WHITNEY EXTENSION THEOREMS FOR CONVEX FUNCTIONS OF THE CLASSES C^1 AND $C^{1,\omega}$.

DANIEL AZAGRA AND CARLOS MUDARRA

Abstract. Let C be a subset of $\mathbb{R}^n$ (not necessarily convex), $f : C \to \mathbb{R}$ be a function, and $G : C \to \mathbb{R}^n$ be a uniformly continuous function, with modulus of continuity ω. We provide a necessary and sufficient condition on f, G for the existence of a *convex* function $F \in C^{1,\omega}(\mathbb{R}^n)$ such that $F = f$ on C and $\nabla F = G$ on C, with a good control of the modulus of continuity of ∇F in terms of that of G. On the other hand, assuming that C is compact, we also solve a similar problem for the class of C^1 convex functions on $\mathbb{R}^n$, with a good control of the Lipschitz constants of the extensions (namely, $\text{Lip}(F) \lesssim \|G\|_\infty$). Finally, we give a geometrical application concerning interpolation of compact subsets K of $\mathbb{R}^n$ by boundaries of C^1 or $C^{1,1}$ convex bodies with prescribed outer normals on K.

1. Introduction and main results

Throughout this paper, by a modulus of continuity ω we understand a concave, strictly increasing function $\omega : [0,\infty) \to [0,\infty)$ such that $\omega(0^+) = 0$. In particular, ω has an inverse $\omega^{-1} : [0,\beta) \to [0,\infty)$ which is convex and strictly increasing, where $\beta > 0$ may be finite or infinite (according to whether ω is bounded or unbounded). Furthermore, ω is subadditive, and satisfies $\omega(\lambda t) \leq \lambda\omega(t)$ for $\lambda \geq 1$, and $\omega(\mu t) \geq \mu\omega(t)$ for $0 \leq \mu \leq 1$. It is well-known that for every uniformly continuous function $f : X \to Y$ between two metric spaces there exists a modulus of continuity ω such that $d_Y(f(x), f(z)) \leq \omega\left(d_X(x,z)\right)$ for every $x, z \in X$. Slightly abusing terminology, we will say that a mapping $G : X \to Y$ has modulus of continuity ω (or that G is ω-continuous) if there exists $M \geq 0$ such that

$$d_Y\left(G(x), G(y)\right) \leq M\omega\left(d_X(x,y)\right)$$

for all $x, y \in X$.

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Let C be a subset of $\mathbb{R}^n$, $f : C \to \mathbb{R}$ a function, and $G : C \to \mathbb{R}^n$ a uniformly continuous mapping with modulus of continuity ω. Assume also that the pair (f, G) satisfies

$$|f(x) - f(y) - \langle G(y), x - y\rangle| \leq M|x - y|\, \omega\left(|x - y|\right) \qquad (W^{1,\omega})$$

for every $x, y \in C$ (in the case that $\omega(t) = t$ we will also denote this condition by $(W^{1,1})$). Then a well-known version of the Whitney extension theorem for the class $C^{1,\omega}$ due to Glaeser holds true (see [12, 19]), and we get the existence of a function $F \in C^{1,\omega}(\mathbb{R}^n)$ such that $F = f$ and $\nabla F = G$ on C.

It is natural to ask what further assumptions on f, G would be necessary and sufficient to ensure that F can be taken to be convex. In a recent paper [3], we solved a similar problem for the class of C^∞, under the much more stringent assumptions that C be convex and compact. We refer to the introduction of [3] and the references therein for background; see in particular [8, 4] for an account of the spectacular progress made on Whitney extension problems in the last decade. Here we will only mention that results of this nature for the special class of convex functions have interesting applications in differential geometry, PDE theory (such as Monge-Ampère equations), nonlinear dynamics, and quantum computing, see [9, 10, 11, 22] and the references therein. We should also note here that, in contrast with the classical Whitney extension theorem [21] (concerning jets) and with the solutions [12, 5, 6, 7] to the Whitney extension problem (concerning functions), which are both of a local character, the nature of our problem is global. Indeed, consider the following example: take any four numbers $a, b, c, d \in \mathbb{R}$ with $a < b < 0 < c < d$, and define $C = \{a, b, 0, c, d\}$ and $f(x) = |x|$ for $x \in C$. Since C is a five-point set it is clear that there are infinitely many C^1 functions (even infinitely many polynomials) F with $F = f$ on C. However, none of these F can be convex on $\mathbb{R}$, because, as is easily checked, any convex extension g of f to $\mathbb{R}$ must satisfy $g(x) = |x|$ for every $x \in [a, d]$, and therefore g cannot be differentiable at 0.

Let us introduce one global condition which, we have found, is necessary and sufficient for a function $f : C \to \mathbb{R}$ (and a mapping $G : C \to \mathbb{R}^n$ with modulus of continuity ω) to have a convex extension F of class $C^{1,\omega}(\mathbb{R}^n)$ such that $\nabla F = G$ on C. For a mapping $G : C \subset \mathbb{R}^n \to \mathbb{R}^n$ we will denote

$$M(G, C) := \sup_{x,y \in C,\, x \neq y} \frac{|G(x) - G(y)|}{\omega(|x - y|)}. \qquad (1.1)$$

We will say that f and G satisfy the property $(CW^{1,\omega})$ on C if either G is constant, or else $0 < M(G, C) < \infty$ and there exists a constant $\eta \in (0, 1/2]$ such that

$$f(x) - f(y) - \langle G(y), x - y\rangle \geq \eta\, |G(x) - G(y)|\, \omega^{-1}\left(\frac{1}{2M}|G(x) - G(y)|\right)$$

$$\text{for all } x, y \in C, \text{ where } M = M(G, C). \qquad (CW^{1,\omega})$$

Throughout this paper we will assume that $0 < M < \infty$, or equivalently that G is nonconstant and has modulus of continuity ω. We may of course do so, because if $M = 0$ then our problem has a trivial solution (namely, the function $x \mapsto f(x_0) + \langle G(x_0), x - x_0 \rangle$ defines an affine extension of f to $\mathbb{R}^n$, for any $x_0 \in C$).

In the case that $\omega(t) = t$, we will also denote this condition by $(CW^{1,1})$. That is, (f, G) satisfies $(CW^{1,1})$ if and only if there exists $\delta > 0$ such that

$$f(x) - f(y) - \langle G(y), x - y \rangle \geq \delta \, |G(x) - G(y)|^2, \qquad (CW^{1,1})$$

for all $x, y \in C$.

Remark 1.1. *If (f, G) satisfies condition $(CW^{1,\omega})$ and G has modulus of continuity ω, then (f, G) satisfies condition $(W^{1,\omega})$.*

Proof. Condition $(CW^{1,\omega})$ implies that

$$0 \leq f(x) - f(y) - \langle G(y), x - y \rangle \leq \langle G(y) - G(x), y - x \rangle,$$

for all $x, y \in C$ with $x \neq y$, hence,

$$0 \leq \frac{f(x) - f(y) - \langle G(y), x - y \rangle}{|x - y| \omega(|x - y|)} \leq \frac{\langle G(y) - G(x), y - x \rangle}{|x - y| \omega(|x - y|)} \leq M.$$

□

It is also easy to show (without even assuming that G is continuous) that if the pair (f, G) satisfies $(CW^{1,\omega})$ for some M and with $\eta = 1/2$ then G is ω-continuous and

$$\sup_{x, y \in C} \frac{|G(x) - G(y)|}{\omega(|x - y|)} \leq 2M.$$

The first of our main results is the following.

Theorem 1.2. *Let ω be a modulus of continuity. Let C be a (not necessarily convex) subset of $\mathbb{R}^n$. Let $f : C \to \mathbb{R}$ be an arbitrary function, and $G : C \to \mathbb{R}^n$ be continuous, with modulus of continuity ω. Then f has a convex, $C^{1,\omega}$ extension F to all of $\mathbb{R}^n$, with $\nabla F = G$ on C, if and only if (f, G) satisfies $(CW^{1,\omega})$ on C.*

In particular, for the most important case that $\omega(t) = t$, we have the following.

Corollary 1.3. *Let C be a (not necessarily convex) subset of $\mathbb{R}^n$. Let $f : C \to \mathbb{R}$ be an arbitrary function, and $G : C \to \mathbb{R}^n$ be a Lipschitz function. Then f has a convex, $C^{1,1}$ extension F to all of $\mathbb{R}^n$, with $\nabla F = G$ on C, if and only if (f, G) satisfies $(CW^{1,1})$ on C.*

It is worth noting that our proofs provide good control of the modulus of continuity of the gradients of the extensions F, in terms of that of G. In fact, assuming $\eta = 1/2$ in $(CW^{1,\omega})$, which we can fairly do (see Proposition 3.2

below), there exists a constant $k(n)>0$, depending only on the dimension n, such that

$$M(\nabla F, \mathbb{R}^n) := \sup_{x,y\in\mathbb{R}^n,\, x\neq y} \frac{|\nabla F(x)-\nabla F(y)|}{\omega(|x-y|)} \leq k(n)\, M(G,C). \tag{1.2}$$

Because convex functions on $\mathbb{R}^n$ are not bounded (unless they are constant), the most usual definitions of norms in the space $C^{1,\omega}(\mathbb{R}^n)$ are not suited to estimate convex functions. In this paper, for a differentiable function $F:\mathbb{R}^n\to\mathbb{R}$ we will denote

$$\|F\|_{1,\omega} = |F(0)| + |\nabla F(0)| + \sup_{x,y\in\mathbb{R}^n,\, x\neq y} \frac{|\nabla F(x)-\nabla F(y)|}{\omega(|x-y|)}. \tag{1.3}$$

With this notation, and assuming $0\in C$ and $\eta=1/2$ in ($CW^{1,\omega}$), equation (1.2) implies that

$$\|F\|_{1,\omega} \leq k(n)\left(|f(0)|+|G(0)|+M(G,C)\right). \tag{1.4}$$

In particular, the norm of the extension F of f that we construct is nearly optimal, in the sense that

$$\|F\|_{1,\omega} \leq k(n)\, \inf\{\|\varphi\|_{1,\omega} : \varphi\in C^{1,\omega}(\mathbb{R}^n), \varphi_{|_C}=f, (\nabla\varphi)_{|_C}=G\} \tag{1.5}$$

for a constant $k(n)\geq 1$ only depending on n.

Let us now consider a similar extension problem for the class of C^1 convex functions: given a continuous mapping $G:C\to\mathbb{R}^n$ and a function $f:C\to\mathbb{R}$, how can we decide whether there is a convex function $F\in C^1(\mathbb{R}^n)$ such that $F_{|_C}=f$ and $(\nabla F)_{|_C}=G$? There is evidence suggesting that, if C is not assumed to be compact or G is not uniformly continuous, this problem does not have a solution which is simple enough to use; see [18, Example 4], [20, Example 3.2], and [3, Example 4.1]. These examples show in particular that there exists a closed convex body V in $\mathbb{R}^2$ and a C^∞ function $f:\mathbb{R}^2\to\mathbb{R}$ so that f is convex on an open convex neighborhood of V and yet there is no convex function $F:\mathbb{R}^2\to\mathbb{R}$ such that $F=f$ on V. However, we will show that there cannot be any such examples with V compact (see Theorem 1.4 below).

Since for a function $\varphi\in C^1(\mathbb{R}^n)$ and a compact set $C\subset\mathbb{R}^n$ there always exists a modulus of continuity for the restriction $(\nabla\varphi)_{|_C}$, Theorem 1.2 also provides a solution to our C^1 convex extension problem when C is compact. However, given such a 1-jet (f,G) on a compact set C, unless $\omega(t)=t$ or one has a clue about what ω might do the job, in practice it may be difficult to find a modulus of continuity ω such that (f,G) satisfies ($CW^{1,\omega}$), and for this reason it is also desirable to have a criterion for C^1 convex extendibility which does not involve dealing with moduli of continuity. We next study this question.

