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A maximum principle for evolution Hamilton–Jacobi equations on Riemannian manifolds

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Abstract

We establish a maximum principle for viscosity subsolutions and supersolutions of equations of the form $u_t + F(t, d_x u) = 0$, $u(0, x) = u_0(x)$, where $u_0 : M \to \mathbb{R}$ is a bounded uniformly continuous function, M is a Riemannian manifold, and $F : [0, \infty) \times T^*M \to \mathbb{R}$. This yields uniqueness of the viscosity solutions of such Hamilton–Jacobi equations.

Keywords: Hamilton–Jacobi equations; Viscosity solutions; Riemannian manifolds

First order Hamilton–Jacobi equations are partial differential equations of the form

$$F\big(x, u(x), du(x)\big) = 0$$

in the stationary case, and of the form,

$$F\big(t, x, u(x, t), du(t, x)\big) = 0$$

in the evolution case. Such equations arise, for instance, in optimal control theory and differential games. It is very well known that, even in the simplest case where $u : \mathbb{R} \to \mathbb{R}$, there are examples

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of such equations which possess no classical solutions. Nevertheless, a class of weak solutions, the so-called viscosity solutions, do exist under very general hypotheses, see for instance [3–11] and the references therein. Up to very recently, the literature on viscosity solutions to Hamilton–Jacobi equations dealt with equations defined on (subsets of) $\mathbb{R}^n$ or of Banach spaces. However, many examples of Hamilton–Jacobi equations arise naturally in the setting of Riemannian manifolds [1].

In [2] we developed a calculus of viscosity subdifferentials and we established a smooth variational principle for functions defined on Riemannian manifolds (either finite-dimensional or infinite-dimensional), which allowed us to prove existence and uniqueness of viscosity solutions to stationary HJ equations defined on Riemannian manifolds. We also showed that the distance function $d(.,\partial\Omega)$ is the unique viscosity solution of the eikonal equation $\|du(x)\|_x = 1$ on Ω, $u = 0$ on $\partial\Omega$, where Ω is an open subset of a Riemannian manifold. The latter result was independently proved by Mantegazza and Menucci [13] in the finite-dimensional case.

In this note we will apply some of the tools developed in [2] to establish a maximum principle for evolution HJ equations of the form

$$\begin{cases} u_t + F(t, d_x u) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{$*$}$$

where $u_0 : M \to \mathbb{R}$ is an initial condition which we assume to be bounded and uniformly continuous, M is a Riemannian manifold (with some weak restrictions, see below), and $F : [0,\infty) \times T^*M \to \mathbb{R}$ is a Hamiltonian satisfying a uniform continuity-type condition which we will introduce below, let us just now say that if M is compact and we regarded M as embedded in $\mathbb{R}^n$, then every function $F : [0,\infty) \times T^*M \subset \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$ which is uniformly continuous with respect to the usual metric of $\mathbb{R}^{2n+1}$ meets that condition.

In particular we will deduce from our main result that if u and v are bounded viscosity solutions of $(*)$ then $u = v$.

Next we restate some definitions and key results from [2] that we will use in our proof. We refer the reader to [2] and [12] for the definitions of injectivity and convexity radii, exponential mapping, parallel transport and other standard terms of differential geometry. Recall that, for a given curve $\gamma : I \to M$, numbers $t_0, t_1 \in I$ and a vector $V_0 \in TM_{\gamma(t_0)}$ there exists a unique parallel vector field $V(t)$ along γ such that $V(t_0) = V_0$, and the mapping defined by $V_0 \mapsto V(t_1)$ is a linear isometry between the tangent spaces $TM_{\gamma(t_0)}$ and $TM_{\gamma(t_1)}$. In the case when γ is a minimizing geodesic and $x = \gamma(t_0)$, $y = \gamma(t_1)$ we denote this mapping by $L_{xy} : TM_x \to TM_y$, the parallel transport from TM_x to TM_y along γ. The parallel transport allows us to measure the length of the "difference" between vectors (or forms) which are in different tangent spaces (or in duals of tangent spaces, that is, different fibers of the cotangent bundle), and to do so in a natural way. Indeed, let γ be a minimizing geodesic connecting two points $x, y \in M$, and let L_{xy} the parallel transport along γ. For any two vectors $v \in TM_x$, $w \in TM_y$ we can define a natural distance between v and w as the number

