SMOOTH APPROXIMATIONS WITHOUT CRITICAL POINTS OF CONTINUOUS MAPPINGS BETWEEN BANACH SPACES, AND DIFFEOMORPHIC EXTRACTIONS OF SETS

D. AZAGRA, TADEUSZ DOBROWOLSKI, AND MIGUEL GARCÍA-BRAVO

Abstract. Let E, F be separable Hilbert spaces, and assume that E is infinite-dimensional. We show that for every continuous mapping $f : E \to F$ and every continuous function $\varepsilon : E \to (0, \infty)$ there exists a C^∞ mapping $g : E \to F$ such that $\|f(x) - g(x)\| \leq \varepsilon(x)$ and $Dg(x) : E \to F$ is a surjective linear operator for every $x \in E$. We also provide a version of this result where E can be replaced with a Banach space from a large class (including all the classical spaces with smooth norms, such as c_0, ℓ_p or L^p, $1 < p < \infty$), and F can be taken to be any Banach space such that there exists a bounded linear operator from E onto F. In particular, for such E, F, every continuous mapping $f : E \to F$ can be uniformly approximated by smooth open mappings. Part of the proof provides results of independent interest that improve some known theorems about diffeomorphic extractions of closed sets from Banach spaces or Hilbert manifolds.

1. Introduction and main results

The main purpose of this paper is to show the following two results.

Theorem 1.1. *Let E, F be separable Hilbert spaces, and assume that E is infinite-dimensional. Then, for every continuous mapping $f : E \to F$ and every continuous function $\varepsilon : E \to (0, \infty)$ there exists a C^∞ mapping $g : E \to F$ such that $\|f(x) - g(x)\| \leq \varepsilon(x)$ and $Dg(x) : E \to F$ is a surjective linear operator for every $x \in E$.*

Theorem 1.2. *Let E be one of the classical Banach spaces c_0, ℓ_p or L^p, $1 < p < \infty$. Let F be a Banach space, and assume that there exists a bounded linear operator from E onto F. Then, for every continuous mapping $f : E \to F$ and every continuous function $\varepsilon : E \to (0, \infty)$ there exists a C^k mapping $g : E \to F$ such that $\|f(x) - g(x)\| \leq \varepsilon(x)$ and $Dg(x) : E \to F$ is a surjective linear operator for every $x \in E$.*

Here k denotes the order of smoothness of the space E, defined as follows: $k = \infty$ if $E \in \{c_0\} \cup \{\ell_{2n} : n \in \mathbb{N}\} \cup \{L^{2n} : n \in \mathbb{N}\}$; $k = 2n+1$ if $E \in \{\ell_{2n+1} : n \in \mathbb{N}\} \cup \{L^{2n+1} : n \in \mathbb{N}\}$, and k is equal to the integer part of p if $E \in \{\ell_p\} \cup \{L^p\}$ and $p \notin \mathbb{N}$. The Sobolev spaces $W^{k,p}(\mathbb{R}^n)$ with $1 < p < \infty$ are also included in Theorem 1.2 since they are isomorphic to $L^p(\mathbb{R}^n)$ (see [56, Theorem 11]).
Notice that the assumption that there exists a bounded linear operator from E onto F is necessary, as otherwise all points of E are critical for all functions $g \in C^1(E, F)$.
Of course Theorem 1.1 is a particular case of Theorem 1.2 (also note that if E is a separable Hilbert space, F is a Banach space, and there exists a continuous linear surjection $T : E \to F$, then F must be isomorphic to $\mathbb{R}^n$ or to E). In general, note that a continuous linear surjection $T : E \to F$ between Banach spaces exists if and only if F is isomorphic to a quotient space of E.
We will also establish more technical results (see Theorems 1.6 and 1.7 below) that generalize the preceding theorems to much larger classes of Banach spaces (especially in the case that E is reflexive).

Date: November 2018.
Key words and phrases. Banach space, Morse-Sard theorem, approximation, critical point, diffeomorphic extraction.
D. Azagra and M. García-Bravo were partially supported by grant MTM2015-65825-P.

Part of the motivation for this kind of results is in their connection with the Morse-Sard theorem, a fundamental result in Differential Geometry and Analysis. Throughout this paper, for a C^k smooth mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, C_f stands for the set of critical points of f (that is, the points $x \in \mathbb{R}^n$ at which the differential $Df(x)$ is not surjective), and $f(C_f)$ is thus the set of critical values of f; the same terminology applies to smooth mappings between manifolds, both finite and infinite-dimensional. The Morse-Sard theorem [54, 62] states that if $k \geq \max\{n-m+1, 1\}$ then $f(C_f)$ is of Lebesgue measure zero in $\mathbb{R}^m$. This result also holds true for C^k smooth mappings $f : N \longrightarrow M$ between two smooth manifolds of dimensions n and m respectively.

Given the crucial applications of the Morse-Sard theorem in several branches of mathematics, it is natural both to try to extend this result for other classes of mappings, and also to ask what happens in the case that M and N are infinite-dimensional manifolds. Regarding the first issue, many refinements of the Morse-Sard theorem for other classes of mappings (notably Hölder, Sobolev, and BV mappings) have appeared in the literature; see for instance [68, 69, 55, 11, 12, 53, 24, 35, 18, 19, 45, 41, 40, 9, 10] and the references therein.