Given a 1-jet (f,G) on C (where $f:C\to\mathbb{R}$ is a function and $G:C\to\mathbb{R}^n$ is a continuous mapping), a necessary condition for the existence of a convex function $F \in C^1(\mathbb{R}^n)$ with $F_{|_C} = f$ and $(\nabla F)_{|_C} = G$ is given by

$$\lim_{|z-y|\to 0^+} \frac{f(z) - f(y) - \langle G(y), z-y\rangle}{|z-y|} = 0 \text{ uniformly on } C, \qquad (W^1)$$

which is equivalent to Whitney's classical condition for C^1 extendibility. If a 1-jet (f, G) satisfies condition (W^1), Whitney's extension theorem [21] provides us with a function $F \in C^1(\mathbb{R}^n)$ such that $F_{|_C} = f$ and $(\nabla F)_{|_C} = G$. In the special case that C is a convex body, if $f : C \to \mathbb{R}$ is convex and (f, G) satisfies (W^1), we will see that, without any further assumptions on (f, G), f *always* has a convex C^1 extension to all of $\mathbb{R}^n$ with $(\nabla F)_{|_C} = G$.

Theorem 1.4. *Let C be a compact convex subset of $\mathbb{R}^n$ with non-empty interior. Let $f : C \to \mathbb{R}$ be a convex function, and $G : C \to \mathbb{R}^n$ be a continuous mapping satisfying Whitney's extension condition (W^1) on C. Then there exists a convex function $F \in C^1(\mathbb{R}^n)$ such that $F_{|_C} = f$ and $(\nabla F)_{|_C} = G$.*

If C and f are convex but $\text{int}(C)$ is empty then, in order to obtain differentiable convex extensions of f to all of $\mathbb{R}^n$ we will show that it is enough to complement (W^1) with the following global geometrical condition:

$$f(x) - f(y) = \langle G(y), x-y\rangle \implies G(x) = G(y), \text{ for all } x, y \in C. \quad (CW^1)$$

Theorem 1.5. *Let C be a compact convex subset of $\mathbb{R}^n$. Let $f : C \to \mathbb{R}$ be a convex function, and $G : C \to \mathbb{R}^n$ be a continuous mapping. Then f has a convex, C^1 extension F to all of $\mathbb{R}^n$, with $\nabla F = G$ on C, if and only if f and G satisfy (W^1) and (CW^1) on C.*

In the general case of a non-convex compact set C, we will just have to add another global geometrical condition to (CW^1):

$$f(x) - f(y) \geq \langle G(y), x-y\rangle \text{ for all } x, y \in C. \qquad (C)$$

Remark 1.6. *If (f, G) satisfies condition (C) and G is continuous, then (f, G) satisfies Whitney's condition (W^1).*

This is easily shown by an obvious modification of the proof of Remark 1.1.

Theorem 1.7. *Let C be a compact (not necessarily convex) subset of $\mathbb{R}^n$. Let $f : C \to \mathbb{R}$ be an arbitrary function, and $G : C \to \mathbb{R}^n$ be a continuous mapping. Then f has a convex, C^1 extension F to all of $\mathbb{R}^n$, with $\nabla F = G$ on C, if and only if (f, G) satisfies the conditions (C) and (CW^1) on C.*

Similarly to the $C^{1,\omega}$ case, we will see that the proof of the above result provides good control of the Lipschitz constant of the extension F in terms of $\|G\|_\infty$. Namely, we will see that

$$\sup_{x\in\mathbb{R}^n} |\nabla F(x)| \leq k(n) \sup_{y\in C} |G(y)| \qquad (1.6)$$

for a constant $k(n) \geq 1$ only depending on n. Interestingly, this kind of control of $\text{Lip}(F)$ in terms of $\|G\|_\infty$ cannot be obtained, in general, for jets

(f, G) not satisfying (C), as is easily seen by examples, and the proof of Whitney's extension theorem only permits to obtain extensions $(F, \nabla F)$ (of jets (f, G) on C) which satisfy estimations of the type

$$\sup_{x\in\mathbb{R}^n} |\nabla F(x)| \leq k(n) \left(\sup_{z,y\in C, z\neq y} \frac{|f(z)-f(y)|}{|z-y|} + \sup_{y\in C} |G(y)| \right).$$

or of the type

$$\sup_{x\in\mathbb{R}^n} |\nabla F(x)| \leq k(n) \left(\sup_{y\in C} |f(y)| + \sup_{y\in C} |G(y)| \right).$$

It is condition (C) that allows us to get finer control of $\mathrm{Lip}(F)$ in the convex case; see the proof of Claim 2.3 below. In particular, assuming $0 \in C$ and defining

$$\|F\|_1 := |F(0)| + \sup_{x\in\mathbb{R}^n} |\nabla F(x)|, \tag{1.7}$$

we obtain an extension F of f such that

$$\|F\|_1 \leq k(n) \inf\{\|\varphi\|_1 : \varphi \in C^1(\mathbb{R}^n), \varphi_{|_C} = f, (\nabla\varphi)_{|_C} = G\}, \tag{1.8}$$

so the norm of our extension is nearly optimal in this case too.

In the particular case when C is finite, Theorem 1.7 provides necessary and sufficient conditions for interpolation of finite sets of data by C^1 convex functions.

Corollary 1.8. *Let S be a finite subset of $\mathbb{R}^n$, and $f : S \to \mathbb{R}$ be a function. Then there exists a convex function $F \in C^1(\mathbb{R}^n)$ with $F = f$ on S if and only if there exists a mapping $G : S \to \mathbb{R}^n$ such that f and G satisfy conditions (C) and (CW^1) on S.*

In [16, Theorem 14] it is proved that, for every finite set of *strictly convex data* in $\mathbb{R}^n$ there always exists a C^∞ convex function (or even a convex polynomial) that interpolates the given data. However, in the case that the data are convex but not strictly convex, the above corollary seems to be new.

Let us conclude this introduction with two geometrical applications of Corollary 1.3 and Theorem 1.7 concerning characterizations of compact subsets K of $\mathbb{R}^n$ which can be interpolated by boundaries of $C^{1,1}$ or C^1 convex bodies (with prescribed unit outer normals on K). Namely, if K is a compact subset of $\mathbb{R}^n$ and we are given an M-Lipschitz (resp. continuous) map $N : K \to \mathbb{R}^n$ such that $|N(y)| = 1$ for every $y \in K$, it is natural to ask what conditions on K and N are necessary and sufficient for K to be a subset of the boundary of a $C^{1,1}$ (resp. C^1) convex body V such that $0 \in \mathrm{int}(V)$ and $N(y)$ is outwardly normal to ∂V at y for every $y \in K$. A suitable set of conditions in the $C^{1,1}$ case is:

$$(\mathcal{O}) \qquad \langle N(y), y\rangle > 0 \text{ for all } y \in K;$$

$$(\mathcal{KW}^{1,1}) \qquad \langle N(y), y-x\rangle \geq \frac{\eta}{2M}|N(y)-N(x)|^2 \text{ for all } x,y \in K,$$

for some $\eta \in (0, \frac{1}{2}]$. Our main result in this direction is as follows.

Theorem 1.9. *Let K be a compact subset of $\mathbb{R}^n$, and let $N : K \to \mathbb{R}^n$ be an M-Lipschitz mapping such that $|N(y)| = 1$ for every $y \in K$. Then the following statements are equivalent:*

1. *There exists a $C^{1,1}$ convex body V with $0 \in int(V)$ and such that $K \subseteq \partial V$ and $N(y)$ is outwardly normal to ∂V at y for every $y \in K$.*
2. *K and N satisfy conditions $(\mathcal{O})$ and $(\mathcal{KW}^{1,1})$.*

This result may be compared to [9], where M. Ghomi showed how to construct C^m smooth strongly convex bodies V with prescribed strongly convex submanifolds and tangent planes. In the same spirit, the above Theorem allows us to deal with arbitrary compacta instead of manifolds, and to drop the strong convexity assumption, in the particular case of $C^{1,1}$ bodies. Similarly, for interpolation by C^1 bodies, the pertinent conditions are:

$$(\mathcal{O}) \qquad \langle N(y), y\rangle > 0 \text{ for all } y \in K;$$

$$(\mathcal{K}) \qquad \langle N(y), x-y\rangle \leq 0 \text{ for all } x,y \in K;$$

$$(\mathcal{KW}^1) \qquad \langle N(y), x-y\rangle = 0 \implies N(x) = N(y) \text{ for all } x,y \in K,$$

and our result for the class C^1 then reads as follows.

Theorem 1.10. *Let K be a compact subset of $\mathbb{R}^n$, and let $N : K \to \mathbb{R}^n$ be a continuous mapping such that $|N(y)| = 1$ for every $y \in K$. Then the following statements are equivalent:*

1. *There exists a C^1 convex body V with $0 \in int(V)$ and such that $K \subseteq \partial V$ and $N(y)$ is outwardly normal to ∂V at y for every $y \in K$.*
2. *K and N satisfy conditions $(\mathcal{O})$, $(\mathcal{K})$, and $(\mathcal{KW}^1)$.*

The rest of this paper is devoted to the proofs of the above results. Most of the main ideas in the proofs of Theorem 1.2 and 1.7 are similar, but the case $C^{1,\omega}$ is considerably more technical, so, in order to convey these ideas more easily, we will begin by proving Theorems 1.4, 1.5, 1.7 and 1.10 in Section 2. The proofs of Theorems 1.2 and 1.9 will be provided in Section 3.

2. Proofs of the C^1 results

Theorem 1.4 is a consequence of Theorem 1.5 and of the following result.

Lemma 2.1. *Let $f \in C^1(\mathbb{R}^n)$, $C \subset \mathbb{R}^n$ be a compact convex set with nonempty interior, $x_0, y_0 \in C$. Assume that f is convex on C and*

$$f(x_0) - f(y_0) = \langle \nabla f(y_0), x_0 - y_0 \rangle.$$

Then $\nabla f(x_0) = \nabla f(y_0)$.

Proof. **Case 1.** Suppose first that $f(x_0) = f(y_0) = 0$. We may of course assume that $x_0 \neq y_0$ as well. Then we also have $\langle \nabla f(y_0), x_0 - y_0 \rangle = 0$. If we consider the C^1 function $\varphi(t) = f(y_0 + t(x_0 - y_0))$, we have that φ is convex on the interval $[0,1]$ and $\varphi'(0) = 0$, hence $0 = \varphi(0) = \min_{t\in[0,1]} \varphi(t)$, and because $\varphi(0) = \varphi(1)$ and the set of minima of a convex function on a convex set is convex, we deduce that $\varphi(t) = 0$ for all $t \in [0,1]$. This shows that f is constant on the segment $[x_0, y_0]$ and in particular we have

$$\langle \nabla f(z), z_0 - z_0' \rangle = 0 \text{ for all } z, z_0, z_0' \in [x_0, y_0].$$

Now pick a point a_0 in the interior of C and a number $r_0 > 0$ so that $B(a_0, r_0) \subset \text{int}(C)$. Since C is a compact convex body, every ray emanating from a point $a \in B(a_0, r_0)$ intersects the boundary of C at exactly one point. This implies that (even though the segment $[x_0, y_0]$ might entirely lie on the boundary ∂C), for every $a \in B(a_0, r_0)$, the interior of the triangle Δ_a with vertices x_0, a, y_0, relative to the affine plane spanned by these points, is contained in the interior of C; we will denote $\text{relint}(\Delta_a) \subset \text{int}(C)$.

Let p_0 be the unique point in $[x_0, y_0]$ such that $|a_0 - p_0| = d(a_0, [x_0, y_0])$ (the distance to the segment $[x_0, y_0]$), set $w_0 = a_0 - p_0$, and denote $v_a := a - p_0$ for each $a \in B(a_0, r_0)$. Thus for every $a \in B(a_0, r_0)$ we can write $v_a = u_a + w_0$, where $u_a := a - a_0 \in B(0, r_0)$, and in particular we have $\{v_a : a \in B(a_0, r_0)\} = B(w_0, r_0)$.

Claim 2.2. *For every z_0, z_0' in the relative interior of the segment $[x_0, y_0]$, we have $\nabla f(z_0) = \nabla f(z_0')$.*

Let us prove our claim. It is enough to show that $\langle \nabla f(z_0) - \nabla f(z_0'), v_a \rangle = 0$ for every $a \in B(a_0, r_0)$ (because if a linear form vanishes on a set with nonempty interior, such as $B(w_0, r_0)$, then it vanishes everywhere). So take $a \in B(a_0, r_0)$. Since z_0 and z_0' are in the relative interior of the segment $[x_0, y_0]$ and $\text{relint}(\Delta_a) \subset \text{int}(C)$, there exists $t_0 > 0$ such that $z_0 + tv_a, z_0' + tv_a \in \text{int}(C)$ for every $t \in (0, t_0]$.