$$\left\| v - L_{yx}(w) \right\|_x = \left\| w - L_{xy}(v) \right\|_y$$

(this equality holds because L_{xy} is a linear isometry between the two tangent spaces, with inverse L_{yx}). Since the spaces TM_x and T^*M_x are isometrically identified by the formula $v \equiv \langle v, \cdot \rangle$, we can obviously use the same method to measure distances between forms $\zeta \in T^*M_x$ and $\eta \in T^*M_y$ lying on different fibers of T^*M.

In order to establish a smooth variational principle for complete Riemannian manifolds we had to assume in [2] that the manifold is *uniformly bumpable*:

Definition 1. We say that a Riemannian manifold M is *uniformly bumpable* provided there are numbers $R > 1$ (possibly large) and $r > 0$ (small) such that for every $p \in M$ and $\delta \in (0, r)$ there exists a C^1 smooth function $b : M \to [0, 1]$ such that:

(1) $b(p) = 1$,
(2) $b(x) = 0$ if $d(x, p) \geqslant \delta$,
(3) $\sup_{x \in M} \|db(x)\|_x \leqslant R/\delta$.

The class of uniformly bumpable is very large. In fact we do not know whether every complete Riemannian manifold is uniformly bumpable. The following result, proved in [2], provide us with some sufficient conditions for a Riemannian manifold M to be uniformly bumpable.

Proposition 2. *Let M be a Riemannian manifold. Consider the following conditions*:

(1) *M is compact.*
(2) *M is finite-dimensional, complete, and is uniformly locally convex.*
(3) *M is uniformly locally convex and has a strictly positive injectivity radius.*
(4) *There is a constant $r > 0$ such that for every $x \in M$ the exponential mapping* $\exp_x$ *is defined on $B(0_x, r) \subset TM_x$,* $\exp_x : B(0_x, r) \to B(x, r)$ *is a C^∞ diffeomorphism, and the distance function is given by*

$$d(y, x) = \left\|\exp_x^{-1}(y)\right\|_x \quad \text{for all } y \in B(x, r).$$

(5) *There is a constant $r > 0$ such that for all $x \in M$ the distance function to $\{x\}$, $y \in M \mapsto d(y, x)$, is of class C^∞ on $B(x, r) \setminus \{x\}$.*
(6) *M is uniformly bumpable.*

Then we have (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6).

As we have said, the following smooth variational principle from [2] will play a key role in the proof below.

Theorem 3. *Let M be a uniformly bumpable complete Riemannian manifold, and let $F : M \to (-\infty, +\infty]$ be a lower semicontinuous function which is bounded below, $F \not\equiv +\infty$. Then, for each $\delta > 0$ there exists a bounded C^1 smooth and Lipschitz function $\varphi : M \to \mathbb{R}$ such that*:

(1) *$F - \varphi$ attains its strong minimum on M,*
(2) *$\|\varphi\|_\infty := \sup_{p \in M} |\varphi(p)| < \delta$, and $\|d\varphi\|_\infty := \sup_{p \in M} \|d\varphi(p)\|_p < \delta$.*

Let us now recall the definition of the subdifferential and superdifferential sets of a function $f : M \to \mathbb{R}$.

