\rho_2-1};
\]

hence, \( B^\alpha_q(\Gamma) \hookrightarrow L_{ps}(\Gamma) \) and \( B^\alpha_q(\Gamma) \hookrightarrow L_{ps\rho_2-1}(\Gamma) \). These embeddings and (5.8) imply (5.7).

ii) Since the trace operator \( \gamma: H^\beta_q(\Omega) \to B^\beta_q(\Gamma) \), \( \beta > 1/q \), and its right inverse \( T \) are continuous linear operators and \( G(v) = \gamma \tilde{G}(Tv) \), where \( \tilde{G}(u) = g(\cdot, t, u(\cdot)) \) is the Nemyskii operator defined for functions \( u : \Omega \to \mathbb{R} \), the estimate (5.7) will follow from

\[
|\tilde{G}(u_1) - \tilde{G}(u_2)|_{\theta,q} \leq C|u_1 - u_2|_{\alpha,q} (1 + |u_1|_{\alpha,q}^{\rho_2-1} + |u_2|_{\alpha,q}^{\rho_2-1}). \quad (5.9)
\]

Let \( p \) be defined by \( 1 - n/p = \theta - n/q \). Then \( 1 < p \leq q \) and \( H^\theta_1(\Omega) \hookrightarrow H^\theta_q(\Omega) \); hence,

\[
|\tilde{G}(u_1) - \tilde{G}(u_2)|_{\theta,q} \leq C|\tilde{G}(u_1) - \tilde{G}(u_2)|_{1,p}
\]
\begin{equation}
\leq C(|\tilde{G}(u_1) - \tilde{G}(u_2)|_p + |\nabla \tilde{G}(u_1) - \nabla \tilde{G}(u_2)|_p)
\leq C(|g(\cdot, u_1(\cdot)) - g(\cdot, u_2(\cdot))|_p + |\partial x g(\cdot, u_1(\cdot)) - \partial x g(\cdot, u_2(\cdot))|_p
+ |\partial \xi g(\cdot, u_1(\cdot)) \nabla u_1 - \partial \xi g(\cdot, u_2(\cdot)) \nabla u_2|_p) = C(I_1 + I_2 + I_3),
\end{equation}

where we suppressed the dependence of \( g \) on \( t \) in our notation. From (1.3), (5.6) and Hölder's inequality, we have

\begin{equation}
I_1 + I_2 \leq c(|u_1 - u_2|(1 + |u_1|^{\rho_2-1} + |u_2|^{\rho_2-1})_p
\leq c|u_1 - u_2|_p(s(1 + |u_1|^{\rho_2-1}) + |u_2|^{\rho_2-1}_{pr_1(\rho_2-1)}),
\end{equation}

\begin{equation}
I_3 \leq |(\partial \xi g(\cdot, u_1(\cdot)) - \partial \xi g(\cdot, u_2(\cdot))) \nabla u_1|_p + |\partial \xi g(\cdot, u_2(\cdot))(\nabla u_1 - \nabla u_2)|_p
\leq |(\partial \xi g(\cdot, u_1(\cdot)) - \partial \xi g(\cdot, u_2(\cdot))) \nabla u_1|_p + c(1 + |u_2|^{\rho_2-1}_{pt(\rho_2-1)})|\nabla u_1 - \nabla u_2|_{pt'},
\end{equation}

for \( s, t > 1 \). If \( \rho_2 > 2 \), then we have

\begin{equation}
|\nabla u_1|_p \leq c(1 + |u_1|^{\rho_2-2}_{pr_1(\rho_2-2)} + |u_2|^{\rho_2-2}_{pr_2(\rho_2-2)})|u_1 - u_2|_{pr_2}|\nabla u_1|_{pr'},
\end{equation}

where \( 1 = 1/r_1 + 1/r_2 + 1/r_3 \).

The estimates above imply (5.9) provided we find \( s, r_1, r_2, r_3, t \) in such a way that the following embeddings hold:

\begin{equation}
H^\alpha_q \hookrightarrow L_{ps}, L_{ps'(-1)}, L_{pr_1(\rho_2-2)}, L_{pr_2}, H^1_{pr_3}, L_{pt(\rho_2-1)}, H^1_{pt'}.
\end{equation}

These embeddings are true as long as

\begin{equation}
\alpha - \frac{n}{q} \geq \max\left\{ \frac{-n}{ps}, \frac{-n}{ps'(-1)}, \frac{-n}{pr_1(\rho_2-2)}, \frac{-n}{pr_2}, \frac{-n}{pr_3}, \frac{-n}{pt(\rho_2-1)}, 1 - \frac{n}{pt'} \right\},
\end{equation}

which can be achieved by the following choice:

\begin{equation}
\frac{n + q(1 - \theta)}{n + q(1 - \theta)} \leq s \leq \frac{n + q(1 - \theta)}{n - q\alpha}, \quad t = \frac{n + q(1 - \theta)}{q(\alpha - \theta)},
\end{equation}

\begin{equation}
r_1 = \frac{n + q(1 - \theta)}{(\rho_2 - 2)(n - q\alpha)}, \quad r_2 = \frac{n + q(1 - \theta)}{n - q\alpha}, \quad r_3 = \frac{n + q(1 - \theta)}{n + q(1 - \alpha)}.
\end{equation}

This concludes the proof for the case \( \rho_2 > 2 \).

