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Riemann integrability and Lebesgue measurability of the composite function

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ABSTRACT

If f is continuous on the interval $[a,b]$, g is Riemann integrable (resp. Lebesgue measurable) on the interval $[\alpha,\beta]$ and $g([\alpha,\beta]) \subset [a,b]$, then $f \circ g$ is Riemann integrable (resp. measurable) on $[\alpha,\beta]$. A well-known fact, on the other hand, states that $f \circ g$ might not be Riemann integrable (resp. measurable) when f is Riemann integrable (resp. measurable) and g is continuous. If $\mathfrak{c}$ stands for the continuum, in this paper we construct a $2^{\mathfrak{c}}$-dimensional space V and a $\mathfrak{c}$-dimensional space W of, respectively, Riemann integrable functions and continuous functions such that, for every $f \in V \setminus \{0\}$ and $g \in W \setminus \{0\}$, $f \circ g$ is not Riemann integrable, showing that *nice* properties (such as continuity or Riemann integrability) can be lost, in a linear fashion, via the composite function. Similarly we construct a $\mathfrak{c}$-dimensional space W of continuous functions such that for every $g \in W \setminus \{0\}$ there exists a $\mathfrak{c}$-dimensional space V of measurable functions such that $f \circ g$ is not measurable for all $f \in V \setminus \{0\}$.

1. Introduction

In the last years the study of large algebraic structures enjoying certain special properties has become a trend in mathematical analysis. Many examples of functions such as continuous nowhere differentiable functions, everywhere surjective functions, differentiable nowhere monotone functions, or continuous functions with non-convergent Fourier series have been constructed in the past. Coming up with a concrete example of such a function can be difficult. In fact, it may seem so difficult that if you succeed, you may think that there cannot be too many objects of that kind. Moreover, probably one cannot find infinite dimensional vector spaces of such functions. This is, however, exactly what has happened, suggesting that some of these special properties are not, at all, isolated phenomena. A set M enjoying some special property is said to be *lineable* if $M \cup \{0\}$ contains an infinite dimensional vector space and *spaceable* if $M \cup \{0\}$ contains a closed infinite dimensional vector space. More specifically, we will say that M is μ*-lineable* if $M \cup \{0\}$ contains a vector space of dimension μ, where μ is a cardinal number. These notions of lineability and spaceability were coined by Gurariy and first introduced in [1,2,11].

Some of the earliest results on this direction come from 1966, when Gurariy proved in [10] that the set of continuous nowhere differentiable functions on the interval $[0,1]$ is lineable. He also showed that the set of everywhere differentiable functions is lineable, but not spaceable, in $\mathcal{C}[0,1]$. More recently it was shown that the set of everywhere surjective functions is $2^{\mathfrak{c}}$-lineable (where $\mathfrak{c}$ denotes the cardinality of $\mathbb{R}$) and that the set of differentiable functions on $\mathbb{R}$ which

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are nowhere monotone is lineable in $\mathcal{C}(\mathbb{R})$ (see [2]). Some of these behaviors occur in very interesting ways. For instance, in [12], Hencl showed that any separable Banach space is isometrically isomorphic to a subspace of $\mathcal{C}[0,1]$ whose non-zero elements are nowhere approximately differentiable and nowhere Hölder. Also, in [3], the authors showed that, if $E \subset \mathbb{T}$ is a set of measure zero, and if $\mathcal{F}(\mathbb{T})$ denotes the subset of $\mathcal{C}(\mathbb{T})$ of continuous functions whose Fourier series expansion diverges at every point of E, then $\mathcal{F}(\mathbb{T})$ contains an infinitely generated and dense subalgebra. We refer the interested reader to [4,5,7,14,15] for more results on this direction.