If we had $\langle \nabla f(z_0') - \nabla f(z_0), v_a \rangle > 0$ then, because f is convex on C and $f(z_0) = f(z_0') = 0$, $\langle \nabla f(z_0'), z_0 - z_0' \rangle = 0$, we would get

$$f(z_0 + tv_a) = f(z_0' + z_0 - z_0' + tv_a) \geq \langle \nabla f(z_0'), z_0 - z_0' + tv_a \rangle = \langle \nabla f(z_0'), tv_a \rangle,$$

hence

$$\lim_{t\to 0^+} \frac{f(z_0 + tv_a)}{t} \geq \langle \nabla f(z_0'), v_a \rangle > \langle \nabla f(z_0), v_a \rangle = \lim_{t\to 0^+} \frac{f(z_0 + tv_a)}{t},$$

a contradiction. By interchanging the roles of z_0, z_0', we see that the inequality $\langle \nabla f(z_0') - \nabla f(z_0), v_a \rangle < 0$ also leads to a contradiction. Therefore $\langle \nabla f(z_0') - \nabla f(z_0), v_a \rangle = 0$ and the Claim is proved.

Now, by using the continuity of ∇f, we easily conclude the proof of the Lemma in Case 1.

Case 2. In the general situation, let us consider the function h defined by

$$h(x) = f(x) - f(y_0) - \langle \nabla f(y_0), x - y_0 \rangle, \quad x \in \mathbb{R}^n.$$

It is clear that h is convex on C, and $h \in C^1(\mathbb{R}^n)$. We also have

$$\nabla h(x) = \nabla f(x) - \nabla f(y_0),$$

and in particular $\nabla h(y_0) = 0$. Besides, using the assumption that $f(x_0) - f(y_0) = \langle \nabla f(y_0), x_0 - y_0 \rangle$, we have $h(x_0) = 0 = h(y_0)$, and $h(x_0) - h(y_0) = \langle \nabla h(y_0), x_0 - y_0 \rangle$. Therefore we can apply Case 1 with h instead of f and we get that $\nabla h(x_0) = \nabla h(y_0) = 0$, which implies that $\nabla f(x_0) = \nabla f(y_0)$. $\square$

From the above Lemma it is clear that (CW^1) is a necessary condition for a convex function $f : C \to \mathbb{R}$ (and a mapping $G : C \to \mathbb{R}^n$) to have a convex, C^1 extension F to all of $\mathbb{R}^n$ with $\nabla F = G$ on C, and also that if the jet (f, G) satisfies (W^1) and $\text{int}(C) \neq \emptyset$ then (f, G) automatically satisfies (CW^1) on C as well. It is also obvious that Theorem 1.5 is an immediate consequence of Theorem 1.7, and that the condition (C) is also necessary in Theorem 1.7. Thus, in order to prove Theorems 1.4, 1.5, and 1.7 it will be sufficient to establish the *if* part of Theorem 1.7.

2.1. **Proof of Theorem 1.7.** Because f satisfies (C) and G is continuous, by Remark 1.6 we know that (f, G) satisfies (W^1). Then, according to Whitney's Extension Theorem, there exists $\widetilde{f} \in C^1(\mathbb{R}^n)$ such that, on C, we have $\widetilde{f} = f$ and $\nabla \widetilde{f} = G$.

Claim 2.3. *If f satisfies (C) then we can further assume that there exists a constant $k(n)$, only depending on n, such that*

$$Lip(\widetilde{f}) = \sup_{x \in \mathbb{R}^n} |\nabla \widetilde{f}(x)| \leq k(n) \sup_{y \in C} |G(y)|. \tag{2.1}$$

Proof. Let us recall the construction of the function $\tilde{f}$. Consider a Whitney's partition of unity $\{\varphi_j\}_j$ associated to the family of Whitney's cubes $\{Q_j, Q_j^*\}_j$ decomposing $\mathbb{R}^n \setminus C$ (see the following section or [19, Chapter VI] for notation). Define polynomials $P_z(x) = f(z) + \langle G(z), x - z \rangle$ for all $x \in \mathbb{R}^n$ and all $z \in C$. For every j, find a point $p_j \in C$ such that $d(C, Q_j) = d(p_j, Q_j)$. Then define the function $\tilde{f}$ by

$$\tilde{f}(x) = \begin{cases} \sum_j P_{p_j}(x)\varphi_j(x) & \text{if } x \in \mathbb{R}^n \setminus C \\ f(x) & \text{if } x \in C. \end{cases} \tag{2.2}$$

As a particular case of the proof of the Whitney extension theorem, we know that this function $\tilde{f}$ is of class $C^1(\mathbb{R}^n)$, extends f to $\mathbb{R}^n$ and satisfies $\nabla \tilde{f} = G$ on C. By the definition of $\tilde{f}$ we can write, for $x \in \mathbb{R}^n \setminus C$,

$$\nabla \tilde{f}(x) = \sum_j \nabla P_{p_j}(x)\varphi_j(x) + \sum_j P_{p_j}(x)\nabla\varphi_j(x). \tag{2.3}$$

Since $\nabla P_{p_j} = G(p_j)$ for all j, the first sum is bounded above by $\sup\{|G(y)| : y \in C\} := \|G\|_\infty$. In order to estimate the second sum, recall that $\sum_j \nabla\varphi_j = 0$ and find a point $b \in C$ such that $|b - x| = d(x, C)$. Then we write

$$\sum_j P_{p_j}(x)\nabla\varphi_j(x) = \sum_j \left(P_{p_j}(x) - P_b(x)\right)\nabla\varphi_j(x), \tag{2.4}$$

and observe that

$$\begin{aligned} P_{p_j}(x) - P_b(x) &= f(p_j) + \langle G(p_j), x - p_j\rangle - f(b) - \langle G(b), x - b\rangle \\ &= f(p_j) - f(b) - \langle G(b), p_j - b\rangle + \langle G(p_j) - G(b), x - p_j\rangle. \end{aligned}$$

By the same argument used in Remark 1.6 involving condition (C), we have that

$$\begin{aligned} &0 \le f(p_j) - f(b) - \langle G(b), p_j - b\rangle \\ &\le \langle G(b) - G(p_j), b - p_j\rangle \le 2\|G\|_\infty |b - p_j|. \end{aligned}$$

On the other hand,

$$|\langle G(p_j) - G(b), x - p_j\rangle| \le 2\|G\|_\infty |x - p_j|.$$

These inequalities lead us to

$$|P_{p_j}(x) - P_b(x)| \le 2\|G\|_\infty (|b - p_j| + |x - p_j|). \tag{2.5}$$

For those integers j such that $x \in Q_j^*$, the results exposed in [19, Chapter VI] show that $|b - p_j| \le 8|x - p_j|$, and that $|x - p_j|$ is of the same order as $\operatorname{diam}(Q_j)$ (with constants not even depending on n). Hence

$$|P_{p_j}(x) - P_b(x)| \lesssim \|G\|_\infty \operatorname{diam}(Q_j) \quad \text{if} \quad x \in Q_j^*$$

(by $A \lesssim B$ we mean that $A \le KB$, where K is a constant only depending on the dimension n). Also, by the properties of the Whitney's partition of unity $\{\varphi_j\}_j$ we know that

$$\left|\nabla\varphi_j(x)\right| \lesssim \operatorname{diam}(Q_j)^{-1},$$

and because all these sums has at most $N = (12)^n$ nonzero terms, we obtain

$$\sum_{Q_j^* \ni x} |P_{p_j}(x) - P_b(x)|\left|\nabla\varphi_j(x)\right| \lesssim \|G\|_\infty \sum_{Q_j^* \ni x} \operatorname{diam}(Q_j)\operatorname{diam}(Q_j)^{-1} \lesssim \|G\|_\infty,$$

which together with (2.4) allows us to control the second sum in (2.3) as required. □

Thus we may and do assume in what follows, for simplicity of notation, that f is of class $C^1(\mathbb{R}^n)$, with $\nabla f = G$ on C, and that f satisfies conditions (C) and (CW^1) on C. Occasionally, if the distinction between $\widetilde{f}$ and f matters (e.g. in the estimations of Lipschitz constants involving $\widetilde{f}$), we will nevertheless write $\widetilde{f}$ instead of f in order to prevent any misinterpretation. Let us consider the function $m(f) : \mathbb{R}^n \to \mathbb{R}$ defined by

$$m(f)(x) = \sup_{y \in C}\{f(y) + \langle \nabla f(y), x - y\rangle\}. \tag{2.6}$$

Since C is compact and the function $y \mapsto f(y)+\langle \nabla f(y), x-y\rangle$ is continuous, it is obvious that $m(f)(x)$ is well defined, and in fact the sup is attained, for every $x \in \mathbb{R}^n$. Furthermore, if we set

$$K := \max_{y\in C} |\nabla f(y)| = \max_{y\in C} |G(y)| \tag{2.7}$$

then each affine function $x \mapsto f(y) + \langle \nabla f(y), x-y\rangle$ is K-Lipschitz, and therefore $m(f)$, being a sup of a family of convex and K-Lipschitz functions, is convex and K-Lipschitz on $\mathbb{R}^n$. Note also that

$$K \leq \mathrm{Lip}(\widetilde{f}). \tag{2.8}$$

Moreover, we have

$$m(f) = f \text{ on } C. \tag{2.9}$$

Indeed, if $x \in C$ then, because f satisfies (C) on C, we have $f(x) \geq f(y) + \langle \nabla f(y), x-y\rangle$ for every $y \in C$, hence $m(f)(x) \leq f(x)$. On the other hand, we also have $f(x) \leq m(f)(x)$ because of the definition of $m(f)(x)$ and the fact that $x \in C$.

(In the case when C is convex and has nonempty interior, it is easy to see that if $h : \mathbb{R}^n \to \mathbb{R}$ is convex and $h = f$ on C, then $m(f) \leq h$. Thus, in this case, $m(f)$ is the minimal convex extension of f to all of $\mathbb{R}^n$, which accounts for our choice of notation. However, if C is convex but has empty interior then there is no minimal convex extension operator. We refer the interested reader to [18] for necessary and sufficient conditions for $m(f)$ to be finite everywhere, in the situation when $f : C \to \mathbb{R}$ is convex but not necessarily everywhere differentiable.)

If the function $m(f)$ were differentiable on $\mathbb{R}^n$, there would be nothing else to say. Unfortunately, it is not difficult to construct examples showing that $m(f)$ need not be differentiable outside C (even when C is convex and f satisfies (CW^1), see Example 2.9 at the end of this section). Nevertheless, a crucial step in our proof is the following fact: $m(f)$ is differentiable on C, provided that f satisfies conditions (C) and (CW^1) on C.