If \( \rho_2 = 2 \) then we have

\begin{equation}
|(\partial \xi g(u_1) - \partial \xi g(u_2)) \nabla u_1|_p \leq c|u_1 - u_2|_{pr}|\nabla u_1|_{pr'}.
\end{equation}

Similar to the case \( \rho_2 > 2 \), it is sufficient to find \( s, r, \) and \( t \) in such a way that the following embeddings hold:

\begin{equation}
H^\alpha_q \hookrightarrow L_{ps}, L_{ps'(-1)}, L_{pr}, H^1_{pr'}, L_{pt(\rho_2-1)}, H^1_{pt'}.
\end{equation}
and this is possible with the same choice of $t$ and $s$ as above and with 

$$r = \frac{n-q(\theta-1)}{n-q\alpha}.$$

\[ \square \]

**Remark 5.3.** Notice that if $n = 1$ and $g$ satisfies (Hg) for some $\rho_2 \geq 1$ then (5.7) is trivially true for any $\alpha, \theta, q$.

We also have the following related result:

**Lemma 5.4.** Let $g$ satisfy (Hg+) for some $\rho_2 \geq 2$, let $q \geq n$ and $\varepsilon > 0$. Then the Nemytskii operator $G: B^{1+\varepsilon}_q(\Gamma) \to B^1_q(\Gamma): v \mapsto g(\cdot, t, v(\cdot))$ satisfies

$$
\|G(v_1) - G(v_2)\|_{1,q} \leq C(\varepsilon)\|v_1 - v_2\|_{1+\varepsilon,q}(1 + \|v_1\|^{\rho_2-1}_{1+\varepsilon,q} + \|v_2\|^{\rho_2-1}_{1+\varepsilon,q}). \ (5.10)
$$

**Proof.** The proof is very similar to that of Lemma 5.2 ii). We estimate

$$
|\hat{G}(u_1) - \hat{G}(u_2)|_{1,q} \leq C(|g(\cdot, u_1(\cdot)) - g(\cdot, u_2(\cdot))|_{q} + |\partial_x g(\cdot, u_1(\cdot))\partial_x u_1 - \partial_x g(\cdot, u_2(\cdot))\partial_x u_2|_q) \\
= C(I_1 + I_2 + I_3).
$$

Using (1.3), (5.6) and Hölder’s inequality, we obtain

$$
I_1 + I_2 \leq c(|u_1 - u_2|(1 + |u_1|^{\rho_2-1} + |u_2|^{\rho_2-1})|_{\infty} \\
\leq c|u_1 - u_2|_{\infty}(1 + |u_1|^{\rho_2-1} + |u_2|^{\rho_2-1}),
$$

$$
I_3 \leq \|(\partial_x g(\cdot, u_1(\cdot)) - \partial_x g(\cdot, u_2(\cdot)))\partial_x u_1\|_q + |\partial_x g(\cdot, u_2(\cdot))(\partial_x u_1 - \partial_x u_2)|_q \\
\leq c(1 + |u_1|^{\rho_2-2} + |u_2|^{\rho_2-2})|u_1 - u_2|_{\infty}|\partial_x u_1|_q \\
+ c(1 + |u_2|^{\rho_2-1})|\partial_x u_1 - \partial_x u_2|_q.
$$

Since $B^{1+\varepsilon}_q(\Gamma) \hookrightarrow L_\infty(\Gamma)$ for all $\varepsilon > 0$, we obtain the result from the previous estimates on $I_1, I_2$ and $I_3$. \[ \square \]

5.1. **Case of initial data in $F_1 = L_q(\Omega) \times B^1_q(\Gamma)$.** In this case we use the scale

$$X^{1+\eta} := X^{1+\eta}_1 := [E_1, F_1]_\eta, \ X^{\eta} := X^{\eta}_1 := [F_1, G_1]_\eta, \ 0 \leq \eta \leq 1.$$

In Propositions 5.5 and 5.6 below, we shall verify (4.3) for the operators $F_2$ and $F_1$, respectively.