This paper is related to the loss of Riemann integrability (and Lebesgue measurability) when considering the composite function. It is well known that the composition of any two continuous functions is also continuous and, therefore, Riemann integrable. Moreover, if f is continuous on the interval $[a,b]$ and g is Riemann integrable (resp. Lebesgue measurable) on the interval $[\alpha,\beta]$ with $g([\alpha,\beta]) \subset [a,b]$, then $f \circ g$ is Riemann integrable (resp. measurable) on $[\alpha,\beta]$. On the other hand, it is not always true that $f \circ g$ is Riemann integrable (or even measurable) when f is Riemann integrable (resp. measurable) and g is continuous (see, e.g. [9,13], where examples of such f and g are given). Lineability and Riemann integrability have been, very recently, studied in [6]. Our contribution here is the explicit construction of infinite dimensional spaces V and W (with the largest possible dimensions) of, respectively, Riemann integrable functions and continuous functions such that, for every $f \in V \setminus \{0\}$ and $g \in W \setminus \{0\}$, $f \circ g$ is not Riemann integrable. We also obtain a weaker result concerning the loss of Lebesgue measurability via composition with continuous functions. To be more precise we construct a c-dimensional space W of continuous functions such that for every $g \in W \setminus \{0\}$ there exists a c-dimensional space V of measurable functions such that $f \circ g$ is not measurable for all $f \in V \setminus \{0\}$.

2. The main results

In the following we construct a 2^c-dimensional linear space V of Riemann integrable functions on $\mathbb{R}$ and a c-dimensional linear space W of continuous functions on $[0,1]$, such that for every $f \in V$ and every $g \in W$, we have that $f \circ g$ is not Riemann integrable on $[0,1]$. As mentioned in the introduction, in [13] we can find the definition of a Riemann integrable function $f_0 : [0,1] \to \mathbb{R}$ and a continuous mapping $g_0 : [0,1] \to [0,1]$ such that $f_0 \circ g_0$ is not Riemann integrable on $[0,1]$. The construction of our space W is based upon the definition of g_0, which we reproduce below for completeness in the following result (see [13] for a detailed proof).

Proposition 2.1. *There exists a continuous function $g_0 : [0,1] \to [0,1]$ such that g_0 is not identically zero on any subinterval of $[0,1]$ and the total length of all the intervals in which g_0 is not zero is $1/2$. Moreover, g_0 satisfies the following property: If $[a,b] \subset [0,1]$ and there exists $x_0 \in (a,b)$ such that $g_0(x_0) = 0$, then there exists $[a',b'] \subset (a,b)$ such that $g_0([a',b']) = [0, 1/2^n]$ for some $n \in \mathbb{N}$.*

Proof. Let $\{g_n\}_{n=1}^{\infty}$ be the sequence of continuous functions $g_n : [0,1] \to [0,1]$ defined as follows:

First, let g_1 be a continuous function on $[0,1]$ such that $g_1(x) = 0$ for all $x \in [0,1/3] \cup [2/3,1]$, $g_1(x) > 0$ for all $x \in (1/3,2/3)$ and

$$\max\{g_1(x)\colon\ x \in (1/3,2/3)\} = 1/2.$$

Once g_{n-1} is defined, we construct g_n as follows. First, divide all the intervals on which g_{n-1} is always zero into three sections such that the centre of the middle section is the center of the original interval and the length of the middle section is $3^{-n} \cdot 2^{1-n}$. Second, modify the values of g_{n-1} only on the middle sections and obtain a continuous function g_n such that in the interior of each modified interval is always greater than zero and its maximum is 2^{-n}.

Then the sequence $\{g_n\}_{n=1}^{\infty}$ converges uniformly to a continuous function $g_0 : [0,1] \to [0,1]$ such that g_0 is not identically zero on any subinterval of $[0,1]$ and the total length of all the intervals in which g_0 is not zero is $1/2$. Furthermore, g_0 satisfies the property of the statement. □

Proposition 2.2. *The space*

$$W = \operatorname{span}\{g_0^{\alpha}\colon\ \alpha > 0\}$$

is an algebra of continuous functions on $[0,1]$.

