Laminations by Riemann surfaces embedded in Kähler surfaces

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Abstract

Our aim is to explain how to prove that a lamination by Riemann surfaces embedded in Kähler surfaces is uniquely ergodic under some assumptions. This result remains true when admitting hyperbolic singularites.

Laminations

Laminations by Riemann surfaces arise naturally from different dynamical systems. They are defined like objects with a special local structure, a product of a complex disk and a topological space.

Rigourously, let X be a topological space endowed with an atlas $\mathcal{U}_\alpha = \{U_\alpha, \phi_\alpha\}$, where

$$\phi_\alpha : \mathbb{D} \times \mathcal{T}_\alpha \to U_\alpha.$$

We say that X is a transversally Lipschitz lamination if $\mathcal{T}_\alpha$ is a metric space and the change of coordinates can be written like

$$\phi_\beta^{-1} \circ \phi_\alpha : \mathbb{D} \times \mathcal{T}_\alpha \to \mathbb{D} \times \mathcal{T}_\beta$$
$$(z,t) \to (\phi^1_{\alpha\beta}(z,t), \phi^2_{\alpha\beta}(t)).$$

with $\phi^1_{\alpha\beta}$ holomorphic in the first variable and Lipschitz in the second one, and $\phi^2_{\alpha\beta}$ Lipschitz.