**Proposition 5.5.** Let $1 < q < \infty$, and let $g$ satisfy (Hg) with $\rho_2 \leq \rho_2^n := n/(n-q)$ if $q < n$ (no restriction on $\rho_2$ if $q \geq n$). Then the operator $F_2$ fulfills (4.3) with $\rho_{2i} = 1$ for $i = 0, 1$, and with $\rho_{22}, \gamma_2, \varepsilon_2$ as follows:

i) If $q \geq n$ and $g$ satisfies (Hg+), then we can choose

$$
\rho_{22} \geq \rho_2, \ \rho_{22} > 1, \ \varepsilon_2 = 0^+, \ \gamma_2 = 1^- > \rho_{22}\varepsilon_2.
$$
ii) If \( n > q > n/2 \) and \( g \) satisfies (Hg+), then we can choose
\[
\rho_{22} = \rho_2^m, \quad \varepsilon_2 = \left[ \frac{1}{\rho_2^m} \right]^- \quad \gamma_2 = \rho_{22}\varepsilon_2 = 1^-.
\]

iii) If \( n/2 \geq q > 1 \), then we can choose
\[
\rho_{22} = \rho_2^m, \quad \varepsilon_2 = \left[ \frac{1}{q\rho_2^m} \right]^- \quad \gamma_2 = \rho_{22}\varepsilon_2 = \left[ \frac{1}{q} \right]^-.
\]
Moreover, if \( q < n \) and \( \rho_2 < \rho_2^m \) then one can choose \( \rho_{22} = (\rho_2^m)^-, \varepsilon_2 \) as above and \( \gamma_2 = (\rho_{22}\varepsilon_2)^+ \).

**Proof.** Due to Lemmas 3.1 and 3.2, we have

\[
X^{1+\varepsilon_2} \hookrightarrow H_q^{2\varepsilon_2}(\Omega) \times B_q^{1+\varepsilon_2}(\Gamma) \quad \text{and} \quad X^{\gamma_2} \hookrightarrow F_{(1+\gamma_2)/2} \hookrightarrow \{0\} \times B_q^{\gamma_2}(\Gamma)
\]

for any \( \varepsilon_2, \gamma_2 \in [0,1] \). Therefore, we just need to check the corresponding properties of the Nemytskii mapping \( G : B_q^{1+\varepsilon_2}(\Gamma) \rightarrow B_q^{\gamma_2}(\Gamma) : v \mapsto g(\cdot, t, v(\cdot)) \).

i) If \( q \geq n \), then Lemma 5.4 guarantees the required properties of \( G \) for any \( \varepsilon_2 < 1/\rho_{22} \) and \( \gamma_2 \in (\varepsilon_2\rho_{22}, 1) \).

ii)-iii) Let \( q < n \). Notice that \( \rho_2^m > 2 \) if \( q > n/2 \). Choose \( \varepsilon_2 = [1/\rho_2^m]^- \) if \( q > n/2 \), \( \varepsilon_2 = [1/(q\rho_2^m)]^- \) if \( q \leq n/2 \) and define
\[
\gamma_2 := \rho_{22}\varepsilon_2 + \frac{n}{q} \left( 1 - \frac{\rho_{22}}{\rho_2^m} \right).
\]

We shall use Lemma 5.2 with \( \theta = \gamma_2 \) and \( \alpha = 1+\varepsilon_2 \). This choice and (5.11) imply \( \rho_{22} = \frac{n-q\theta}{n-q} \). Moreover, fixing \( \rho_{22} = \rho_2^m \) or \( \rho_{22} = (\rho_2^m)^- \), we obtain \( \gamma_2 = \rho_{22}\varepsilon_2 \) or \( \gamma_2 = (\rho_{22}\varepsilon_2)^+ \), respectively, and

\[
0 \vee (1-n+n/q) \leq \theta < 1/q \quad \text{and} \quad 1/q < \alpha < n/q \quad \text{if} \quad q \leq n/2,
\]

\[
1/q < \theta \leq 1 \quad \text{and} \quad 1 \leq \alpha < n/q \quad \text{if} \quad q > n/2,
\]
so that Lemma 5.2 concludes the proof. \( \square \)

**Proposition 5.6.** Let \( 1 < q < \infty \) and let \( f \) fulfill (Hf) with \( \rho_0, \rho_1, \tilde{\rho}_1 \) and \( \rho_1 \) satisfying (5.3) and \( p_0 = \lfloor \rho_0^m(q) \rfloor^- \). Then \( F_1 \) fulfills (4.3) with \( \rho_0 = \rho_0, \rho_01 = \tilde{\rho}_0, \rho_10 = \tilde{\rho}_1, \rho_{11} = \rho_1, \rho_02 = \rho_2 = 1 \) and \( \varepsilon_0 = (1/\rho_0)^-, \varepsilon_1 = (1/\rho_1)^-, \varepsilon_2 = (1/\rho_2)^- \), \( 1 > \gamma_0 = \gamma_1 > \max_{i,j=0,1} \{ \varepsilon_i + (\rho_j - 1)\varepsilon_j \} = 1^- \). More precisely, if \( \eta := \rho_0^m(q) - \rho_0 \) is small and \( a > 0 \) is such that \( a\rho_0^m(q) + 1/\rho_0^m(q) \in (n/2q, n/q) \), then we may choose
\[
\varepsilon_0 = \frac{1}{\rho_0^m(q)} - a\eta, \quad \varepsilon_1 = \frac{n+q}{n}\varepsilon_0 \quad \text{and} \quad \gamma_0 = \gamma_1 = 1 + 2\rho_0\varepsilon_0 - (\rho_0 - 1) \frac{n}{q}.
\]
Proof. First let us show that the choice of $\varepsilon_i, \gamma_i$ in the proposition guarantees

$$1 > \gamma_0 > \varepsilon_0 \rho_0 = 1^- \quad \text{if } \eta > 0 \text{ is small enough.} \quad (5.12)$$