Lemma 2.4. *Let $f \in C^1(\mathbb{R}^n)$, let C be a compact subset of $\mathbb{R}^n$ (not necessarily convex), and assume that f satisfies (C) and (CW^1) on C. Then, for each $x_0 \in C$, the function $m(f)$ is differentiable at x_0, with $\nabla m(f)(x_0) = \nabla f(x_0)$.*

Proof. Notice that, by definition of $m(f)$ we have, for every $x \in \mathbb{R}^n$,

$$\langle \nabla f(x_0), x-x_0\rangle + m(f)(x_0) = \langle \nabla f(x_0), x-x_0\rangle + f(x_0) \leq m(f)(x).$$

Since $m(f)$ is convex, this means that $\nabla f(x_0)$ belongs to $\partial m(f)(x_0)$ (the subdifferential of $m(f)$ at x_0). If $m(f)$ were not differentiable at x_0 then there would exist a number $\varepsilon > 0$ and a sequence (h_k) converging to 0 in $\mathbb{R}^n$ such that

$$\frac{m(f)(x_0+h_k) - m(f)(x_0) - \langle \nabla f(x_0), h_k\rangle}{|h_k|} \geq \varepsilon \quad \text{for every } k \in \mathbb{N}. \tag{2.10}$$

Because the sup defining $m(f)(x_0+h_k)$ is attained, we obtain a sequence $(y_k)\subset C$ such that

$$m(f)(x_0+h_k)=f(y_k)+\langle\nabla f(y_k),x_0+h_k-y_k\rangle,$$

and by compactness of C we may assume, up to passing to a subsequence, that (y_k) converges to some point $y_0\in C$. Because $f=m(f)$ on C, and by continuity of f, ∇f, and $m(f)$ we then have

$$f(x_0)=m(f)(x_0)=\lim_{k\to\infty}m(f)(x_0+h_k)=$$
$$\lim_{k\to\infty}\left(f(y_k)+\langle\nabla f(y_k),x_0+h_k-y_k\rangle\right)=f(y_0)+\langle\nabla f(y_0),x_0-y_0\rangle,$$

that is, $f(x_0)-f(y_0)=\langle\nabla f(y_0),x_0-y_0\rangle$. Since $x_0,y_0\in C$ and f satisfies (CW^1), this implies that $\nabla f(x_0)=\nabla f(y_0)$. And because $m(f)(x_0)\geq f(y_k)+\langle\nabla f(y_k),x_0-y_k\rangle$ by definition of $m(f)$, we then have

$$\frac{m(f)(x_0+h_k)-m(f)(x_0)-\langle\nabla f(x_0),h_k\rangle}{|h_k|}\leq$$
$$\frac{f(y_k)+\langle\nabla f(y_k),x_0+h_k-y_k\rangle-f(y_k)-\langle\nabla f(y_k),x_0-y_k\rangle-\langle\nabla f(x_0),h_k\rangle}{|h_k|}=$$
$$\frac{\langle\nabla f(y_k)-\nabla f(x_0),h_k\rangle}{|h_k|}\leq|\nabla f(y_k)-\nabla f(x_0)|=|\nabla f(y_k)-\nabla f(y_0)|,$$

from which we deduce, using the continuity of ∇f, that

$$\limsup_{k\to\infty}\frac{m(f)(x_0+h_k)-m(f)(x_0)-\langle\nabla f(x_0),h_k\rangle}{|h_k|}\leq 0,$$

in contradiction with (2.10). □

Now we proceed with the rest of the proof of Theorem 1.7. Our strategy will be to use the differentiability of $m(f)$ on ∂C in order to construct a (not necessarily convex) differentiable function g such that $g=f$ on C, $g\geq m(f)$ on $\mathbb{R}^n$, and $\lim_{|x|\to\infty}g(x)=\infty$. Then we will define F as the convex envelope of g, which will be of class $C^1(\mathbb{R}^n)$ and will coincide with f on C.

For each $\varepsilon>0$, let $\theta_\varepsilon:\mathbb{R}\to\mathbb{R}$ be defined by

$$\theta_\varepsilon(t)=\begin{cases}0 & \text{if } t\leq 0\\ t^2 & \text{if } t\leq\frac{K+\varepsilon}{2}\\ (K+\varepsilon)\left(t-\frac{K+\varepsilon}{2}\right)+\left(\frac{K+\varepsilon}{2}\right)^2 & \text{if } t>\frac{K+\varepsilon}{2}\end{cases}$$

(recall that $K=\max_{y\in C}\|\nabla f(y)\|\leq\text{Lip}(\widetilde{f})$). Observe that $\theta_\varepsilon\in C^1(\mathbb{R})$, $\text{Lip}(\theta_\varepsilon)=K+\varepsilon$. Now set

$$\Phi_\varepsilon(x)=\theta_\varepsilon\left(d(x,C)\right),$$

where $d(x,C)$ stands for the distance from x to C, notice that $\Phi_\varepsilon(x)=d(x,C)^2$ on an open neighborhood of C, and define

$$H_\varepsilon(x)=|f(x)-m(f)(x)|+2\Phi_\varepsilon(x).$$

Note that $\mathrm{Lip}(\Phi_\varepsilon) = \mathrm{Lip}(\theta_\varepsilon)$ because $d(\cdot, C)$ is 1-Lipschitz, and therefore

$$\mathrm{Lip}(H_\varepsilon) \leq \mathrm{Lip}(\widetilde{f}) + K + 2(K+\varepsilon) \leq 4\,\mathrm{Lip}(\widetilde{f}) + 2\varepsilon. \tag{2.11}$$

Claim 2.5. *H_ε is differentiable on C, with $\nabla H_\varepsilon(x_0) = 0$ for every $x_0 \in C$.*

Proof. The function $d(\cdot, C)^2$ is obviously differentiable, with a null gradient, at x_0, hence we only have to see that $|f - m(f)|$ is differentiable, with a null gradient, at x_0. Since $\nabla m(f)(x_0) = \nabla f(x_0)$ by Lemma 2.4, the Claim boils down to the following easy exercise: if two functions h_1, h_2 are differentiable at x_0, with $\nabla h_1(x_0) = \nabla h_2(x_0)$, then $|h_1 - h_2|$ is differentiable, with a null gradient, at x_0. □

Now, because Φ_ε is continuous and positive on $\mathbb{R}^n \setminus C$, using mollifiers and a partition of unity, one can construct a function $\varphi_\varepsilon \in C^\infty(\mathbb{R}^n \setminus C)$ such that

$$|\varphi_\varepsilon(x) - H_\varepsilon(x)| \leq \Phi_\varepsilon(x) \quad \text{for every } x \in \mathbb{R}^n \setminus C, \tag{2.12}$$

and

$$\mathrm{Lip}(\varphi_\varepsilon) \leq \mathrm{Lip}(H_\varepsilon) + \varepsilon \tag{2.13}$$

(see for instance [13, Proposition 2.1] for a proof in the more general setting of Riemannian manifolds, or [2] even for possibly infinite-dimensional Riemannian manifolds). Let us define $\widetilde{\varphi} = \widetilde{\varphi_\varepsilon} : \mathbb{R}^n \to \mathbb{R}$ by

$$\widetilde{\varphi} = \begin{cases} \varphi_\varepsilon(x) & \text{if } x \in \mathbb{R}^n \setminus C \\ 0 & \text{if } x \in C. \end{cases}$$

Claim 2.6. *The function $\widetilde{\varphi}$ is differentiable on $\mathbb{R}^n$, and it satisfies $\nabla \widetilde{\varphi}(x_0) = 0$ for every $x_0 \in C$.*

Proof. It is obvious that $\widetilde{\varphi}$ is differentiable on $\mathrm{int}(C) \cup (\mathbb{R}^n \setminus C)$. We also have $\nabla \widetilde{\varphi} = 0$ on $\mathrm{int}(C)$, trivially. We only have to check that $\widetilde{\varphi}$ is differentiable, with a null gradient, on ∂C. If $x_0 \in \partial C$ we have (recalling that $\Phi_\varepsilon(x) = d(x, C)^2$ on a neighborhood of C) that

$$\frac{|\widetilde{\varphi}(x) - \widetilde{\varphi}(x_0)|}{|x - x_0|} = \frac{|\widetilde{\varphi}(x)|}{|x - x_0|} \leq \frac{|H_\varepsilon(x)| + d(x, C)^2}{|x - x_0|} \to 0$$

as $|x - x_0| \to 0^+$, because both H_ε and $d(\cdot, C)^2$ vanish at x_0 and are differentiable, with null gradients, at x_0. Therefore $\widetilde{\varphi}$ is differentiable at x_0, with $\nabla \widetilde{\varphi}(x_0) = 0$. □

Note also that

$$\mathrm{Lip}(\widetilde{\varphi}) = \mathrm{Lip}(\varphi_\varepsilon) \leq \mathrm{Lip}(H_\varepsilon) + \varepsilon \leq 4\,\mathrm{Lip}(\widetilde{f}) + 3\varepsilon. \tag{2.14}$$

Next we define

$$g = g_\varepsilon := f + \widetilde{\varphi}. \tag{2.15}$$

The function g is differentiable on $\mathbb{R}^n$, and coincides with f on C. Moreover, we also have $\nabla g = \nabla f$ on C (because $\nabla\widetilde{\varphi} = 0$ on C). And, for $x \in \mathbb{R}^n \setminus C$, we have

$$g(x) \geq f(x) + H(x) - \Phi_\varepsilon(x) = f(x) + |f(x) - m(f)(x)| + \Phi_\varepsilon(x) \geq m(f)(x) + \Phi_\varepsilon(x).$$

This shows that $g \geq m(f)$. On the other hand, because $m(f)$ is K-Lipschitz, we have

$$m(f)(x) \geq m(f)(0) - K|x|,$$

and because C is bounded, say $C \subset B(0,R)$ for some $R > 0$, also

$$\begin{aligned}\Phi_\varepsilon(x) &= (K+\varepsilon)d(x,C) - \frac{(K+\varepsilon)^2}{4} \\ &\geq (K+\varepsilon)d(x,B(0,R)) - \frac{(K+\varepsilon)^2}{4} = (K+\varepsilon)\left(|x| - R - \frac{K+\varepsilon}{4}\right)\end{aligned}$$

for $|x| \geq R + \frac{K+\varepsilon}{2}$. Hence

$$g(x) \geq m(f)(x) + \Phi_\varepsilon(x) \geq m(f)(0) - K|x| + (K+\varepsilon)\left(|x| - R - \frac{K+\varepsilon}{4}\right),$$

for $|x|$ large enough, which implies

$$\lim_{|x|\to\infty} g(x) = \infty. \tag{2.16}$$

Also, notice that according to (2.14) and the definition of g, we have

$$\operatorname{Lip}(g) \leq \operatorname{Lip}(\widetilde{f}) + \operatorname{Lip}(\widetilde{\varphi}) \leq 5\operatorname{Lip}(\widetilde{f}) + 3\varepsilon. \tag{2.17}$$

Now we will use a differentiability property of the convex envelope of a function $\psi : \mathbb{R}^n \to \mathbb{R}$, defined by

$$\operatorname{conv}(\psi)(x) = \sup\{h(x) \,:\, h \text{ is convex }, h \leq \psi\}$$

(another expression for $\operatorname{conv}(\psi)$, which follows form Carathéodory's Theorem, is

$$\operatorname{conv}(\psi)(x) = \inf\left\{\sum_{j=1}^{n+1}\lambda_j\psi(x_j) \,:\, \lambda_j \geq 0, \sum_{j=1}^{n+1}\lambda_j = 1, x = \sum_{j=1}^{n+1}\lambda_j x_j\right\},$$

see [17, Corollary 17.1.5] for instance). The following result is a restatement of a particular case of the main theorem in [15]; see also [14].

Theorem 2.7 (Kirchheim-Kristensen)**.** *If $\psi : \mathbb{R}^n \to \mathbb{R}$ is differentiable and $\lim_{|x|\to\infty}\psi(x) = \infty$, then conv$(\psi) \in C^1(\mathbb{R}^n)$.*

Although not explicitly stated in that paper, the proof of [15] also shows that

$$\operatorname{Lip}(\operatorname{conv}(\psi)) \leq \operatorname{Lip}(\psi).$$

If we define $F = \text{conv}(g)$ we thus get that F is convex on $\mathbb{R}^n$ and $F \in C^1(\mathbb{R}^n)$, with

$$\text{Lip}(F) \leq \text{Lip}(g) \leq 5\text{Lip}(\widetilde{f}) + 3\varepsilon \leq 5\,k(n) \sup_{y \in C} |G(y)| + 3\varepsilon. \tag{2.18}$$

Let us now check that $F = f$ on C. Since $m(f)$ is convex on $\mathbb{R}^n$ and $m(f) \leq g$, we have that $m(f) \leq F$ on $\mathbb{R}^n$ by definition of $\text{conv}(g)$. On the other hand, since $g = f$ on C we have, for every convex function h with $h \leq g$, that $h \leq f$ on C, and therefore, for every $y \in C$,

$$F(y) = \sup\{h(y) \,:\, h \text{ is convex }, h \leq g\} \leq f(y) = m(f)(y).$$

This shows that $F(y) = f(y)$ for every $y \in C$.

Next let us see that we also have $\nabla F(y) = \nabla f(y)$ for every $y \in C$. In order to do so we use the following well known criterion for differentiability of convex functions, whose proof is straightforward and can be left to the interested reader.