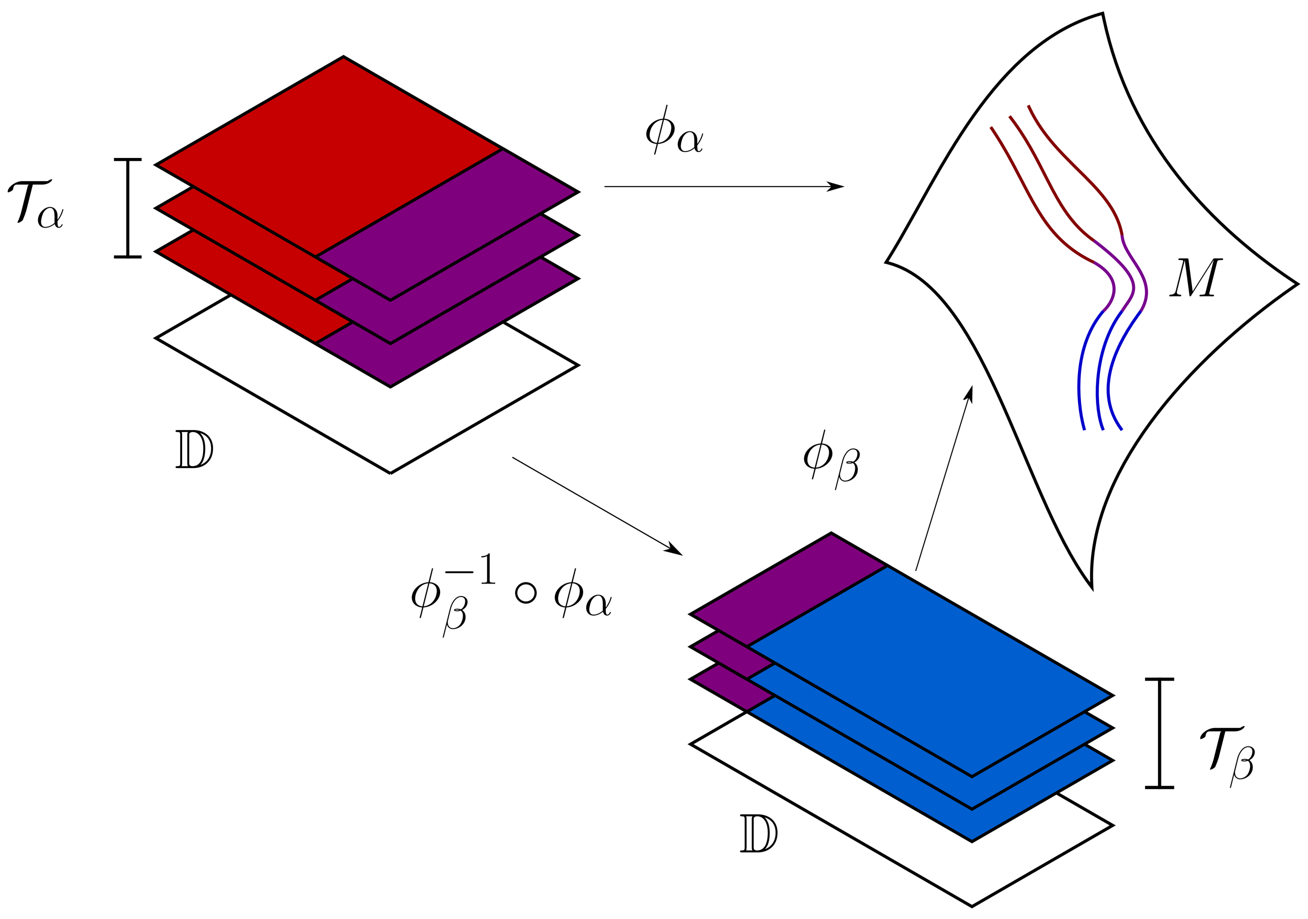

Figure 1: Flow boxes and changes of coordinates

These open coordinated sets are called flow boxes, the disks and their images are plaques -we will not make deep diferences between both in our treatment- and, since the change of coordinates is holomorphic in the variable z, these plaques can be analytically continued. This analytical continuations are the leaves of the laminations. The map between transversal is, roughly speaking, what we will call holonomy.

Since, we are interested in embedding these objects in Kähler surfaces our treatment of the flow boxes will be slightly different. Suppose X is a lamination embedded in a compact homogeneous Kähler surface (M, ω). We can take local coordinates of M, $\varphi : \mathbb{D}^2 \to U$ such that the plaques are holomorphic local graphs of one of these coordinates, namely a plaque can be written like $\{\Gamma_\alpha = \varphi(x,y), \quad y = f_\alpha(x)\}$ for certain f_α holomorphic function. In this case, the Lipschitz transversality condition says that there is a constant $C > 1$ depending on the flow box, such that

$$\frac{d(t,t')}{C} \leq d(f_t(z), f_{t'}(z)) \leq C d(t,t')$$

for every $z \in \Delta_\delta$. In the expression above d is the distance in the transversal coming from the metric defined in the surfaces.

Currents

The space of $(1,1)$-forms on M can be endowed with the supremum norm, in this way the space of $(1,1)$-forms gets a structure of Banach space. A $(1,1)$-current T of order 0 is a $\mathbb{C}$ linear functional T on this space. We will deal with positive harmonic currents directed by a lamination. It means that $T(\alpha) \geq 0$ if α is a positive $(1,1)$-form, $T_{|U}(\gamma) = 0$ where γ is a $(1,0)$-form in U holding that the plaques of the lamination in U are integral varieties of γ and $T(\partial\bar{\partial}u) = 0$ for every real function u.

This kind of currents can be decomposed in regular flow boxes as

$$T = \int_{t \in \mathcal{T}} h_t [\Gamma_t] d\mu(t).$$

where h_t is a harmonic function on each plaque depending on the point in the transversal $t \in \mathcal{T}$, $[\Gamma_t]$ is the integration current over the plaque Γ_t and μ is a local transversal measure. In the specific case of $dT = 0$, the harmonic functions are constants and the transversal measure holonomy invariant.

Main theorem [3]

Let $\mathcal{M}$ be a homogeneous compact Kähler surface containing a minimal Lipschitz lamination $\mathcal{L}$ by Riemann surfaces. If there are no closed currents directed by $\mathcal{L}$, then there is a unique directed harmonic current of mass one.

Sketch of the proof

Intersection theory

This result was proven by Fornæss and Sibony [1] when $M = \mathbb{CP}^2$ developing an intersection theory for currents, which is the cornerstone of the proof of the theorem. Besides $\mathbb{CP}^2$, there are only three other cases to consider in order to prove it [5]: complex tori,$\mathbb{T}^2$; products of elliptic curves by complex lines, $\mathbb{T}^1 \times \mathbb{CP}^1$ and product of two complex lines, $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Intersection theory for currents is a very subtle matter. It is not always well defined in most. But in the case of directed harmonic currents, the intersection Q it is. Assuming non existence of closed currents, if we can prove that $Q(T,T) = 0$ for every $T \in \mathcal{C}$, then there will be only one harmonic current directed by the lamination.