Proof. If $0 < \alpha_1 < \cdots < \alpha_n$ and $\sum_{k=1}^{n} \lambda_k g_0^{\alpha_k} \equiv 0$, then from the fact that $g_0([1/3,2/3]) = [0,1/2]$ it follows that $\sum_{k=1}^{n} \lambda_k x^{\alpha_k} = 0$ for all $x \in [0,1/2]$. Therefore the linear independence of the functions x^{α} on any interval yields $\lambda_1 = \cdots = \lambda_n = 0$. This shows that W is a linear space of continuous functions on $[0,1]$ of dimension c.

Moreover, if we choose the set of α's forming a Hamel basis of $\mathbb{R}$ as a $\mathbb{Q}$-vector space, then it can be seen that W is an algebra with a minimal system of generators, $\{g_0^{\alpha}\colon\ \alpha > 0\}$, of cardinality c (see, for instance, [7] where this argument is also used). □

In order to construct V we will use the fact that the space of all bounded mappings from $\mathbb{R}$ into $\mathbb{R}$, namely $\mathcal{F}_b(\mathbb{R},\mathbb{R})$, is a vector space of dimension $2^{\mathfrak{c}}$. Indeed, consider any bijection $\phi:\mathbb{R}\to\mathbb{R}^{\mathbb{N}}$ and, for every nonempty subset $A\subset\mathbb{R}$, the mapping $H_A:\mathbb{R}^{\mathbb{N}}\to\{0,1\}$ given by

$$H_A\big(\{x_n\}_{n=1}^{\infty}\big)=\prod_{n=1}^{\infty}\chi_A(x_n),$$

where χ_A denotes the characteristic function of the set A. Then, using Lemma 4.1 and Proposition 4.2 from [2], it can be shown that the set

$$\{H_A\circ\phi\colon\ \emptyset\neq A\subset\mathbb{R}\}$$

is a linear independent family of cardinality $2^{\mathfrak{c}}$ of bounded functions. From now on $\mathcal{B}=\{h_i\colon\ i\in I\}$ will be any fixed Hamel basis for $\mathcal{F}_b(\mathbb{R},\mathbb{R})$.

Definition 2.3. If C_n is a Cantor set in $[\frac{1}{n+1},\frac{1}{n}]$ for every $n\in\mathbb{N}$, $\tau_n:C_n\to\mathbb{R}$ is a bijection and for each $i\in I$, $f_i:\mathbb{R}\to\mathbb{R}$ is the even mapping defined for $x\geqslant 0$ by

$$f_i(x)=\begin{cases}0 & \text{if } x\notin\bigcup_{n=1}^{\infty}C_n,\\ (h_i\circ\tau_n)(x) & \text{if } x\in C_n \text{ for some } n\in\mathbb{N},\end{cases}$$

let us define V as the linear space generated by the set $\{f_i\colon\ i\in I\}$.

Remark 2.4. The linearly independency of the functions h_i's implies that the set $\{f_i\colon\ i\in I\}$ is linearly independent in $\mathcal{F}_b(\mathbb{R},\mathbb{R})$. Moreover, for every $i\in I$ the mapping f_i is Riemann integrable since its set of discontinuities is contained in $\bigcup_{n=1}^{\infty}[(-C_n)\cup C_n]\cup\{0\}$, which has measure zero. Hence V is a $2^{\mathfrak{c}}$-dimensional linear space of Riemann integrable functions.

The following two technical lemmas will be used in the proof of the main result.

Lemma 2.5. *Let $h=\lambda_1 h_{i_1}+\cdots+\lambda_n h_{i_n}$ be a non-trivial linear combination of elements of $\mathcal{B}$ and define $f=\lambda_1 f_{i_1}+\cdots+\lambda_n f_{i_n}$. If for every $r>0$ we set*

$$m(f,\pm r)=\inf\big\{f(x)\colon\ x\in\pm[0,r]\big\}\quad\textit{and}\quad M(f,\pm r)=\sup\big\{f(x)\colon\ x\in\pm[0,r]\big\},$$

then there exists $C>0$ such that $M(f,\pm r)-m(f,\pm r)\geqslant C$.