Lemma 2.8. *If ϕ is convex, ψ is differentiable at y, $\phi \leq \psi$, and $\phi(y) = \psi(y)$, then ϕ is differentiable at y, with $\nabla\phi(y) = \nabla\psi(y)$.*

(This fact can also be phrased as: a convex function ϕ is differentiable at y if and only if ϕ is superdifferentiable at y.)

Since we know that $m(f) \leq F$, $m(f)(y) = f(y) = F(y)$ for all $y \in C$, and $F \in C^1(\mathbb{R}^n)$, it follows from this criterion (by taking $\phi = m(f)$ and $\psi = F$), and from Lemma 2.4, that

$$G(y) = \nabla f(y) = \nabla m(f)(y) = \nabla F(y) \text{ for all } y \in C.$$

Finally, note that, equation (2.18) implies (by assuming that $\varepsilon \leq k(n)\|G\|_\infty/3$, which we may do) that

$$\text{Lip}(F) \leq 6\,k(n) \sup_{y \in C} |G(y)| \tag{2.19}$$

and also, assuming $0 \in C$, that

$$\|F\|_1 \leq 6\,k(n)\, \inf\{\|\varphi\|_1 : \varphi \in C^1(\mathbb{R}^n), \varphi_{|_C} = f, (\nabla\varphi)_{|_C} = G\}. \tag{2.20}$$

The proof of Theorem 1.7 is complete.

2.2. **Proof of Theorem 1.10.** (2) $\implies$ (1): Set $C = K \cup \{0\}$, choose a number $\alpha > 0$ sufficiently close to 1 so that

$$0 < 1 - \alpha < \min_{y \in K} \langle N(y), y \rangle$$

(this is possible thanks to condition $(\mathcal{O})$, continuity of N and compactness of K; notice in particular that $0 \notin K$), and define $f : C \to \mathbb{R}$ and $G : C \to \mathbb{R}^n$ by

$$f(y) = \begin{cases} 1 & \text{if } y \in K \\ \alpha & \text{if } y = 0, \end{cases} \quad \text{and } G(y) = \begin{cases} N(y) & \text{if } y \in K \\ 0 & \text{if } y = 0. \end{cases}$$

By using conditions $(\mathcal{K})$ and $(\mathcal{KW}^1)$, it is straightforward to check that f and G satisfy conditions (C) and (CW^1). Therefore, according to Theorem 1.7, there exists a convex function $F \in C^1(\mathbb{R}^n)$ such that $F = f$ and $\nabla F = G$ on C. Moreover, from the proof of Theorem 1.7, it is clear that F can be taken so as to satisfy $\lim_{|x|\to\infty} F(x) = \infty$. If we define $V = \{x \in \mathbb{R}^n : F(x) \leq 1\}$ we then have that V is a compact convex body with $0 \in \text{int}(V)$ (because $F(0) = \alpha < 1$), and $\nabla F(y) = N(y)$ is outwardly normal to $\{x \in \mathbb{R}^n : F(x) = 1\} = \partial V$ at each $y \in K$. Moreover, $F(y) = f(y) = 1$ for each $y \in K$, hence $K \subseteq \partial V$.

(1) $\implies$ (2): Let μ_C be the Minkowski functional of C. By composing μ_C with a C^1 convex function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(t) = |t|$ if and only if $|t| \geq 1/2$, we obtain a function $F(x) := \phi(\mu_C(x))$ which is of class C^1 and convex on $\mathbb{R}^n$ and coincides with μ_C on a neighborhood of ∂C. By Theorem 1.7 we then have that F satisfies conditions (C) and (CW^1) on ∂C. On the other hand $\nabla \mu_C(y)$ is outwardly normal to ∂C at every $y \in \partial C$ and $\nabla F = \nabla \mu_C$ on ∂C, and hence we have $\nabla F(y) = N(y)$ for every $y \in \partial C$. These facts, together with the assumption $K \subseteq \partial V$, are easily checked to imply that N and K satisfy conditions $(\mathcal{K})$ and $(\mathcal{KW}^1)$. Finally, because $0 \in \text{int}(C)$ and $\nabla \mu_C(x)$ is outwardly normal to ∂C for every $x \in \partial C$, we have that $\langle \nabla \mu_C(x), x\rangle > 0$ for every $x \in \partial C$, and in particular condition $(\mathcal{O})$ is satisfied as well. $\square$

Let us conclude this section with a couple of examples. We first observe that $m(f)$ need not be differentiable outside C, even in the case when C is a convex body and f is C^∞ on C.

Example 2.9. Let g be the function $g(x, y) = \max\{x+y-1, -x+y-1, \frac{1}{3}y\}$. Using for instance the smooth maxima introduced in [1], one can smooth away the edges of the graph of g produced by the intersection of the plane $z = \frac{1}{3}y$ with the planes $z = y \pm x - 1$, thus obtaining a smooth convex function f defined on $C := g^{-1}(-\infty, 0] \cap \{(x, y) : y \geq -1\}$. However, $m(f)$ will not be everywhere differentiable, because for $y \geq 2$ we have $m(f)(x, y) = \max\{x + y - 1, -x + y - 1\}$, and this max function is not smooth on the line $x = 0$. We leave the details to the interested reader.

The following example shows that when C has empty interior there are convex functions $f : C \to \mathbb{R}$ and continuous mappings $G : C \to \mathbb{R}^n$ which satisfy (W^1) but do not satisfy (CW^1).

Example 2.10. Let C be the segment $\{0\} \times [0, 1]$ in $\mathbb{R}^2$, and f, G be defined by $f(0, y) = 0$ and $G(0, y) = (y, 0)$. If we define $\widetilde{f}(x, y) = xy$ then it is clear that $\widetilde{f}$ is a C^1 extension of f to $\mathbb{R}^2$ which satisfies $\nabla \widetilde{f}(0, y) = G(0, y)$ for $(0, y) \in C$. Therefore the pair f, G satisfies Whitney's extension condition (W^1). However, since f is constant on the segment C and $G(0, 1) = (1, 0) \neq (0, 0) = G(0, 0)$, it is clear that the pair f, G does not satisfy (CW^1). In particular f does not have any convex C^1 extension F to $\mathbb{R}^n$ with $\nabla F = G$ on C.

3. Proofs of the $C^{1,\omega}$ results

3.1. **Necessity.** Let us prove the necessity of condition $(CW^{1,\omega})$ in Theorem 1.2. We will use the following.

Lemma 3.1. *Let $\omega : [0,\infty) \to [0,\infty)$ be a modulus of continuity, let a, b, η be real numbers with $a > 0$, $\eta \in (0, \frac{1}{2}]$, and define $h : [0,\infty) \to \mathbb{R}$ by*

$$h(s) = -as + b + \omega(s)s.$$

Assume that $b < \eta a\, \omega^{-1}(\eta a)$. Then we have

$$h\left(\omega^{-1}\left(\eta a\right)\right) < 0.$$

Proof. We can write

$$h\left(\omega^{-1}\left(\eta a\right)\right) = -a(1-2\eta)\omega^{-1}(\eta a) + b - \eta a\, \omega^{-1}(\eta a),$$

and the result follows at once. □

Proposition 3.2. *Let $f \in C^{1,\omega}(\mathbb{R}^n)$ be convex. Then*

$$f(x)-f(y)-\langle \nabla f(y), x-y\rangle \geq \frac{1}{2}|\nabla f(x)-\nabla f(y)|\, \omega^{-1}\left(\frac{1}{2M}|\nabla f(x)-\nabla f(y)|\right)$$

for all $x, y \in \mathbb{R}^n$, where

$$M = M(\nabla f, \mathbb{R}^n) = \sup_{x,y\in\mathbb{R}^n,\, x\neq y} \frac{|\nabla f(x)-\nabla f(y)|}{\omega\left(|x-y|\right)}.$$

In particular, the pair $(f, \nabla f)$ satisfies $(CW^{1,\omega})$, with $\eta = 1/2$, on every subset $C \subset \mathbb{R}^n$.

Proof. Suppose that there exist different points $x, y \in \mathbb{R}^n$ such that

$$f(x)-f(y)-\langle \nabla f(y), x-y\rangle < \frac{1}{2}|\nabla f(x)-\nabla f(y)|\, \omega^{-1}\left(\frac{1}{2M}|\nabla f(x)-\nabla f(y)|\right),$$

and we will get a contradiction.

Case 1. Assume further that $M = 1$, $f(y) = 0$, and $\nabla f(y) = 0$. By convexity this implies $f(x) \geq 0$. Then we have

$$0 \leq f(x) < \frac{1}{2}|\nabla f(x)|\, \omega^{-1}\left(\frac{1}{2}|\nabla f(x)|\right).$$

Call $a = |\nabla f(x)| > 0$, $b = f(x)$, set

$$v = -\frac{1}{|\nabla f(x)|}\nabla f(x),$$

and define

$$\varphi(t) = f(x+tv)$$

for every $t \in \mathbb{R}$. We have $\varphi(0) = b$, $\varphi'(0) = -a$, and ω is a modulus of continuity of the derivative φ'. This implies that

$$|\varphi(t) - b + at| \leq t\omega(t)$$

for every $t \in \mathbb{R}^+$, hence also that

$$\varphi(t) \leq h(t) \text{ for all } t \in \mathbb{R}^+,$$

where $h(t) = -at + b + t\omega(t)$. By assumption,

$$b < \frac{1}{2} a \omega^{-1} \left(\frac{1}{2} a \right),$$

and then Lemma 3.1 implies that

$$f\left(x + \omega^{-1}\left(\frac{1}{2} a \right) v \right) = \varphi\left(\omega^{-1}\left(\frac{1}{2} a \right) \right) \leq h\left(\omega^{-1}\left(\frac{1}{2} a \right) \right) < 0,$$

which is in contradiction with the assumptions that f is convex, $f(y) = 0$, and $\nabla f(y) = 0$. This shows that

$$f(x) \geq \frac{1}{2} |\nabla f(x)| \, \omega^{-1}\left(\frac{1}{2} |\nabla f(x)| \right).$$

Case 2. Assume only that $M = 1$. Define

$$g(z) = f(z) - f(y) - \langle \nabla f(y), z - y \rangle$$

for every $z \in \mathbb{R}^n$. Then $g(y) = 0$ and $\nabla g(y) = 0$. By Case 1, we get

$$g(x) \geq \frac{1}{2} |\nabla g(x)| \, \omega^{-1}\left(\frac{1}{2} |\nabla g(x)| \right),$$

and since $\nabla g(x) = \nabla f(x) - \nabla f(y)$ the Proposition is thus proved in the case when $M = 1$.

Case 3. In the general case, we may assume $M > 0$ (the result is trivial for $M = 0$). Consider $\psi = \frac{1}{M} f$, which satisfies the assumption of Case 2. Therefore

$$\psi(x) - \psi(y) - \langle \nabla \psi(y), x - y \rangle \geq \frac{1}{2} |\nabla \psi(x) - \nabla \psi(y)| \, \omega^{-1}\left(\frac{1}{2} |\nabla \psi(x) - \nabla \psi(y)| \right),$$

which is equivalent to the desired inequality. □

3.2. **Sufficiency.** Conversely, let us now show that condition $(CW^{1,\omega})$ is sufficient in Theorem 1.2. In the rest of this section C will be an arbitrary subset of $\mathbb{R}^n$, and for $f : C \to \mathbb{R}$ and $G : C \to \mathbb{R}^n$ satisfying $(CW^{1,\omega})$ with $\eta = 1/2$, we will denote

$$m_C(f, G)(x) = \sup_{y \in C} \{ f(y) + \langle G(y), x - y \rangle \}$$

for every $x \in \mathbb{R}^n$.