Definition 4. Let M be a Riemannian manifold, and $f : M \to (-\infty, \infty]$ a proper function. We say that f is subdifferentiable at the point $p \in \mathrm{dom}(f) = \{x \in M\colon\ f(x) < \infty\}$ provided there is a C^1 function $\varphi : M \to \mathbb{R}$ so that $f - \varphi$ attains a local minimum at the point p. In such case we will say that $\zeta = d\varphi(p) \in (TM_p)^* \simeq H^* = H$ is a (viscosity) subdifferential (or subgradient) of f at p. In general we define the subdifferential set of f at p by

$$D^- f(p) = \{d\varphi(p):\ \varphi \in C^1(M, \mathbb{R}),\ f - \varphi \text{ attains a local minimum at } p\},$$

a (possibly empty) subset of T^*M_p. Similarly, we define

$$D^+ f(p) = \{d\varphi(p):\ \varphi \in C^1(M, \mathbb{R}),\ f - \varphi \text{ attains a local maximum at } p\},$$

and we say that f is superdifferentiable at p whenever this set is nonempty. For every $\zeta \in D^- f(p) \cup D^+ f(p)$ the norm of ζ is defined as

$$\|\zeta\|_p = \sup\{|\zeta(h)|:\ h \in TM_p,\ \|h\|_p = 1\}.$$

We refer the reader to [2] for a list of properties and nonsmooth calculus with these subdifferentials; for instance it can be shown that bounded below functions are subdifferentiable on dense subsets of their domains, and that convex functions (that is, functions which are convex along geodesics) are everywhere subdifferentiable. In this note we will only need to use the following fuzzy sum rule from [2].

Theorem 5. *Let M be a Riemannian manifold. Let $f_1, f_2 : M \to \mathbb{R}$ be functions such that f_1 is lower semicontinuous and f_2 is uniformly continuous. Then, for every $x \in M$, $\zeta \in D^-(f_1 + f_2)(x)$ and $\varepsilon > 0$, there exist points x_1 and x_2 in M, and subdifferentials $\zeta_1 \in D^- f_1(x_1)$ and $\zeta_2 \in D^- f_2(x_2)$ such that*:

(1) $d(x_1, x) < \varepsilon$ *and* $d(x_2, x) < \varepsilon$,
(2) $|f_1(x_1) - f_1(x)| < \varepsilon$ *and* $|f_2(x_2) - f_2(x)| < \varepsilon$,
(3) $\|L_{x_2x}(\zeta_2) + L_{x_1x}(\zeta_1) - \zeta\|_x < \varepsilon$.

Here L_{x_ix} stands for the parallel transport from the point x_i to the point x along the unique geodesic connecting these points, $i = 1, 2$. The number ε can always be assumed to be small enough so that this unique geodesic and L_{x_ix} are well defined.

Definition 6. Let M be a Riemannian manifold, $u_0 : M \to \mathbb{R}$, and $F : [0, \infty) \times T^*M \to \mathbb{R}$. We say that a function $u : [0, +\infty) \times M \to \mathbb{R}$ is a viscosity subsolution of the equation

$$\begin{cases} u_t + F(t, d_x u) = 0, \\ u(0, x) = u_0(x), \end{cases} \tag{$*$}$$

provided u is upper semicontinuous and for every $(t, x) \in \mathbb{R}^+ \times M$ and every $(a, \zeta) \in D^+u(t, x)$ we have that

$$\begin{cases} a + F(t, x, \zeta) \leqslant 0, \\ u(0, x) \leqslant u_0(x). \end{cases}$$

We say that a function $v : [0, +\infty) \times M \to \mathbb{R}$ is a viscosity supersolution of $(*)$ if v is lower semicontinuous and for every $(t, x) \in \mathbb{R}^+ \times M$ and $(a, \zeta) \in D^-u(t, x)$ we have that

$$\begin{cases} a + F(t, x, \zeta) \geqslant 0, \\ u(0, x) \geqslant u_0(x). \end{cases}$$

A continuous function $u : [0, \infty) \times M \to \mathbb{R}$ is said to be a viscosity solution of $(*)$ provided it is both a viscosity supersolution and subsolution of $(*)$.