Proof. Since f is even, we just need to find $C>0$ so that $M(f,r)-m(f,r)\geqslant C$ for all $r>0$. Let $m(h)$ and $M(h)$ be defined as

$$m(h)=\inf\big\{h(x)\colon\ x\in\mathbb{R}\big\}\quad\text{and}\quad M(h)=\sup\big\{h(x)\colon\ x\in\mathbb{R}\big\}.$$

Fix $r>0$ and choose $n\in\mathbb{N}$ so that $\frac{1}{n}<r$. If $M(h)-m(h)=0$ then h is constant, and since h is non-null, then $h\equiv\pm C$ with $C>0$. Using the latter together with the definition of f and the fact that $C_n\subset[0,r]$, we obtain $f([0,r])=\{0,\pm C\}$, from which $M(f,r)-m(f,r)=C$. On the other hand, if $M(h)-m(h)>0$ we have

$$h(\mathbb{R})=h\big(\tau_n(C_n)\big)=f(C_n)\subset f\big([0,r]\big),$$

and the result follows with $C=M(h)-m(h)$. $\square$

Lemma 2.6. *Let $g\in W\setminus\{0\}$ and $[a,b]\subset[0,1]$ such that $g_0(x_0)=0$ for some $x_0\in(a,b)$. Then $g([a,b])$ contains an interval of the form $[0,r]$ or $[-r,0]$ for some $r>0$.*

Proof. Since $g(x_0)=0$ and g is continuous, it suffices to prove that g is not null on $[a,b]$. By Proposition 2.1 there exists $[a',b']\subset(a,b)$ such that $g_0([a',b'])=[0,1/2^n]$ for some $n\in\mathbb{N}$. Now let us consider $g=\sum_{k=1}^{n}\lambda_k g_0^{\alpha_k}$ with $0<\alpha_1<\cdots<\alpha_n$. If g was null on $[a',b']$ then

$$\sum_{k=1}^{n}\lambda_k x^{\alpha_k}=0,$$

for every $x\in[0,1/2^n]$ and hence $\lambda_1=\cdots=\lambda_n=0$. This contradicts the fact that $g\neq 0$. $\square$

Theorem 2.7. *If $f\in V\setminus\{0\}$ and $g\in W\setminus\{0\}$ then $f\circ g$ is not Riemann integrable on $[0,1]$.*

Proof. Let f and h as in Lemma 2.5. Now let $\epsilon > 0$ and suppose that P is the partition of $[0,1]$ given by $0 = x_0 < x_1 < \cdots < x_n = 1$. For each $k = 1, \ldots, n$ define

$$m_k = \inf\{(f \circ g)(x):\ x \in [x_{k-1}, x_k]\},$$
$$M_k = \sup\{(f \circ g)(x):\ x \in [x_{k-1}, x_k]\}.$$

If A is the set of indices k such that there exists $x_0 \in (x_{k-1}, x_k)$ with $g_0(x_0) = 0$, it is obvious that $A \neq \emptyset$ and that for each $k \in A$, according to Lemma 2.6, $g([x_{k-1}, x_k])$ contains an interval of the form $[-r, 0]$ or $[0, r]$ with $r > 0$. Using Lemma 2.5 the latter shows that $M_k - m_k \geqslant C$ for every $k \in A$ for some $C > 0$, and hence

$$\begin{aligned} U(f \circ g, P) - L(f \circ g, P) &= \sum_{k \in A} (M_k - m_k)(x_k - x_{k-1}) + \sum_{k \notin A} (M_k - m_k)(x_k - x_{k-1}) \\ &\geqslant \sum_{k \in A} (M_k - m_k)(x_k - x_{k-1}) \\ &\geqslant C \sum_{k \in A} (x_k - x_{k-1}) \\ &\geqslant \frac{C}{2} > 0. \end{aligned}$$