Lemma 3.3. *Under the above assumptions $m_C(f, G)(x)$ is finite for every $x \in \mathbb{R}^n$.*

Proof. Given $x \in \mathbb{R}^n$ we take a point $z \in C$ for which $|x-z| = d(x, C)$. Making use of condition $(CW)^{1,\omega}$ we obtain, for every $y \in C$, that

$$\begin{aligned} & f(z) + \langle G(z), x-z\rangle - (f(y) + \langle G(y), x-y\rangle) \\ & = f(z) - f(y) - \langle G(y), z-y\rangle + \langle G(z)-G(y), x-z\rangle \\ & \geq \frac{1}{2}|G(z)-G(y)|\omega^{-1}\left(\frac{1}{2M}|G(z)-G(y)|\right) - |G(z)-G(y)|d(z,C). \end{aligned}$$

This leads us to the inequality

$$\begin{aligned} f(y) + & \langle G(y), x-y\rangle \leq f(z) + \langle G(z), x-z\rangle \\ & + |G(z)-G(y)|\left(d(x,C) - \frac{1}{2}\omega^{-1}\left(\frac{1}{2M}|G(z)-G(y)|\right)\right). \end{aligned}$$

The first term in the last sum does not depend on y. In the case that ω is bounded, we also have that G is bounded, and this implies that the second term is also bounded by a constant only dependent on x and z. On the other hand, if ω is unbounded, then ω^{-1} is a nonconstant convex function defined on $(0, +\infty)$, hence $\lim_{t\to+\infty}\omega^{-1}(t) = +\infty$. This implies that the second term in the above sum is bounded above by a constant only dependent on x and z in either case. Because y is arbitrary on C, we have shown that $m_C(f, G)(x)$ is finite. $\square$

Lemma 3.4. *If $x \in \mathbb{R}^n$, $x_0, y \in C$ are such that*

$$f(x_0) + \langle G(x_0), x-x_0\rangle \leq f(y) + \langle G(y), x-y\rangle,$$

then $|G(y) - G(x_0)| \leq 4M\omega(|x-x_0|)$.

Proof. From the hyphotesis we easily have

$$f(x_0) - f(y) - \langle G(y), x_0-y\rangle \leq \langle G(y)-G(x_0), x-x_0\rangle \leq |G(y)-G(x_0)||x-x_0|.$$

Applying the inequality $(CW)^{1,\omega}$ to the left-side term we see that

$$\frac{1}{2}|G(y)-G(x_0)|\omega^{-1}\left(\frac{1}{2M}|G(y)-G(x_0)|\right) \leq |G(y)-G(x_0)||x-x_0|,$$

which inmediately implies

$$|G(y)-G(x_0)| \leq 2M\omega(2|x-x_0|) \leq 4M\omega(|x-x_0|).$$

$\square$

As we have already mentioned, condition $(CW)^{1,\omega}$ implies condition $(W^{1,\omega})$ and therefore we may use Glaeser's $C^{1,\omega}$ version of the Whitney Extension Theorem in order to extend f to $\mathbb{R}^n$ as a $C^{1,\omega}$ function. That is, we may assume that f is extended to a $C^{1,\omega}(\mathbb{R}^n)$ function such that $\nabla f = G$ on C. In fact, as a consequence of the proof of the $C^{1,\omega}$ version of the Whitney Extension Theorem (see [12] or [19, Chapter (VI)] for instance) the extension f can be taken so that

$$M(\nabla f, \mathbb{R}^n) := \sup_{x\neq y,\, x,y\in\mathbb{R}^n} \frac{|\nabla f(x) - \nabla f(y)|}{\omega(|x-y|)} \leq c(n)\max\{\tilde{M}, M\}.$$

where
$$\widetilde{M} = \sup_{x\neq y,\, x,y\in C} \left\{ \frac{|f(x)-f(y)-\langle G(y), x-y\rangle|}{|x-y|\omega(|x-y|)} \right\}$$
and $c(n)>0$ is a constant only depending on n. In our problem, thanks to the condition $(CW)^{1,\omega}$, we additionally know that $\widetilde{M}\leq M$ (see the proof of Remark 1.1), and therefore

$$M(\nabla f, \mathbb{R}^n) := \sup_{x\neq y,\, x,y\in \mathbb{R}^n} \frac{|\nabla f(x)-\nabla f(y)|}{\omega(|x-y|)} \leq c(n) M. \tag{3.1}$$

From now on we will denote
$$m_C(f)(x) := m_C(f, \nabla f)(x) = \sup_{y\in C}\{f(y)+\langle \nabla f(y), x-y\rangle\}, \quad x\in\mathbb{R}^n.$$
Note that according to Lemma 3.3 the function $m_C(f)$ is well defined on $\mathbb{R}^n$ and, being the supremum of a family of convex functions, is convex on $\mathbb{R}^n$ as well. We also see that, thanks to condition $(CW)^{1,\omega}$, we have $f(x)\geq f(y)+\langle \nabla f(y), x-y\rangle$ for every $x,y\in C$, and hence $m_C(f)=f$ on C.

Proposition 3.5. *For the function $m_C(f)$, the following property holds. For every $x\in\mathbb{R}^n$, $x_0\in C$,*
$$m_C(f)(x)-m_C(f)(x_0)-\langle \nabla f(x_0), x-x_0\rangle \leq 4M\omega(|x-x_0|)|x-x_0|.$$

Proof. Given $x\in\mathbb{R}^n\setminus C$ and $x_0\in C$, by definition of $m_C(f)(x)$, we can find a sequence $\{y_k\}_k\subset C$ such that $\lim_k(f(y_k)+\langle\nabla f(y_k), x-y_k\rangle)=m_C(f)(x)$. We may assume that
$$f(y_k)+\langle \nabla f(y_k), x-y_k\rangle \geq f(x_0)+\langle\nabla f(x_0), x-x_0\rangle$$
for $k\geq k_0$ large enough (indeed, if $m_C(f)(x)=f(x_0)+\langle\nabla f(x_0), x-x_0\rangle$ we may take $y_k=x_0$ for every k; otherwise we have $m_C(f)(x)>f(x_0)+\langle\nabla f(x_0), x-x_0\rangle$ and the inequality follows by definition of y_k). Hence Lemma 3.4 gives

$$|\nabla f(y_k)-\nabla f(x_0)|\leq 4M\omega(|x-x_0|) \quad \text{for} \quad k\geq k_0. \tag{3.2}$$

On the other hand,
$$\begin{aligned}
0 &\leq m_C(f)(x)-f(x_0)-\langle\nabla f(x_0), x-x_0\rangle \\
&= \lim_k \left(f(y_k)+\langle\nabla f(y_k), x-y_k\rangle - f(x_0)-\langle\nabla f(x_0), x-x_0\rangle\right) \\
&\leq \liminf_k \left(f(y_k)+\langle\nabla f(y_k), x-y_k\rangle - f(y_k)-\langle\nabla f(y_k), x_0-y_k\rangle-\langle\nabla f(x_0), x-x_0\rangle\right) \\
&= \liminf_k \langle\nabla f(y_k)-\nabla f(x_0), x-x_0\rangle \leq \liminf_k |\nabla f(y_k)-\nabla f(x_0)||x-x_0|.
\end{aligned}$$
By (3.2), the last term is smaller than or equal to $4M\omega(|x-x_0|)|x-x_0|$. We thus have proved the required inequality. □

Now we are going to use the following result, which is implicit in [1] and describes the (rather rigid) global geometrical behaviour of convex functions defined on $\mathbb{R}^n$.

Proposition 3.6. *Let $g:\mathbb{R}^n\to\mathbb{R}$ be a convex function, and assume g is not affine. Then there exist a linear function $\ell:\mathbb{R}^n\to\mathbb{R}$, a positive integer $k\leq n$, a linear subspace $X\subseteq\mathbb{R}^n$ of dimension k, and a convex function $c:X\to\mathbb{R}$ such that*

$$\lim_{|x|\to\infty} c(x)=\infty \text{ and } g=\ell+c\circ P,$$

where $P:\mathbb{R}^n\to X$ is the orthogonal projection.

Proof. We use the terminology of [1, Section 4]. If g is supported by an $(n+1)$-dimensional corner function then it is clear that $f=\ell+c$, with c convex and $\lim_{|x|\to\infty}c(x)=\infty$, so we may take $X=\mathbb{R}^n$ and P as the identity. Otherwise, by the proof of [1, Lemma 4.2] (see also the proof of [1, Proposition 1.6] on page 813 to know why [1, Lemma 4.2] also holds true for nonsmooth convex functions), there exist a positive integer $k_1<n$, a linear subspace $X_1\subseteq\mathbb{R}^n$ of dimension k_1, and a convex function $c_1:X_1\to\mathbb{R}$ such that

$$g=\ell_1+c_1\circ P_1,$$

where $P_1:\mathbb{R}^n\to X_1$ is the orthogonal projection and ℓ_1 is linear. If $\lim_{|x|\to\infty}c_1(x)=\infty$ we are done. Otherwise, we apply the same argument to the convex function $c_1:X_1\to\mathbb{R}$ in order to obtain a subspace $X_2\subset X_1$ of dimension $k_2<k_1$, an orthogonal projection $P_2:X_1\to X_2$, a convex function $c_2:X_2\to\mathbb{R}$, and a linear function $\ell_2:X_1\to\mathbb{R}$ such that $c_1=\ell_2+c_2\circ P_2$; in particular we have

$$g=(\ell_1+\ell_2\circ P_1)+c_2\circ P_2\circ P_1:=q_2+c_2\circ Q_2,$$

where q_2 is linear and Q_2 is still an orthogonal projection. Because g is not affine, by iterating this argument at most $n-1$ times we arrive at an expression $g=q+c\circ P$, where q is linear, $P:\mathbb{R}^n\to X$ is an orthogonal projection onto a subspace X of dimension at least 1, and c is convex with $\lim_{|x|\to\infty}c(x)=\infty$. □

By applying the preceding Proposition to $m_C(f)$ we may write $m_C(f)=\ell+c\circ P$, with ℓ and P as in the statement of the Proposition. Then, in the case that $k<n$, by taking coordinates with respect to an appropriate orthonormal basis of $\mathbb{R}^n$, we may assume without loss of generality that $X=\mathbb{R}^k\times\{0\}$ (which we identify with $\mathbb{R}^k$) and in particular that $P(x_1,...,x_n)=(x_1,...,x_k)$. Furthermore, since every linear function ℓ is convex, of class $C^{1,1}$, and satisfies $M(\nabla\ell,\mathbb{R}^n)=0$ and, besides,

$$m_C(f)(x)-\ell(x)=\sup_{y\in C}\{f(y)-\ell(y)+\langle\nabla f(y)-\ell,x-y\rangle\}=m_C(f-\ell,\nabla(f-\ell))(x),$$

we see that the addition or subtraction of a linear function does not affect our extension problem, and therefore we may also assume that $\ell=0$.

Hence, from now on, we assume that $m_C(f)$ is of the form

$$m_C(f)=c\circ P, \tag{3.3}$$

where $c : \mathbb{R}^k \to \mathbb{R}$ is a convex function on $\mathbb{R}^k$ with $k \leq n$, $\lim_{|x|\to\infty} c(x) = +\infty$, and $P : \mathbb{R}^n \to \mathbb{R}^k$ is defined by $P(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$ for all $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Of course, in the case that $k = n$, P is the identity map. In the case $k < n$, from (3.3) it is clear that $\frac{\partial m_C(f)}{\partial x_j}(x) = 0$ for every $j > k$ and $x \in C$. Recall that, by Proposition 3.5, the function $m_C(f)$ is differentiable at every $x \in C$, and $\nabla m_C(f)(x) = \nabla f(x)$.

Lemma 3.7. *The function c is ω-differentiable on $P(C)$ and for each $x_0 \in P(C)$ we have $\nabla c(x_0) = P(\nabla f(z_0))$ for every $z_0 \in C$ with $P(z_0) = x_0$. In fact, we have that*

$$0 \leq c(x) - c(x_0) - \langle P(\nabla f(z_0)), x - x_0 \rangle \leq 4M\omega\left(|x - x_0|\right)|x - x_0|.$$

for every $x \in \mathbb{R}^k$, $x_0 \in P(C)$, and $z_0 \in C$ with $P(z_0) = x_0$.