Remark 7. Since every Fréchet differentiable function u satisfies

$$D^+u(p) = D^-u(p) = \{du(p)\},$$

it is clear that any Fréchet differentiable viscosity solution of $(*)$ is also a classical solution of $(*)$.

In the maximum principle we are going to establish we will require that M is a complete Riemannian manifold with strictly positive convexity and injectivity radii, and $u_0 : M \to \mathbb{R}$ is an initial condition which we assume to be uniformly continuous. We will also demand that $F : [0, \infty) \times T^*M \to \mathbb{R}$ satisfies the following uniform continuity condition.

Definition 8. Let M be a complete Riemannian manifold with strictly positive convexity and injectivity radii (greater than, say, $r_M > 0$). We will say that a function $F : [0, \infty) \times T^*M \to \mathbb{R}$ is intrinsically uniformly continuous provided that, for every $\varepsilon > 0$ there exists $\delta \in (0, r_M)$ such that if

$$\zeta \in T^*M_x, \quad \xi \in T^*M_y, \quad d(x, y) \leqslant \delta, \quad |t - s| \leqslant \delta, \quad \big\|\zeta - L_{yx}(\xi)\big\|_x \leqslant \delta$$

then

$$\big|F(t, x, \zeta) - F(s, y, \xi)\big| \leqslant \varepsilon.$$

Remark 9. When M is compact and is regarded as embedded in $\mathbb{R}^n$, every function $F : [0, \infty) \times T^*M \to \mathbb{R} \subset \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$ which is uniformly continuous with respect to the usual Euclidean metric of $\mathbb{R}^{2n}$ is intrinsically uniformly continuous as well, see [2].

Now we can state and prove our main result.

Theorem 10. *Let M be a complete Riemannian manifold with strictly positive convexity and injectivity radii, let $u_0, v_0 : M \to \mathbb{R}$ be two bounded, uniformly continuous functions, and let $F : [0, \infty) \times T^*M \to \mathbb{R}$ be an intrinsically uniformly continuous function. Assume that u is a bounded viscosity subsolution of*

$$\begin{cases} u_t + F(t, d_x u) = 0, \\ u(0, x) = u_0(x), \end{cases}$$

and that v is a bounded viscosity supersolution of

$$\begin{cases} v_t + F(t, d_x v) = 0, \\ v(0, x) = v_0(x). \end{cases}$$

Then $\sup_{[0,\infty)\times M}(u - v) \leqslant \sup_M(u_0 - v_0)$.

Proof. Assume, on the contrary, that

$$\sup_{[0,\infty)\times M} (u - v) > \sup_M (u_0 - v_0).$$

Let us fix an ε_1 such that $0 < \varepsilon_1 < \sup_{[0,\infty)\times M}(u - v) - \sup_M(u_0 - v_0)$. Then there exists $(t_0, x_0) \in (0, \infty) \times M$, such that for every $\varepsilon < \varepsilon_1$

$$u(t_0, x_0) - v(t_0, x_0) > \sup_M (u_0 - v_0) + \varepsilon.$$

Next pick $\delta > 0$ such that $\varepsilon_1 > \delta(1/(2t_0) + 1)$ and fix any $\varepsilon_0 < \delta/(2t_0)$. Then, for every $\varepsilon \leqslant \varepsilon_0$ we have that

$$u(t_0, x_0) - v(t_0, x_0) - \delta > \sup_M (u_0 - v_0) + \varepsilon. \tag{1}$$

Let us now consider the function Φ defined on $\mathbb{R} \times M$ by

$$\Phi = \begin{cases} u(t,x) - v(t,x) - \frac{\delta t}{t_0} & \text{if } (t,x) \in [0,\infty) \times M, \\ -\infty & \text{otherwise.} \end{cases}$$

The function Φ is upper semicontinuous and bounded above. If we apply the smooth variational principle (Theorem 3) to the function $-\Phi$ then we get a C^1 smooth function $g : \mathbb{R} \times M \to \mathbb{R}$ such that:

(i) $\Phi + g$ attains its maximum at a point $(s, y) \in [0, \infty) \times M$;
(ii) $\|g\|_\infty = \sup\{|g(t,x)|: (t,x) \in \mathbb{R} \times M\} < \varepsilon/2$ and $\|g'\|_\infty = \sup\{|g'(t,x)|: (t,x) \in \mathbb{R} \times M\} < \varepsilon/2$.