This Q is defined in a very analytic way. However if we define the geometric self intersection of a current $T = \int h_\alpha [\Gamma_\alpha] d\mu(\alpha)$ to be 0 like

$$T \wedge_g T(\psi) = \lim_{\epsilon \to 0} T \wedge_g F_{\epsilon *} T(\psi) = \int \sum_{p \in J^\epsilon_{\alpha,\beta}} \psi(p) h_\alpha(p) h^\epsilon_\beta(p) d\mu(\alpha) d\mu(\beta) = 0$$

where F_ϵ is a continuous family of automorphisms of the surface satisfying $F_0 = id$ and $J^\epsilon_{\alpha,\beta}$ are the intersection points between the plaque Γ_α and the moved one Γ^ϵ_β. The fact of $T \wedge_g T = 0$ is independent on the choice of F_ϵ, so if we find a family of automorphisms F_ϵ verifying that $J^\epsilon_{\alpha,\beta}$ is uniformly bounded for $|\epsilon| < \epsilon_0$ and for every pair of plaques, we obtain that $T \wedge_g T = 0$. In addition, by a smoothing process, we can prove that $Q(T,T) = 0$ when $T \wedge_g T = 0$. So, we have reduced the uniqueness problem to a problem of finding a good family of automorphisms and computing intersection points between plaques.

Counting intersection points

Therefore, our aim is seeking for a family of automorphisms on each surface, and bounding the intersection points between plaques. These families are as follows

$\mathbb{CP}^2$, $F_\epsilon([z:w:t]) = [z : w + \epsilon t : t]$

$\mathbb{T}^2$, $F_\epsilon([(z,w)]) = [(z + \epsilon v_1, w + \epsilon v_2)]$

$\mathbb{T}^1 \times \mathbb{CP}^1$, $F_\epsilon([z],[w:t]) = ([z + \epsilon v_1], [w + \epsilon t v_2 : t])$

$\mathbb{CP}^1 \times \mathbb{CP}^1$, $F_\epsilon([z_0 : z_1],[w_0 : w_1]) = ([z_0 + \epsilon v_1 z_1 : z_1], [w_0 + \epsilon v_1 w_1 : w_1])$.

The proof for $\mathbb{CP}^2$ is based on the existence of a invariant line for the family of automorphisms, where we know explicetely the behaviour of the lamination. On the other hand, for the rest of the surfaces we can use the fact that we can find a large open relatively compact set, where these families behave like translations. In this way, we can make the difference between some plaques following the direction of the automorphisms and some plaques which are transversal to these motions. Thus, when assuming absence of closed currents, we are also assuming absence of compact leaves, so we can cover these large open set by transversal and longitudinal flow boxes. In longitudinal flow boxes, the absence of compact leaves, Lipchitzness of the lamination and Hurwitz's Theorem give us that intersection between plaques is bounded by above. In transversal flow boxes, plaques with a lot of intersection points have to be very close, but they happen to be far away at the same time, getting a contradiction. The behaviour of the points outside this large set have to be controlled too.

Generalization, application and meaning

In the sketch of the proof, we include (v_1, v_2), but we would not need it. However, since it is not proven that these laminations can exist, by making a good choice of (v_1, v_2) and arguing in the same way, we can obtain that this theorem is true for foliations on these surfaces with only hyperbolic singularities [2]. And, on the contrary to the non singular case, this happens to be the generic case of foliations. Hence, we can rewrite the theorem as follows.

Main Theorem Generalized [4]

Let $\mathcal{M}$ be a homogeneous compact Kähler surface and $\mathcal{F}$ a holomorphic foliation with only hyperbolic singularities. If there are no closed currents directed by $\mathcal{F}$, then there is a unique directed harmonic current of mass one.