Proof. In the case $k = n$, our result is precisely Proposition 3.5. We now consider the case $k < n$. Fix $x \in \mathbb{R}^k$, $x_0 \in P(C)$, and $z_0 \in C$ such that $P(z_0) = x_0$. We denote $\overline{x_0} = Q(z_0)$, where $Q : \mathbb{R}^n \to \mathbb{R}^{n-k}$ is given by $Q(y_1, \ldots, y_n) = (y_{k+1}, \ldots, y_n)$ for every $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. By equation (3.3) and the fact that $z_0 = (x_0, \overline{x_0})$ we see that

$$\begin{aligned} & c(x) - c(x_0) - \langle P(\nabla f(z_0)), x - x_0 \rangle \\ & \qquad = m_C(f)(x, \overline{x_0}) - m_C(f)(z_0) - \langle \nabla f(z_0), (x, \overline{x_0}) - z_0 \rangle. \end{aligned}$$

From Proposition 3.5 we obtain that the last term is less than or equal to $4M\omega\left(|(x, \overline{x_0}) - z_0|\right)|(x, \overline{x_0}) - z_0| = 4M\omega\left(|x - x_0|\right)|x - x_0|$. □

Lemma 3.8. *The pair $(c, \nabla c)$ satisfies condition $(CW)^{1,\omega}$ on $P(C)$ with constant M. That is,*

$$c(x) - c(y) - \langle \nabla c(y), x - y \rangle \geq \frac{1}{2}|\nabla c(x) - \nabla c(y)|\omega^{-1}\left(\frac{1}{2M}|\nabla c(x) - \nabla c(y)|\right)$$

for every $x, y \in P(C)$. In particular, recalling Remark 1.1 we have

$$0 \leq c(x) - c(y) - \langle \nabla c(y), x - y \rangle \leq 2M|x - y|\omega(|x - y|) \quad \textit{and}$$

$$|\nabla c(x) - \nabla c(y)| \leq 2M\omega(|x - y|) \quad \textit{for all} \quad x, y \in P(C).$$

Proof. Given two points $x, y \in P(C)$, we find $z, w \in C$ such that $P(z) = x$ and $P(w) = y$. Proposition 3.7 shows that $\nabla c(x) = P(\nabla f(z))$ and $\nabla c(y) = P(\nabla f(w))$ and of course $c(x) = f(z)$, $c(y) = f(w)$. Recall also that, in the case $k < n$, $\nabla f(z) = (P(\nabla f(z)), 0)$ and similarly for w. This proves that

$$c(x) - c(y) - \langle \nabla c(y), x - y \rangle = f(z) - f(w) - \langle \nabla f(w), z - w \rangle \quad \text{and}$$

$$|\nabla c(x) - \nabla c(y)| = |\nabla f(z) - \nabla f(w)|.$$

Because the pair $(f, \nabla f)$ satisfies property $(CW)^{1,\omega}$ on C we have

$$f(z) - f(w) - \langle \nabla f(w), z - w \rangle \geq \frac{1}{2}|\nabla f(z) - \nabla f(w)|\omega^{-1}\left(\frac{1}{2M}|\nabla f(z) - \nabla f(w)|\right),$$

which inmediately implies

$$c(x)-c(y)-\langle \nabla c(y), x-y\rangle \geq \frac{1}{2}|\nabla c(x)-\nabla c(y)|\omega^{-1}\left(\frac{1}{2M}|\nabla c(x)-\nabla c(y)|\right).$$

□

Proposition 3.9. *The function c is differentiable on $\overline{P(C)}$ and the pair $(c, \nabla c)$ satisfies condition $(CW)^{1,\omega}$ on $\overline{P(C)}$ with constant M. In particular the pair $(c,\nabla c)$ satisfies Whitney's condition $W^{1,\omega}$ on $\overline{P(C)}$ with*

$$0 \leq c(x)-c(y)-\langle \nabla c(y), x-y\rangle \leq 2M|x-y|\omega(|x-y|) \quad \text{and}$$
$$|\nabla c(x)-\nabla c(y)| \leq 2M\omega(|x-y|) \quad \text{for all} \quad x,y\in \overline{P(C)}.$$

In addition,

(3.4)
$$0 \leq c(x)-c(y)-\langle \nabla c(y), x-y\rangle \leq 4M|x-y|\omega(|x-y|) \quad x\in\mathbb{R}^k,\ y\in\overline{P(C)}.$$

Proof. By Lemma 3.8 we have that $(c,\nabla c)$ satisfies $(CW)^{1,\omega}$ on $P(C)$ with constant M. Then a routine density argument immediately shows that ∇c has a unique ω-continuous extension H to $\overline{P(C)}$ and that the pair $(c,\nabla c)$ also satisfies $(CW)^{1,\omega}$ on $\overline{P(C)}$ with the same constant M. In particular, the following inequalities hold:

$$0 \leq c(x)-c(y)-\langle H(y), x-y\rangle \leq 2M|x-y|\omega(|x-y|) \quad \text{and}$$
$$|H(x)-H(y)| \leq 2M\omega(|x-y|) \quad \text{for all} \quad x,y\in \overline{P(C)}.$$

Now, given $x\in\mathbb{R}^k$ and $y\in\overline{P(C)}$, by Lemma 3.7 we have

$$c(x)-c(y_p)-\langle \nabla c(y_p), x-y_p\rangle \leq 4M|x-y_p|\omega(|x-y_p|) \quad p\in\mathbb{N},$$

for every sequence $\{y_p\}_p$ in $P(C)$ converging to y. Passing to the limit as $p\to\infty$ in the above inequality and bearing in mind that c and H are continuous on $\overline{P(C)}$ we obtain

$$0 \leq c(x)-c(y)-\langle H(y), x-y\rangle \leq 4M|x-y|\omega(|x-y|) \quad x\in\mathbb{R}^k,\ y\in\overline{P(C)},$$

which in particular implies that c is ω-differentiable on $\overline{P(C)}$ with $\nabla c = H$. □

Thanks to Proposition 3.9 we may apply Whitney's Extension Theorem to extend c from $\overline{P(C)}$ to a function $\widetilde{c}\in C^{1,\omega}(\mathbb{R}^k)$ such that $\nabla\widetilde{c}=\nabla c$ on $\overline{P(C)}$ and

$$M(\nabla\widetilde{c},\mathbb{R}^k) := \sup_{x\neq y,\, x,y\in\mathbb{R}^k} \frac{|\nabla\widetilde{c}(x)-\nabla\widetilde{c}(y)|}{\omega(|x-y|)} \leq \gamma(n)M,$$

for a constant $\gamma(n)>0$ depending only on n. Note that for every $x\in\mathbb{R}^k$, if we pick a point $x_0\in\overline{P(C)}$ with $d(x,\overline{P(C)})=|x-x_0|$, we obtain, by inequality (3.4) in Proposition 3.9 and the facts that $\widetilde{c}(x_0)=c(x_0)$ and $\nabla\widetilde{c}(x_0)=\nabla c(x_0)$, that

$$|c(x)-\widetilde{c}(x)| \leq (4+\gamma(n))M\omega(d(x,\overline{P(C)}))d(x,\overline{P(C)}). \tag{3.5}$$

Our next step is constructing a function φ of class $C^{1,\omega}(\mathbb{R}^k)$ which vanishes on the closed set $E := \overline{P(C)}$ and is greater than or equal to the function $|\widetilde{c}-c|$ on $\mathbb{R}^k \setminus E$. To this purpose we need to use a Whitney decomposition of an open set into cubes and the corresponding partition of unity, see [19, Chapter VI] for an exposition of this technique. So let $\{Q_j\}_j$ be a decomposition of $\mathbb{R}^k \setminus E$ into Whitney cubes, and for a fixed number $0 < \varepsilon_0 < 1/4$ (for instance take $\varepsilon_0 = 1/8$) consider the corresponding cubes $\{Q_j^*\}_j$ with the same center as Q_j and dilated by the factor $1 + \varepsilon_0$. We next sum up some of the most important properties of this cubes.

Proposition 3.10. *The families $\{Q_j\}_j$ and $\{Q_j^*\}_j$ are sequences of compact cubes for which:*
(i) $\bigcup_j Q_j = \mathbb{R}^k \setminus E$.
(ii) *The interiors of Q_j are mutually disjoint.*
(iii) $\operatorname{diam}(Q_j) \leq d(Q_j, E) \leq 4 \operatorname{diam}(Q_j)$ *for all j.*
(iv) *If two cubes Q_l and Q_j touch each other, that is $\partial Q_l \cap \partial Q_j \neq \emptyset$, then* $\operatorname{diam}(Q_l) \approx \operatorname{diam}(Q_j)$.
(v) *Every point of $\mathbb{R}^k \setminus E$ is contained in an open neighbourhood which intersects at most $N = (12)^k$ cubes of the family $\{Q_j^*\}_j$.*
(vi) *If $x \in Q_j$, then $d(x, E) \leq 5 \operatorname{diam}(Q_j)$.*
(vii) *If $x \in Q_j^*$, then $\frac{3}{4} \operatorname{diam}(Q_j) \leq d(x, E) \leq (6 + \varepsilon_0) \operatorname{diam}(Q_j)$ and in particular $Q_j^* \subset \mathbb{R}^k \setminus E$.*
(viii) *If two cubes Q_l^* and Q_j^* are not disjoint, then* $\operatorname{diam}(Q_l) \approx \operatorname{diam}(Q_j)$.

Here the notation $A_j \approx B_l$ means that there exist positive constants γ, Γ, depending only on the dimension k, such that $\gamma A_j \leq B_l \leq \Gamma A_j$ for all j, l satisfying the properties specified in each case. The following Proposition summarizes the basic properties of the Whitney partition of unity associated to these cubes.

Proposition 3.11. *There exists a sequence of functions $\{\varphi_j\}_j$ defined on $\mathbb{R}^k \setminus E$ such that*

(i) $\varphi_j \in C^\infty(\mathbb{R}^k \setminus E)$.
(ii) $0 \leq \varphi_j \leq 1$ *on $\mathbb{R}^k \setminus E$ and* $\operatorname{supp}(\varphi_j) \subseteq Q_j^*$.
(iii) $\sum_j \varphi_j = 1$ *on $\mathbb{R}^k \setminus E$.*
(vi) *For every multiindex α there exists a constant $A_\alpha > 0$, depending only on α and on the dimension k, such that*

$$|D^\alpha \varphi_j(x)| \leq A_\alpha \operatorname{diam}(Q_j)^{-|\alpha|},$$

for all $x \in \mathbb{R}^k \setminus E$ and for all j.

The statements contained in the following Proposition must look fairly obvious to those readers well acquainted with Whitney's techniques, but we have not been able to find an explicit reference, so we include a proof for the general reader's convenience.

Proposition 3.12. *Consider the family of cubes* $\{Q_j\}_j$ *asociated to* $\mathbb{R}^k \setminus E$ *and its partition of unity* $\{\varphi_j\}_j$ *as in the preceding Propositions. Suppose that there is a sequence of nonnegative numbers* $\{p_j\}_j$ *and a positive constant* $\lambda > 0$ *such that* $p_j \leq \lambda\, \omega(\mathrm{diam}(Q_j))\, \mathrm{diam}(Q_j)$*, for every* j*. Then, the function defined by*

$$\varphi(x) = \begin{cases} \sum_j p_j\, \varphi_j(x) & \text{if} \quad x \in \mathbb{R}^k \setminus E, \\ 0 & \text{if} \quad x \in E \end{cases}$$

is of class $C^{1,\omega}(\mathbb{R}^k)$ *and there is a constant* $\gamma(k) > 0$*, depending only on the dimension* k*, such that*

$$M(\nabla\varphi, \mathbb{R}^k) := \sup_{x,y\in\mathbb{R}^k,\, x\neq y} \frac{|\nabla\varphi(x) - \nabla\varphi(y)|}{\omega\left(|x-y|\right)} \leq \gamma(k)\lambda. \tag{3.6}$$

Proof. Let us start with the proof of (i). Since every point in $\mathbb{R}^k \setminus E$ has an open neighbourhood which intersects at most $N = (12)^k$ cubes, and all the functions φ_j are of class C^∞, it is clear that $\varphi \in C^\infty(\mathbb{R}^k \setminus E)$. Given a point $x \in \mathbb{R}^k$, by our assumptions on the sequence $\{p_j\}_j$ we easily have

$$\varphi(x) = \sum_{Q_j^* \ni x} p_j \varphi_j(x) \leq \lambda \sum_{Q_j^* \ni x} \omega(\mathrm{diam}(Q_j))\, \mathrm{diam}(Q_j)\varphi_j(x).$$

Using Proposition 3.10 we have that $\mathrm{diam}(Q_j) \leq \frac{4}{3} d(x,E)$ for those j such that $x \in Q_j^*$. Then, we can write

$$\varphi(x) \leq \lambda \sum_{Q_j^* \ni x} \omega\left(\frac{4}{3}d(x,E)\right)\frac{4}{3}d(x,E)\varphi_j(x) \leq$$

$$\lambda\left(\frac{4}{3}\right)^2 \omega(d(x,E))d(x,E) \sum_{Q_j^* \ni x} \varphi_j(x) = \left(\frac{4}{3}\right)^2 \lambda\omega(d(x,E))d(x,E).$$