Consider $\rho_0, \varepsilon_0$, and $\gamma_0$ as functions of $\eta \geq 0$. Then $\rho'_0 = -1$, $\varepsilon'_0 = -a$, $\gamma_0(0) = \rho_0(0)\varepsilon_0(0) = 1$, $\gamma'_0(0) = -2(a\rho'_0(q) + 1/\rho'_0(q)) + n/q < 0$, and $(\gamma_0 - \rho_0\varepsilon_0)'(0) = -(a\rho'_0(q) + 1/\rho'_0(q)) + n/q > 0$, which implies (5.12). This estimate and (5.3) imply also $\gamma_0 > \max_{i,j} \{ \varepsilon_i + (\rho_{ij} - 1)\varepsilon_j \} = 1^-$. According to Lemmas 3.1 and 3.2, $X^{1+\varepsilon} \hookrightarrow H^{2\varepsilon}_q(\Omega) \times B^{1+\varepsilon}_{q,\infty}(\Gamma)$ and $X^{-} \hookrightarrow F_{(1+\gamma)/2} \hookrightarrow [H^{-\gamma}_q(\Omega)]' \times \{ 0 \} = H^{-1}_q(\Omega) \times \{ 0 \}$ if $\gamma > 1/q$, so that (4.3) will follow for $i = 0, 1$ if we show

$$|f_i(t, u_1, u_2)|_{\gamma_{i-1},q} \leq C|u_1 - u_2|_{2\varepsilon_i,q} \left( 1 + \sum_{j,k=0}^1 |u_k|_{\rho_{ij}^{-1},q} \right), \quad i = 0, 1.$$

These estimates follow from Lemma 5.1 with $\theta = \gamma_0 - 1$ and $\alpha_i = 2\varepsilon_i$. The assumptions in Lemma 5.1 are satisfied due to $n/(n+q) < 2/\rho_0(q) < n/q$ and $\gamma_0 = 1^-$. □

Theorem 5.7. Let $1 < q < \infty$, $u_0 \in L_q(\Omega)$, $v_0 \in B^{1}_{q}(\Gamma)$ and assume that conditions (HF) and (Hg) are satisfied with

$$\rho_0 < \rho_0(q), \quad \rho_1 < \rho_1(q), \quad \rho_0 < \rho_0(q), \quad \rho_1 < \rho_1(q), \quad (n-q)\rho_2 \leq n.$$  

Assume that one of the following two conditions holds:

i) $q > n/2$ and $g$ satisfies (Hg+).

ii) $1 < q \leq 1 + \frac{1}{2}(\sqrt{n+2} - 4 - n)$.

Then for any $\tau$ small enough there exists a unique solution

$$(u, v) \in C([0, \tau], L_q(\Omega)) \times B^{1}_{q}(\Gamma) \cap C((0, \tau], H^{2\theta}_{q}(\Omega) \times B^{1+\theta}_{q,\infty}(\Gamma))$$

of (4.4), where $\theta = 1^-$ in case i) and $\theta = [1/q]^-$ in case ii). Moreover, $v(t) = v(t)$ for $t > 0$.

Proof. Increasing $\rho_0, \rho_1, \rho_0, \rho_1$ if necessary, we may assume that (5.3) is true and $\rho_0 = [\rho_0(q)]^-$. We apply Theorem 4.1 to prove the assertion. For this we need to check that hypothesis (H) holds. Notice that Propositions 5.5 and 5.6 determine the values of $\varepsilon_j$ and $\rho_{ij}$ in (4.2). In all the possible cases we have the following values: $\varepsilon_0 = [1/\rho_0(q)]^-$, $\varepsilon_1 = [1/\rho_1(q)]^-$, $\rho_{00} = [\rho_0(q)]^-$, $\rho_{01} = [\rho_0(q)]^-$, $\rho_{10} = [\rho_1(q)]^-$, $\rho_{11} = [\rho_2(q)]^-$, and $\rho_{02} = \rho_{12} = \rho_{20} = \rho_{21} = 1$. Propositions 5.5 and 5.6 guarantee that the first condition of (4.2) is satisfied. Therefore, we just need to check $\gamma > \bar{\varepsilon}$. Since $\gamma_0$ and $\gamma_1$ are always close to 1, it is sufficient to fulfill $\gamma_2 > \bar{\varepsilon}$. 


i) If $q > n/2$, then we have $\gamma_2 = 1^{-} > \bar{\varepsilon}$.

ii) If $1 < q \leq n/2$, then we have $\gamma_2 = [1/q]^{-}$, and therefore we need

$$
\frac{1}{q} \geq \max \left\{ \frac{1}{\rho_0(q)}, \frac{1}{\rho_1(q)} \right\} = \max \left\{ \frac{n}{n + 2q}, \frac{n + q}{n + 2q} \right\},
$$

which is equivalent to $1 < q < 1 + \frac{1}{2}\sqrt{n^2 + 4 - n}$.