The number s must be strictly positive (otherwise (1) would be contradicted). If we set

$$A = A(\varepsilon) := \frac{\delta}{t_0} - g_t(s, y), \quad \text{and} \quad \zeta = D_x g(s, y)$$

then we have that $(A, \zeta) \in D^+(u(s, y) - v(s, y))$, with the condition that

$$A > \varepsilon_0 \quad \text{for all } \varepsilon \in (0, \varepsilon_0).$$

Now, by Theorem 5 (applied to v and $-u$) we can find points $(t_1, x_1), (t_2, x_2) \in (0, \infty) \times M$ and $(b_1, \zeta_1) \in D^+u(t_1, x_1)$ and $(b_2, \zeta_2) \in D^-v(t_2, x_2)$ such that:

(i) $|t_i - s| < \varepsilon$ and $d(x_i, y) < \varepsilon$ for $i = 1, 2$.
(ii) $|u(t_1, x_1) - u(s, y)| < \varepsilon$ and $|v(t_2, x_2) - v(s, y)| < \varepsilon$.
(iii) $|b_1 - b_2 - A| < \varepsilon$ and $\|L_{x_1 y}(\zeta_1) - L_{x_2 y}(\zeta_2) - \zeta\|_y < \varepsilon$.

Since u is a viscosity subsolution and v is a viscosity supersolution, we get that $b_1 + F(t_1, x_1, \zeta_1) \leqslant 0$ and $b_2 + F(t_2, x_2, \zeta_2) \geqslant 0$. We can then deduce that

$$b_1 - b_2 + F(t_1, x_1, \zeta_1) - F(t_2, x_2, \zeta_2) \leqslant 0,$$

hence

$$A - \varepsilon + F(t_1, x_1, \zeta_1) - F(t_2, x_2, \zeta_2) \leqslant 0. \tag{2}$$

Besides, we have

$$|t_1 - t_2| \leqslant |t_1 - s| + |s - t_2| < 2\varepsilon,$$
$$d(x_1, x_2) \leqslant d(x_1, y) + d(y, x_2) < 2\varepsilon$$

and

$$\|L_{x_1 y}(\zeta_1) - L_{x_2 y}(\zeta_2)\|_y \leqslant \|L_{x_1 y}(\zeta_1) - L_{x_2 y}(\zeta_2) - \zeta\|_y + \|\zeta\|_y < \varepsilon + \frac{\varepsilon}{2}.$$

Finally, if we let go $\varepsilon \to 0$ and we bear in mind that F is intrinsically uniformly continuous, we obtain that the number A, which depends on ε, should go to a number less than or equal to 0, a contradiction with the fact that $A > \varepsilon_0$ for all $\varepsilon \in (0, \varepsilon_0)$. □

In the case when $u_0 = v_0$ one can immediately deduce the following uniqueness result.

Corollary 11. *Let M be a complete Riemannian manifold with strictly positive convexity and injectivity radii, let $u_0 : M \to \mathbb{R}$ be a bounded, uniformly continuous function, and let $F : [0,\infty) \times T^*M \to \mathbb{R}$ be an intrinsically uniformly continuous function. Assume that v and w are bounded viscosity solutions of*

$$\begin{cases} u_t + F(t, d_x u) = 0, \\ u(0,x) = u_0(x). \end{cases} \tag{$*$}$$

Then $v = w$.