In other words, using the Riemann Integrability Criteria, $f \circ g$ is not Riemann integrable on $[0,1]$. □

We now focus on the construction of a c-dimensional space W of continuous functions with the property that for every $g \in W \setminus \{0\}$ there exists a c-dimensional space V of measurable functions such that $f \circ g$ is not measurable for all $f \in V \setminus \{0\}$. For further information on the construction of Banach spaces of nonmeasurable functions we refer to [8].

We start recalling a standard counterexample concerning composition of continuous and measurable functions (see [9] for details). Let $C \subset [0,1]$ be the standard Cantor set, and $f_C : [0,1] \to [0,1]$ be the corresponding Cantor function, which is continuous, nondecreasing, locally constant on $[0,1] \setminus C$, and maps $[0,1]$ onto $[0,1]$. Define

$$\varphi(x) = x + f_C(x)$$

for $x \in [0,1]$. The function φ is a (strictly increasing) homeomorphism from $[0,1]$ onto $[0,2]$ and, since each open interval removed form $[0,1]$ in the construction of C is mapped by φ onto an interval of equal length, φ maps a set of measure zero onto a set of positive measure (namely, $\mu(\varphi(C)) = 1$, where μ denotes the Lebesgue measure in $\mathbb{R}$). Therefore the function

$$\psi := \varphi^{-1} : [0,2] \to [0,1]$$

is a strictly increasing homeomorphism mapping the set $A = \varphi(C)$ (of positive measure) onto a set of measure zero. If we take any nonmeasurable set $P \subset A$ and we set $h = \chi_{\psi(P)}$ then we come up with a typical example of a continuous function ψ and a measurable function h such that $h \circ \psi = \chi_P$ is not measurable. It is worth noting (and we will use this fact later on) that $\mu(\varphi(C \cap (0,\epsilon))) > 0$, and therefore, for any $\epsilon > 0$, the set P can be taken to be a subset of the interval $(0,\epsilon)$.

We next construct a large space of continuous functions W with the property that for every $g \in W \setminus \{0\}$ there exists a nonmeasurable set P_g and a set Z_g of measure zero such that $g^{-1}(Z_g) = P_g$.

Proposition 2.8. *Let us define W as the subspace of $\mathcal{C}[0,2]$ consisting of all finite linear combinations of functions of the form $g_\alpha(x) = \psi(x)^\alpha$, where $\psi : [0,2] \to [0,1]$ is the homeomorphism defined above, and $\alpha > 1$. Then we have*:

(1) *For every $g \in W \setminus \{0\}$ there exists a partition $0 = t_0 < t_1 < \cdots < t_n = 2$ of $[0,2]$ such that g is strictly monotone on each interval (t_{i-1}, t_i), and one can write locally $\psi(x) = F_i(g(x))$ for $x \in (t_{i-1}, t_i)$, where the F_i are real-analytic diffeomorphisms defined on (t_{i-1}, t_i), $i = 1, \ldots, n$.*
(2) *For every $g \in W \setminus \{0\}$ there exists a subset Z_g of measure zero such that $P_g := g^{-1}(Z_g)$ is not measurable.*
(3) *For every $g \in W \setminus \{0\}$ and every finite set E the set $g^{-1}(E)$ is finite.*
(4) *The functions g_α are linearly independent.*
(5) *The algebraic dimension of W is c.*