It remains to show the trace property $\gamma u(t) = v(t)$. Let $z = [1/q]^{+}$, $(\varphi, \psi) \in H_q^{\pm}(\Omega) \times B_q^{\pm}(\Gamma)$ and $G(\varphi, \psi) := \gamma \varphi - \psi$. Then

$$
G \in \mathcal{L}(H_q^{\pm}(\Omega) \times B_q^{\pm}(\Gamma), L_q(\Gamma)) \quad \text{and} \quad G(E_\gamma) = \{0\} \quad \text{for} \; \gamma > 1/(2q).
$$

Since $e^{-tA}Z_0, e^{-(t-s)A}F(Z(s)) \in E_\gamma$ with $\gamma > 1/(2q)$ for $0 < s < t$ and the integral $\int_0^t e^{-(t-s)A}F(Z(s)) \, ds$ converges in $H_q^{\pm}(\Omega) \times B_q^{\pm}(\Gamma)$, the integral identity (4.4) implies $G(u(t), v(t)) = 0$ for $t > 0$.

**Remark 5.8.** If $1 < q \leq n/2$ and $f$ is independent of $\nabla u$ then the additional assumption $q \leq 1 + \frac{1}{2}\sqrt{n^2 + 4 - n}$ in Theorem 5.7 may be replaced by $1/q \geq 1/\rho_0(q)$; that is, $(n-2)q \leq n$. This follows from the proof of Theorem 5.7.

### 5.2. Case of initial data in $\mathcal{M}(\Omega) \times \mathcal{M}(\Gamma)$.

Recall first that $\mathcal{M}(\Omega) \times \mathcal{M}(\Gamma) \hookrightarrow X_1^{-\delta}(q)$ whenever $\delta q' > n$, $\delta > 0$, $q > 1$ (see (3.5) and Lemma 3.2) and that we want to use the scale $X^n = X_1^{n-\delta}$ with $\delta > 0$ small and $q$ close to 1.

Let us start with an analogue to Proposition 5.6. In this case, we shall need to choose

$$
\delta \in \left( \frac{n}{q'}, \frac{n^2}{2(n+q)} \right). \tag{5.13}
$$

This choice is possible whenever $n/q' < n^2/(2n+2q)$, that is, $2q^2 + q(n-2) < 2n$ (which is true for $q$ close to 1).

**Proposition 5.9.** Let $q > 1$ be close to 1, let $\delta$ satisfy (5.13) and let $f$ fulfill (HF) with $\rho_0, \rho_1, \bar{\rho}_0$, and $\bar{\rho}_1$ satisfying (5.3) and $\rho_0 = [\rho_0(q)^{-}]^{-}$. Then $\mathcal{F}_1$ fulfills (4.3) with $\rho_0 = \rho_0$, $\rho_1 = \bar{\rho}_0$, $\rho_{10} = \bar{\rho}_1$, $\rho_{11} = \bar{\rho}_1$, $\rho_{02} = \rho_{12} = 1$ and $\varepsilon_0 = (1/\rho_0)^{-}$, $\varepsilon_1 = (1/\rho_1)^{-}$, and $1 > \gamma_0 = \gamma_1 > \max\{\delta\}$ for $i = 0, 1, \delta = 1^{-}$.

**Proof.** The proof is similar to that of Proposition 5.6. Exactly as in that proof we obtain $\gamma_0 > \max\{\delta\}$ for $i = 0, 1, \delta = 1^{-}$.

According to Lemmas 3.1 and 3.2, $X^{1+\varepsilon} \hookrightarrow H_q^{2(\varepsilon, \delta)}(\Omega) \times B_q^{1+\varepsilon, \delta}(\Gamma)$ and $X^\gamma \hookrightarrow F(1+\gamma^{-\delta})/2 \hookrightarrow [H_q^{1+\gamma^{-\delta}}(\Omega)]^* \times \{0\} \hookrightarrow H_q^{-1}(\Omega) \times \{0\}$ provided $\gamma > 1/q$,.
so that (4.3) will follow for $i = 0, 1$, if we show

$$|f_i(t, u_1, u_2)|_{\gamma_i-1,q} \leq C|u_1 - u_2|_{2(\varepsilon_i - \delta),q} \left( 1 + \sum_{j,k=0}^1 |u_k|^{\rho_{j-1}}_{2(\varepsilon_j - \delta),q} \right), \quad i = 0, 1.$$ 

These estimates follow from Lemma 5.1 with $\theta = \gamma_0 - 1$, $\alpha_0 = 2(\varepsilon_0 - \delta)$ and $\alpha_1 = \alpha_0(n + q)/n$, because $2(\varepsilon_1 - \delta) > \alpha_1$. The assumptions on $\alpha_0$ in Lemma 5.1 require $n/(n + q) \leq 2/\rho_0^2(q) - 2\delta < n/q$, which is guaranteed by (5.13). The other assumptions in Lemma 5.1 are obviously satisfied. \qed

Next we prove an analogue to Proposition 5.5.