In particular, due to the fact that $\omega(0^+) = 0$, the above estimation shows that φ is differentiable on E, with $\nabla\varphi = 0$ on C. We next give an estimation for $\nabla\varphi$. Bearing in mind Proposition 3.11, we set

$$A = A(k) := \max\{\sqrt{k}A_\alpha,\ |\alpha| \leq 2\}. \tag{3.7}$$

Given a point $x \in \mathbb{R}^k$, we have that

$$|\nabla\varphi(x)| \leq \sum_{Q_j^* \ni x} p_j |\nabla\varphi_j(x)| \leq A \sum_{Q_j^* \ni x} p_j\, \mathrm{diam}(Q_j)^{-1} \leq A\lambda \sum_{Q_j^* \ni x} \omega(\mathrm{diam}(Q_j))$$

$$\leq A\,\lambda \sum_{Q_j^* \ni x} \omega\left(\frac{4}{3}d(x,E)\right) \leq \frac{4AN}{3}\lambda\, \omega(d(x,E)).$$

Summing up,

$$|\nabla\varphi(x)| \leq \frac{4AN}{3}\lambda\, \omega(d(x,E)) \quad \text{for every} \quad x \in \mathbb{R}^k. \tag{3.8}$$

Note that the above shows inequality (3.6) when $x \in \mathbb{R}^k$ and $y \in E$. Hence, in the rest of the proof we only have consider the situation where x, y are such that $x \neq y$ and $x, y \in \mathbb{R}^k \setminus E$. However, we will still have to separately consider two cases. Let us denote $L := [x, y]$, the line segment connecting x to y.

Case 1: $d(L, E) \geq |x - y|$. Take a multi-index α with $|\alpha| = 1$. Because in this case the segment L is necessarily contained in $\mathbb{R}^k \setminus E$, and the function φ is of class C^2 on $\mathbb{R}^k \setminus E$, we can write

$$|D^\alpha \varphi(x) - D^\alpha \varphi(y)| \leq \left(\sup_{z \in L} |\nabla (D^\alpha \varphi)(z)| \right) |x - y|. \tag{3.9}$$

Using Proposition 3.11 and the definition of A in (3.7), we may write

$$|\nabla (D^\alpha \varphi)(z)| \leq \sum_{Q_j^* \ni z} p_j |\nabla (D^\alpha \varphi_j)(z)| \leq A \sum_{Q_j^* \ni z} p_j \operatorname{diam}(Q_j)^{-2} \leq A\lambda \sum_{Q_j^* \ni z} \frac{\omega(\operatorname{diam}(Q_j))}{\operatorname{diam}(Q_j)}.$$

By Proposition 3.10, we have that $(6 + \varepsilon_0) \operatorname{diam}(Q_j) \geq d(z, E) \geq d(L, E) \geq |x - y|$ for those j with $Q_j^* \ni z$, and by the properties of ω, we have that

$$A\lambda \sum_{Q_j^* \ni z} \frac{\omega(\operatorname{diam}(Q_j))}{\operatorname{diam}(Q_j)} \leq A\lambda \sum_{Q_j^* \ni z} \frac{\omega\left(\frac{|x-y|}{6+\varepsilon_0}\right)}{\frac{|x-y|}{6+\varepsilon_0}} \leq (6 + \varepsilon_0) AN\lambda \frac{\omega(|x - y|)}{|x - y|}.$$

Therefore, by substituting in (3.9) we find that

$$|\nabla \varphi(x) - \nabla \varphi(y)| \leq (6 + \varepsilon_0) \sqrt{k} AN\lambda \omega(|x - y|).$$

Case 2: $d(L, E) \leq |x - y|$. Take points $x' \in L$ and $y' \in E$ such that $d(L, E) = |x' - y'| \leq |x - y|$. We have that

$$|x - y'| \leq |x - x'| + |y' - x'| \leq |x - y| + |x' - y'| \leq 2|x - y|,$$

and similarly we obtain $|y - y'| \leq 2|x - y|$. Hence, if we use (3.8) we obtain

$$|\nabla \varphi(x) - \nabla \varphi(y)| \leq |\nabla \varphi(x)| + |\nabla \varphi(y)| \leq \frac{4AN}{3} \lambda \left(\omega(d(x, E)) + \omega(d(y, E)) \right)$$
$$\leq \frac{4AN}{3} \lambda \left(\omega(|x - y'|) + \omega(|y - y'|) \right) \leq \frac{8AN}{3} \lambda \omega(2|x - y|) \leq \frac{16AN}{3} \lambda \omega(|x - y|).$$

If we call

$$\gamma(k) = \max \left\{ \frac{4AN}{3}, (6 + \varepsilon_0) \sqrt{k} AN, \frac{16AN}{3} \right\} = (6 + \varepsilon_0) \sqrt{k} AN,$$

we get (3.6). □

Let us continue with the proof of Theorem 1.2. We consider a decomposition of $\mathbb{R}^k \setminus \overline{P(C)}$ into Whitney's cubes $\{Q_j, Q_j^*\}_j$ and its asociated partition of unity $\{\varphi_j\}_j$ (see Propositions 3.10 and 3.11). If we define

$$p_j := \sup_{x \in Q_j^*} |c(x) - \widetilde{c}(x)| \quad \text{for all} \quad \nu,$$

we have, thanks to (3.5), that $p_j \lesssim M\omega(\mathrm{diam}(Q_j))\,\mathrm{diam}(Q_j)$ for all ν. Then, according to the preceding Proposition, the function

$$\varphi(x) := \begin{cases} \sum_j p_j\, \varphi_j(x) & \text{if} \quad x \in \mathbb{R}^k \setminus \overline{P(C)}, \\ 0 & \text{if} \quad x \in \overline{P(C)} \end{cases}$$

is of class $C^{1,\omega}(\mathbb{R}^k)$ with $\varphi = 0$ and $\nabla\varphi = 0$ on $\overline{P(C)}$ and $M(\nabla\varphi, \mathbb{R}^k) \lesssim M$. In addition, by the definition of $\{p_j\}_j$ we easily see that $\varphi \geq |c - \widetilde{c}|$ on $\mathbb{R}^k$. Now we set $\psi := \widetilde{c} + \varphi$ on $\mathbb{R}^k$, which is of class $C^{1,\omega}(\mathbb{R}^k)$ with $\psi = \widetilde{c} = c$ and $\nabla\psi = \nabla\widetilde{c} = \nabla c$ on $\overline{P(C)}$, and also $M(\nabla\psi, \mathbb{R}^k) \lesssim M$. On the other hand, it is clear that $\psi \geq c$ on $\mathbb{R}^k$ and this in particular implies that $\lim_{|x|\to\infty} \psi(x) = +\infty$.

Now we will use the following.

Theorem 3.13 (Kirchheim-Kristensen)**.** *If $H : \mathbb{R}^k \to \mathbb{R}$ is an ω-differentiable function on $\mathbb{R}^k$ such that $\lim_{|x|\to+\infty} H(x) = +\infty$, then the convex envelope $conv(H)$ of H is a convex function of class $C^{1,\omega}(\mathbb{R}^k)$, and*

$$M(\nabla\, conv(H), \mathbb{R}^k) \leq 4(n+1) M(\nabla H, \mathbb{R}^k).$$

As a matter of fact, Theorem 3.13 is a restatement of a particular case of Kirchheim and Kristensen's result in [15], and the estimation on the modulus of continuity of the gradient of the convex envelope does not appear in their original statement, but it does follow, with some easy extra work, from their proof.

Since $\psi \in C^{1,\omega}(\mathbb{R}^k)$ with $M(\nabla\psi, \mathbb{R}^k) \lesssim M$ and $\lim_{|x|\to\infty} \psi(x) = +\infty$, if we define $\widetilde{F} := \mathrm{conv}(\psi)$, Theorem 3.13 implies that our function $\widetilde{F}$ is a convex function of class $C^{1,\omega}(\mathbb{R}^k)$ and satisfies $M(\nabla\widetilde{F}, \mathbb{R}^k) \lesssim M$. Since $\psi \geq c$ and c is convex, we have $\widetilde{F} \geq c$ on $\mathbb{R}^k$. On the other hand, because $\psi = c$ on $\overline{P(C)}$, we must have $\widetilde{F} \leq c$ on $\overline{P(C)}$ and therefore $\widetilde{F} = c$ on $\overline{P(C)}$. Moreover, because $c \leq \widetilde{F}$, c is convex and $c = \widetilde{F}$ on $\overline{P(C)}$, Lemma 2.8 implies that $\nabla\widetilde{F} = \nabla c$ on $\overline{P(C)}$.

Finally we define $F := \widetilde{F} \circ P$ on $\mathbb{R}^n$. We immediately see that F is convex and of class $C^{1,\omega}$ on $\mathbb{R}^n$. Thanks to equation (3.3) and the fact that $\widetilde{F} = c$ and $\nabla\widetilde{F} = \nabla c$ on $P(C)$ we have for all $x \in C$,

$$F(x) = \widetilde{F}(P(x)) = c(P(x)) = m_C(f)(x) = f(x) \quad \text{and}$$

$$\nabla F(x) = \nabla\tilde{F}(P(x)) \circ DP = \nabla c(P(x)) \circ DP = \nabla m_C(f)(x) = \nabla f(x) = G(x).$$

Since $M(\nabla\widetilde{F}, \mathbb{R}^k) \lesssim M$, for every $x, y \in \mathbb{R}^n$, the gradient of F satisfies

$$|\nabla F(x) - \nabla F(y)| = |\nabla\widetilde{F}(P(x)) - \nabla\widetilde{F}(P(y))| \lesssim M\omega(|P(x) - P(y)|) \lesssim M\omega(|x-y|).$$

This allows us to conclude the existence of a constant $\gamma(n) > 0$ depending only on n such that

$$M(\nabla F, \mathbb{R}^n) := \sup_{x \neq y,\, x,y \in \mathbb{R}^n} \frac{|\nabla F(x) - \nabla F(y)|}{\omega(|x-y|)} \leq \gamma(n) M.$$

The proof of Theorem 1.2 is complete.

3.3. **Proof of Theorem 1.9.** (2) $\implies$ (1): Set $C = K \cup \{0\}$. We may assume $\eta > 0$ small enough, and choose $\alpha > 0$ sufficiently close to 1, so that

$$0 < 1 - \alpha + \frac{\eta}{2M} < \min_{y \in K} \langle N(y), y \rangle$$

(this is possible thanks to condition $(\mathcal{O})$, M-Lipschitzness of N and compactness of K; notice in particular that $0 \notin K$). Now define $f : C \to \mathbb{R}$ and $G : C \to \mathbb{R}^n$ by

$$f(y) = \begin{cases} 1 & \text{if } y \in K \\ \alpha & \text{if } y = 0, \end{cases} \quad \text{and } G(y) = \begin{cases} N(y) & \text{if } y \in K \\ 0 & \text{if } y = 0. \end{cases}$$

By using condition $(\mathcal{KW}^{1,1})$, it is straightforward to check that (f, G) satisfies condition $(CW^{1,1})$. Therefore, according to Theorem 1.3, there exists a convex function $F \in C^{1,1}(\mathbb{R}^n)$ such that $F = f$ and $\nabla F = G$ on C, and the proof of Theorem 1.3 indicates that F can be taken so as to satisfy $\lim_{|x| \to \infty} F(x) = \infty$. We now define $V = \{x \in \mathbb{R}^n : F(x) \leq 1\}$. The rest of the proof is similar to that of Theorem 1.10, and we leave it to the reader's care. $\Box$

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ICMAT (CSIC-UAM-UC3-UCM), Departamento de Análisis Matemático, Facultad Ciencias Matemáticas, Universidad Complutense, 28040, Madrid, Spain

E-mail address: azagra@mat.ucm.es

ICMAT (CSIC-UAM-UC3-UCM), Calle Nicolás Cabrera 13-15. 28049 Madrid SPAIN

E-mail address: carlos.mudarra@icmat.es