Proof. (1) The function g is of the form $g(x) = G(\psi(x))$, where $G(y) = \sum_{i=1}^m \lambda_i y^{\alpha_i}$, with $\lambda_i \neq 0$ and $1 < \alpha_1 < \cdots < \alpha_m$. Since G is real-analytic on $(0,\infty)$ and

$$\lim_{y \to 0^+} \frac{G'(y)}{y^{\alpha_1 - 1}} = \alpha_1 \lambda_1 \neq 0,$$

it is immediate that G' has a finite number of zeros s_i on $[0,1]$, say $0 = s_0 < s_1 < \cdots < s_n \leqslant 1$. Therefore G is strictly monotone on each $[s_{i-1}, s_i]$ and has local inverse functions $F_i : [z_{i-1}, z_i] \to [s_{i-1}, s_i]$ which are real-analytic on (z_{i-1}, z_i), where $i = 1, \dots, n$ and $z_i = G(s_i)$ or $z_i = G(s_{i-1})$. Because ψ is strictly increasing, it follows that $g = G \circ \psi$ is strictly monotone on each interval $[t_{i-1}, t_i]$, where $t_i := \psi^{-1}(s_i)$, and $\psi(x) = F_i(g(x))$ for $x \in [t_{i-1}, t_i]$, $i = 1, \dots, n$.

(2) Now take a nonmeasurable subset P of $(0, t_1)$ such that $\psi(P)$ has measure zero. Since G is $\mathcal{C}^1$, $g(P) = G(\psi(P))$ has measure zero. If we set

$$Z_g = g(P), \qquad P_g = g^{-1}(Z_g) = \bigcup_{j=1}^{n} \psi^{-1}\bigl(F_j\bigl(G\bigl(\psi(P)\bigr)\bigr)\bigr)$$

then it is clear that Z_g has measure zero and $P_g \cap (0, t_1) = P$, which implies that P_g is not measurable.

(3) This is clear from (2).

(4) and (5) These are obvious consequences of the fact that the function $g(x) = G(\psi(x))$ has a finite number of zeros whenever $\lambda_i \neq 0$ for $i = 1, \dots, m$. $\square$

Theorem 2.9. *Let $W \subset \mathcal{C}[0,2]$ be the c-dimensional vector space defined in the preceding proposition. For each $g \in W \setminus \{0\}$ there exists a c-dimensional space V of measurable functions such that $f \circ g$ is not measurable for all $f \in V \setminus \{0\}$.*

Proof. Take $g \in W \setminus \{0\}$. According to the preceding proposition there exists a set Z_g of measure zero such that $P_g := g^{-1}(Z_g)$ is not measurable. Let us define V as the set of finite linear combinations of functions of the form

$$f_\alpha(y) = \begin{cases} y^\alpha & \text{if } y \in Z_g, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha > 1$. As in the proof of Proposition 2.8, it is immediate that V is a c-dimensional subspace of $\mathcal{C}(\mathbb{R})$. We claim that for every $f \in V \setminus \{0\}$ the function $f \circ g$ is not measurable. Indeed, as in the preceding proposition it is clear that the restriction of $f = \sum_{i=1}^{m} \lambda_i f_{\alpha_i}$ to Z_g has a finite number of zeros, and of course f is zero everywhere outside Z_g. Therefore

$$f^{-1}\bigl(\mathbb{R} \setminus \{0\}\bigr) = Z_g \setminus E,$$

where E is finite, and consequently

$$(f \circ g)^{-1}\bigl(\mathbb{R} \setminus \{0\}\bigr) = g^{-1}(Z_g) \setminus g^{-1}(E) = P_g \setminus g^{-1}(E).$$

Since, according to part (3) of Proposition 2.8, $g^{-1}(E)$ is finite, and P_g is not measurable, this implies that $(f \circ g)^{-1}(\mathbb{R} \setminus \{0\})$ is not measurable, concluding the proof. $\square$

Remark 2.10. It would be interesting to prove an analogue of Theorem 2.7 for measurable functions, that is: Can one construct vector spaces V and W of, respectively, measurable and continuous functions such that $\dim(V) = 2^c$, $\dim(W) = c$ and $f \circ g$ is nonmeasurable for every $f \in V \setminus \{0\}$ and $g \in W \setminus \{0\}$?

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