**Proposition 5.10.** Let $q$ be close to 1 and $\delta$ close to $n/q'$ ($q > 1$, $\delta > n/q'$). Let $g$ satisfy (Hg) with $\rho_2 < n/(n - 1)$. Then the operator $F_2$ fulfills (4.3) with $\rho_{21} = 1$ for $i = 0, 1$, $\varepsilon_2 = [(n - 1)/n]^{-}$ if $n > 1$, $\varepsilon_2 = \delta^+$ if $n = 1$, $\rho_{22} = [1/(q_2^2)]^{-}$ and $1 > \gamma_2 > \rho_{22} \varepsilon_2 = (1/q)^{-}$.

**Proof.** Due to Lemmata 3.1 and 3.2, we have $X^{1+\varepsilon_2} \hookrightarrow H_q^{2(\varepsilon_2 - \delta)}(\Omega) \times B_q^{1+\varepsilon_2 - \delta}(\Gamma)$ and $X^{\gamma_2} \hookrightarrow F_{1+\gamma_2 - \delta}/2 \hookrightarrow \{0\} \times B_q^{1+\varepsilon_2 - \delta}(\Gamma)$ for any $\varepsilon_2, \gamma_2 \in [\delta, 1]$. In order to verify the corresponding properties of the Nemytskii mapping $G : B_q^{1+\varepsilon_2 - \delta}(\Gamma) \to B_q^{1+\varepsilon_2 - \delta}(\Gamma) : v \mapsto g(\cdot, t, v(\cdot))$, it is sufficient to use Lemma 5.2 i) with $\theta = \gamma_2 - \delta$ and $\alpha = 1 + \varepsilon_2 - \delta$ (or Remark 5.3 if $n = 1$).

If $n > 1$ and $\delta$ is sufficiently close to $n/q'$ then the assumptions of Lemma 5.2 i) are satisfied if $\varepsilon_2 = [(n - 1)/n]^{-}$, $\rho_{22} = n/(q(n - 1))$ and

$$\gamma_2 := \rho_{22}(\varepsilon_2 - \delta) + \delta + \frac{n}{q} \left( 1 - \rho_{22} \frac{n - q}{n} \right).$$

Moreover, this choice guarantees also $1 > \gamma_2 > \varepsilon_2 \rho_{22}$. \qed

**Theorem 5.11.** Assume that the nonlinearities $f$ and $g$ satisfy (HF) and (Hg) with exponents $\rho_0, \rho_1, \bar{\rho}_2, \bar{\rho}_0$ and $\bar{\rho}_1$ satisfying $\rho_0 < \rho_0^1(1)$, $\rho_1 < \rho_0^1(1)$, $\bar{\rho}_0 < \bar{\rho}_0^1(1)$, $\bar{\rho}_1 < \bar{\rho}_0^1(1)$, and $\rho_2 < \rho_0^2(1)$. Then there exist $\delta > 0$ small and $q > 1$, close to 1, such that $\mathcal{M}(\Omega) \times \mathcal{M}(\Gamma) \hookrightarrow X_1^{1-\delta}(q)$ and $F_1, F_2$ satisfy (H) with respect to the scale $X^n = X_1^{n-\delta}(q)$.

In particular, for every $(u_0, v_0) \in \mathcal{M}(\Omega) \times \mathcal{M}(\Gamma)$ there exists a unique $\varepsilon$-regular solution $(u, v) \in C([0, \tau], X^1) \cap C((0, \tau], X^{1+\tilde{\gamma}})$ for some $\tilde{\gamma} = 1^{-}$.

**Proof.** The proof follows from Propositions 5.9, 5.10 and 4.1 similarly to the proof of Theorem 5.7. \qed

5.3. **Case of initial data in** $L_{q\#}(\Omega) \times L_q(\Gamma)$. Let $1 < q < \infty$. As already announced, we shall use the scale $X^q = X_{\xi}^{q-\delta}$, where $\delta > 0$ is small and
Recall also that denoting \( q^# = \frac{nq}{n + q - 1} \), (3.5) and Lemma 3.2 imply

\[
L_{q^#}(\Omega) \times L_q(\Gamma) \hookrightarrow H^{1/q-1}_q(\Omega) \times B^{1/q}_q(\Gamma) \hookrightarrow F_{\xi^{-\delta}/2} \hookrightarrow X^{1-\delta}_\xi.
\]

As in the preceding cases, we first verify the assumptions (H). To simplify the proof, we shall work with the scale \( \tilde{X}^{\eta} := X^{\eta-\delta}_\xi \) in the following Proposition 5.12. Since the assumption (H) for the scale \( \tilde{X}^{\eta} \) (with some \( \tilde{\gamma}_i > \tilde{\eta}_i \) and \( \tilde{\varepsilon}_i \)) implies the same assumption for the scale \( X^{\eta} \) (with \( \gamma_i = \tilde{\gamma}_i + \varepsilon_i \) and \( \varepsilon_i = \tilde{\varepsilon}_i \)), the conclusions of Proposition 5.12 remain true also for the original scale \( X^{\eta} \).