Remark 12. It is worth noting that Theorem 10 also immediately yields a result on continuous dependence on the initial data of the viscosity solutions of $(*)$. Namely, if (u_0^n) is a sequence of bounded, uniformly continuous functions such that $\lim_{n\to\infty} u_0^n = u_0$ uniformly on M, and $u^n : [0,\infty) \times M \to \mathbb{R}$ is a bounded viscosity solution of

$$\begin{cases} u_t + F(t, d_x u) = 0, \\ u(0,x) = u_0^n(x), \end{cases}$$

and $u : [0,\infty) \times M \to \mathbb{R}$ is a bounded viscosity solution of

$$\begin{cases} u_t + F(t, d_x u) = 0, \\ u(0,x) = u_0(x), \end{cases}$$

then $\lim_{n\to\infty} u^n = u$ uniformly on $[0,\infty) \times M$.

Remark 13. It should be noted that the method of the proof of Theorem 10 can be used in more situations than just those which are contemplated in its statement. For instance, the assumption that the Hamiltonian F is intrinsically uniformly continuous might seem rather restrictive, as in principle it does not cover the case of equations of the form

$$u_t + \alpha(x)\|d_x u\|_x = 0, \qquad u(0,x) = u_0(x) \tag{$**$}$$

which are a very important class of Hamilton–Jacobi equations. Nevertheless, the same argument as in the proof of Theorem 10 allows us to conclude that the comparison principle still holds at least within the class of Lipschitz functions for a lot of Riemannian manifolds M and functions α. More precisely, if M is a Riemannian manifold with strictly positive convexity and injectivity radii and such that the parallel transport in M is uniformly continuous (these assumptions are met, for instance, in the cases when M is compact, or $M = \mathbb{R}^n$, or M is the Hilbert space), if $\alpha : M \to \mathbb{R}$ is uniformly continuous and bounded, if $u_0, v_0 : M \to \mathbb{R}$ are two bounded Lipschitz functions, and if we suppose that u is a bounded and Lipschitz viscosity subsolution of

$$\begin{cases} u_t + \alpha(x)\|d_x u\|_x = 0, \\ u(0,x) = u_0(x), \end{cases}$$

and that v is a bounded and Lipschitz viscosity supersolution of

$$\begin{cases} v_t + \alpha(x)\|d_x v\|_x = 0, \\ v(0,x) = v_0(x), \end{cases}$$

then $\sup_{[0,\infty)\times M}(u - v) \leqslant \sup_M(u_0 - v_0)$. The proof goes exactly as in Theorem 10, taking into account that the subdifferentials and superdifferentials ζ_i are uniformly bounded (by the

Lipschitz constants of u and v) and thus we can replace the use of the uniform continuity of the Hamiltonian F with an easy estimation and the uniform continuity of α. Let us provide that estimation in the simpler case when M is $\mathbb{R}^n$ or the Hilbert space, just to get rid of parallel transports. In this case Eq. (2) in the proof of Theorem 10 reads

$$A - \varepsilon + \alpha(x_1)\|\zeta_1\| - \alpha(x_2)\|\zeta_2\| \leqslant 0,$$

and, if C is a common bound for α and the Lipschitz constants of u and v, we can estimate

$$\begin{aligned}\alpha(x_1)\|\zeta_1\| - \alpha(x_2)\|\zeta_2\| &\leqslant \alpha(x_1)\big|\|\zeta_1\| - \|\zeta_2\|\big| + \big|\alpha(x_1) - \alpha(x_2)\big|\|\zeta_2\| \\ &\leqslant C\big|\|\zeta_1\| - \|\zeta_2\|\big| + C\big|\alpha(x_1) - \alpha(x_2)\big| \\ &\leqslant C\frac{3}{2}\varepsilon + C\big|\alpha(x_1) - \alpha(x_2)\big|,\end{aligned}$$

hence, by letting ε go to 0 and using uniform continuity of α, we reach the same contradiction as in the proof of Theorem 10.

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