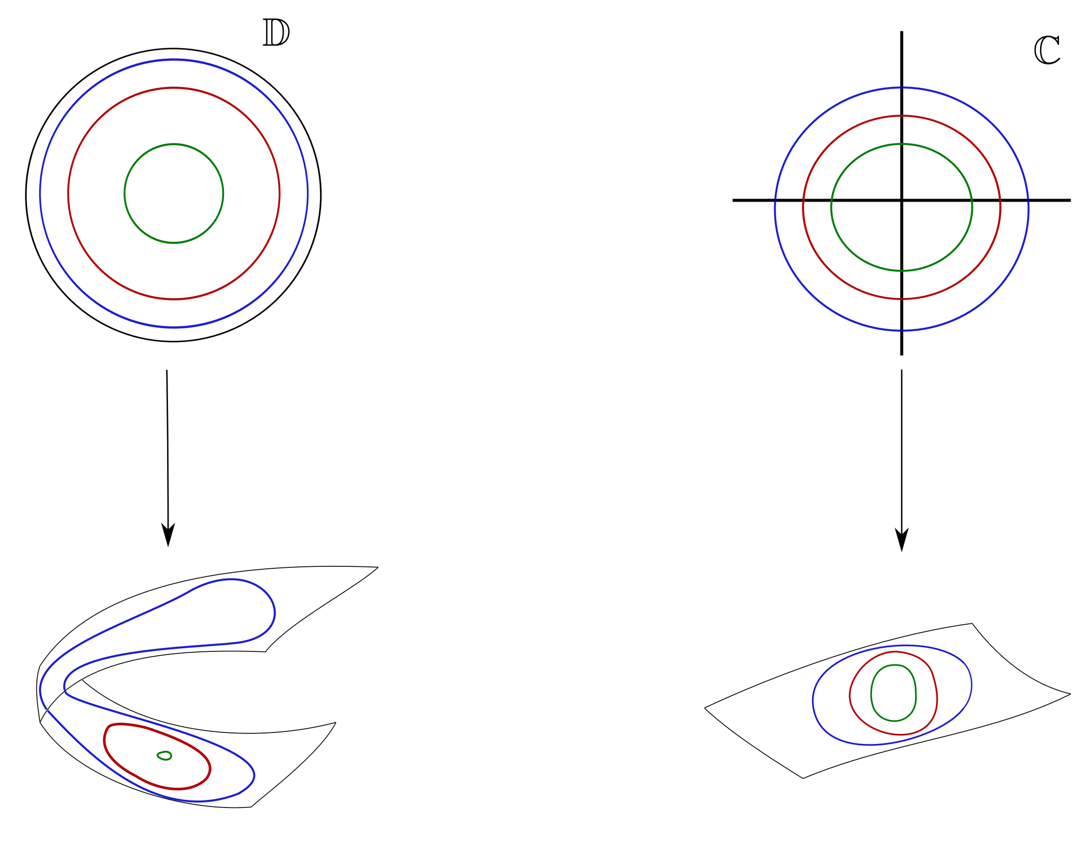

Figure 2: Universal covers of leaves

Best way of understanding this result comes from the understanding of directed harmonic currents. The way we know of create such currents, is a limit averaging proccess of the integration of a exhaustive collection of disks in the universal covering of the leaves. Roughly speaking, if the leaf is parabolic, we can obtain closed currents, and if the leaf is hyperbolic, we would obtain harmonic currents. Therefore, this uniqueness, can be seen as a independent of the initial conditions behaviour. Namely, it does not matter which is the leaf we are averaging on, the limit will always be the same

References

-[1] Fornæss, J.E., Sibony, N.: Harmonic currents of finite energy and laminations. Geom. Funct. Anal.15, (2005)
-[2] Fornæss, J.E., Sibony, N.: Unique ergodicity of harmonic currents on singular foliations of $\mathbb{P}^2$. Geom. Funct. Anal. 19, (2010)
-[3] Pérez-Garrandés, C.: Directed harmonic currents on homogeneous compact Kahler surfaces. Accepted in Internat. J. of Math.
-[4] Pérez-Garrandés, C.: Ergodicity of laminations with singularities in Kahler surfaces. Accepted in Math.Z.
-[5] Tits, J.: Espaces homogénes complexes compactes. Comment. Math. Helv.37, (1962/63)