**Proposition 5.12.** Let \( q > 1 \) and let \( \delta > 0 \) be small enough.

(i) Let \( f \) fulfill (Hf) with \( \rho_0, \rho_1, \rho_0, \) and \( \tilde{\rho}_1 \) satisfying (5.3) and \( \rho_0 = [\rho_0(q^#)]^- \). Then \( \mathcal{F}_1 \) fulfills (4.3) with \( \rho_{00} = \rho_0, \rho_{01} = \tilde{\rho}_1, \rho_{10} = \rho_1, \rho_{11} = \rho_1, \rho_{02} = \rho_{12} = 1 \) and some \( \varepsilon_0 = (1/\rho_0)^-, \varepsilon_1 = (1/\rho_1)^- \) and \( 1 > \gamma_0 = \gamma_1 > \max_i, j, 0 \{ \varepsilon_i + (\rho_{i,j} - 1)\varepsilon_j \} = 1^- \).

(ii) Let \( g \) satisfy (Hg) with \( \rho_2 = [\rho_2(q)]^- \). Then the operator \( \mathcal{F}_2 \) fulfills (4.3) with \( \rho_{20} = 1 \) for \( i = 0, 1, \rho_{22} = \rho_2, \varepsilon_2 = [1/\rho_2]^-, \) and \( 1 > \gamma_2 > \rho_{22} \varepsilon_2 = 1^- \).

**Proof.** In the same way as in the proof of Proposition 5.6, we obtain \( \gamma_0 > \max_i, j \{ \varepsilon_i + (\rho_{i,j} - 1)\varepsilon_j \} = 1^- \).

According to Lemmas 3.1 and 3.2,

\[
\tilde{X}^{1+\varepsilon} = X^{1+\varepsilon-\delta}_\xi \hookrightarrow H^{1/q+2\varepsilon-1-\delta}_q(\Omega) \times B^{1/q+\varepsilon}_q(\Gamma) \hookrightarrow H^{2\varepsilon-\delta}_q(\Omega) \times B^{1/q+\varepsilon}_q(\Gamma),
\]

\[
\tilde{X}^\gamma = X^{\gamma-\delta}_\xi \hookrightarrow F_{\xi^{-(\gamma+\delta)/2}} = F_{(1/q+\gamma)/2} \hookrightarrow [H^{2-1/q-\gamma}_q(\Omega) \times B^{2-1/q-\gamma}_q(\Gamma)]' \hookrightarrow [H^{1-\gamma}_q(\Omega) \times B^{2-1/q-\gamma}_q(\Gamma)]' = H^{1-\gamma}_q(\Omega) \times B^{1/q-\gamma}_q(\Gamma),
\]

for \( \varepsilon > \delta \) and \( \gamma > \delta \vee 1/q^# \).

(i) The above imbeddings show that (4.3) is true for \( i = 0, 1 \) if

\[
|f_i(t, u_1, u_2)|_{\gamma_i-1, q^#} \leq C|u_1 - u_2|_{2\varepsilon_i-\delta, q^#} \left( 1 + \sum_{j, k=0}^{1} |u_k|_{2\varepsilon_j-\delta, q^#}^{-1} \right), \quad i = 0, 1.
\]

These estimates follow from Lemma 5.1 with \( \theta = \gamma_0 - 1, \alpha_0 = 2\varepsilon_0 - \delta, \alpha_1 = \alpha_0(n + q^#)/n \) and \( q \) replaced by \( q^# \) because \( 2\varepsilon_i - \delta > \alpha_1 \). The assumptions in Lemma 5.1 are obviously satisfied if \( \delta \) is small enough.

(ii) In order to verify the corresponding properties of the Nemytskii mapping \( G : B^{1/q+\varepsilon_2}_q(\Gamma) \to B^{1/q+\gamma_2-1}_q(\Gamma) : v \mapsto g(\cdot, t, v(\cdot)) \), it is sufficient to use Lemma 5.2 i) with \( \theta = 1/q + \gamma_2 - 1 \) and \( \alpha = 1/q + \varepsilon_2, \) where \( \varepsilon_2 = [1/\rho_2]^-, \)
\[
\rho_{22} = [\rho_{22}^2(q)]^{-} \quad \text{and} \\
\gamma_2 = \rho_{22}^2 + \frac{n-1}{q}(\rho_{22}^2 - \rho_{22}).
\]

Then the assumptions of Lemma 5.2 i) are satisfied and we are done. \( \square \)

**Theorem 5.13.** Let \( 1 < q < \infty \), and let the nonlinearities \( f \) and \( g \) fulfill (Hf) and (Hg) with exponents \( \rho_0, \rho_1, \rho_2, \hat{\rho}_0 \) and \( \hat{\rho}_1 \) satisfying
\[
\rho_0 < \rho_0^*(q^\#), \quad \rho_1 < \rho_1^*(q^\#), \quad \hat{\rho}_0 < \hat{\rho}_0^*(q^\#), \quad \hat{\rho}_1 < \hat{\rho}_1^*(q^\#), \quad \rho_2 < \rho_2^*(q),
\]
where \( q^\# = \frac{nq}{n+q-1} \). If \( u_0 \in L_{q^\#}(\Omega) \) and \( v_0 \in L_q(\Gamma) \), then there exists a unique \( \varepsilon \)-regular solution \( (u, v) \in C([0, \tau], X_1^{1-\delta}) \cap C((0, \tau], X_2^{-\delta}) \) of (4.4), where \( 2\xi = 1 + 1/q + \delta \) and \( \delta > 0 \) is small.

**Proof.** The proof is a direct consequence of Proposition 5.12 and Proposition 4.1.

6. **Elliptic equation with dynamic boundary conditions**

In this section we consider the following problem with dynamic boundary conditions:

\[
\begin{aligned}
A u &= 0, & x &\in \Omega, \ t > 0, \\
(\gamma u)_t + B u &= g(x, t, \gamma u), & x &\in \Gamma, \ t > 0, \\
u(x, 0) &= u_0(x), & x &\in \Omega, \\
(\gamma u)(x, 0) &= v_0(x), & x &\in \Gamma.
\end{aligned}
\]

(6.1)

In this case one can reduce (6.1) to an evolution problem on \( \Gamma \) as follows. As in Section 2, consider the operator \( T \in \mathcal{L}(B_q^\ast(\Gamma), H_q^s(\Omega)) \) for \( s > 1/q \).

Recall that \( Tg, g \in B_q^s(\Gamma) \), is defined as the unique (generalized) solution of the problem

\[
\begin{aligned}
A u &= 0 \quad \text{in} \ \Omega, \\
u &= g \quad \text{on} \ \Gamma.
\end{aligned}
\]

Then, define the operator, which we still denote \( B \), since no confusion seems likely, as

\[
B u = \frac{\partial(T u)}{\partial \nu}.
\]

Note that for \( s > 1/q \), \( B \) is well defined from \( B_q^{1+s}(\Gamma) \) into \( B_q^s(\Gamma) \). Therefore (6.1) can be written as

\[
u_t + B u = g(x, t, u).
\]

It follows from the results in [9], after a suitable interpolation and extrapolation process, that \( B \) generates an analytic semigroup on each space of the scale \( E^\alpha = B_q^\alpha(\Gamma), \alpha \in \mathbb{R} \), with domain \( E^{\alpha+1} \); see also [13] and [15] for a
Hilbertian setting. On the other hand, we assume that the nonlinear term $g$ is as in (1.3).

Since in this case we are dealing with just one nonlinear term and a suitable scale of spaces, the techniques in [5] can be applied to obtain the following results.

**Theorem 6.1.** Let $1 < q < \infty$, and let $g$ satisfy (Hg) with $\rho_2 \leq \rho_2^*(q)$. Then (6.1) is well-posed in $B^{1/q}_q(\Gamma)$.

**Proof.** In this case we take the scale of spaces $X^n = E^{1/q-1+n}$, so that $X^1 = B^{1/q}_q(\Gamma)$. Using Lemma 5.2, $\rho_2 \leq \rho_2^*(q)$ and some elementary computations we see that the Nemytskii operator $G$ is an $\varepsilon$-regular map relative to the scale $X^n$ (see [5] or [6] for the corresponding definition). Applying the results of [5] or [6] we obtain the result. \qed

**Remark 6.2.** If $\rho_2 < \rho_2^*(q)$, then for each $u_0 \in L^q(\Omega)$ there exists a unique solution of (6.1). This is true since $L^q(\Gamma) \hookrightarrow B^{1/q-\delta}_q(\Gamma)$ for any small $\delta > 0$ and we can consider the scale $X^n = E^{1/q-1-\delta+n}$ for small $\delta > 0$. The condition $\rho_2 < \rho_2^*(q)$ guarantees that $G$ is an $\varepsilon$-regular map relative to the scale $X^n$, and this permits us to apply the abstract result of [5] or [6].

**Theorem 6.3.** Assume that the nonlinearity $g$ satisfies (Hg) with $\rho_2 < \rho_2^*(1)$. Then for each initial function $u_0 \in \mathcal{M}(\Gamma)$, there exists a unique solution of (6.1).

**Proof.** We follow a similar reasoning as that of [6]. For $q > 1$ and $s_0 > n/q'$ we have $\mathcal{M}(\Gamma) \hookrightarrow B^{1-s_0}_q(\Gamma)$. Moreover, if we choose $q$ close to 1 and $s_0 > n/q'$ close to 0, and if we let $X^{1+\eta} = B^{1-s_0+\eta}_q(\Gamma)$, we can prove, using Lemma 5.2, that $G$ is an $\varepsilon$-regular map relative to this scale of spaces as long as $\rho_2 < \rho_2^*(1)$. Applying the results of [5] or [6] we obtain the assertion.

**References**