As for the second issue, which in this paper is of our concern, let us mention the results of several authors who have studied the question as to what extent one can obtain results similar to the Morse-Sard theorem for mappings between infinite-dimensional Banach spaces or manifolds modeled on such spaces.

S. Smale [63] proved that if X and Y are separable connected smooth manifolds modeled on Banach spaces and $f : X \longrightarrow Y$ is a C^r Fredholm mapping (that is, every differential $Df(x)$ is a Fredholm operator between the corresponding tangent spaces) then $f(C_f)$ is meager, and in particular $f(C_f)$ has no interior points, provided that $r > \max\{\text{index}(Df(x)), 0\}$ for all $x \in X$; here $\text{index}(Df(x))$ stands for the index of the Fredholm operator $Df(x)$, that is, the difference between the dimension of the kernel of $Df(x)$ and the codimension of the image of $Df(x)$, both of which are finite. Of course, these assumptions are very restrictive as, for instance, if X is infinite-dimensional then no function $f : X \longrightarrow \mathbb{R}$ is Fredholm.

In general, every attempt to adapt the Morse-Sard theorem to infinite dimensions will have to impose vast restrictions because, as shown by Kupka's counterexample [47], there are C^∞ smooth functions $f : \ell_2 \longrightarrow \mathbb{R}$ so that their sets of critical values $f(C_f)$ contain intervals. Furthermore, as shown by Bates and Moreira in [12, 53], one can take f to be a polynomial of degree 3.

Nevertheless, for many applications of the Morse-Sard theorem, it is often enough to know that any given continuous mapping can be uniformly approximated by a mapping whose set of critical values is small in some sense; therefore it is natural to ask what mappings between infinite-dimensional manifolds will at least have such an approximation property. Going in this direction, Eells and McAlpin established the following theorem [32]: If E is a separable Hilbert space, then every continuous function from E into $\mathbb{R}$ can be uniformly approximated by a smooth function f whose set of critical values $f(C_f)$ is of measure zero. This allowed them to deduce a version of this theorem for mappings between smooth manifolds M and N modeled on E and a Banach space F respectively, which they called an *approximate Morse-Sard theorem*: Every continuous mapping from M into N can be uniformly approximated by a smooth mapping $f : M \longrightarrow N$ so that $f(C_f)$ has empty interior. However, as observed in [32, Remark 3A], we have $C_f = M$ in the case that F is infinite-dimensional (so, even though the set of critical values of f is relatively small, the set of critical points of f is huge, which is somewhat disappointing).

In [6], a much stronger result was obtained by M. Cepedello-Boiso and the first-named author: if M is a C^∞ smooth manifold modeled on a separable infinite-dimensional Hilbert space X, then every continuous mapping from M into $\mathbb{R}^m$ can be uniformly approximated by smooth mappings *with no critical points*. P. Hájek and M. Johanis [37] established a similar result for $m = 1$ in the case that X is a separable Banach space which contains c_0 and admits a C^p-smooth bump function. Finally, in the case that $m = 1$, these results were extended by M. Jiménez-Sevilla and the first-named author [7] for functions $f : X \to \mathbb{R}$, where X is a separable Banach space admitting an equivalent smooth and locally uniformly rotund norm.

In this paper, we will improve these results by showing that the pairs $(\ell_2, \mathbb{R}^m)$ or $(X, \mathbb{R})$ can be replaced with pairs of the form (E, F), where E is a Banach space from a large class (including all the classical spaces with smooth norms such as c_0, ℓ_p or L^p, $1 < p < \infty$), and F can be taken to be any quotient space of E. So we may say that even though an exact Morse-Sard theorem for mappings between classical Banach spaces is false, a stronger approximate version of the Morse-Sard theorem is nonetheless true.

The general plan of the proof of Theorem 1.2 consists in following these steps:

- Step 1: We construct a smooth mapping $\varphi : E \to E$ such that $\|\varphi(x) - f(x)\| \leq \varepsilon(x)/2$ and C_φ, the critical set of φ, is locally contained in the graph of a continuous mapping defined on a complemented subspace of infinite codimension in E and taking values in its linear complement.
- Step 2: We find a diffeomorphism $h : E \to E \setminus C_\varphi$ such that h is sufficiently close to the identity, in the sense that $\{\{x, h(x)\} : x \in E\}$ refines $\mathcal{G}$ (in other words, h is *limited by* $\mathcal{G}$), where $\mathcal{G}$ is an open cover of E by open balls $B(z, \delta_z)$ chosen in such a way that if $x, y \in B(z, \delta_z)$ then
$$\|\varphi(y) - \varphi(x)\| \leq \frac{\varepsilon(z)}{4} \leq \frac{\varepsilon(x)}{2}.$$
The existence of such a diffeomorphism h follows by the results of Section 2.
- Step 3: Then, the mapping $g(x) := \varphi(h(x))$ has no critical point and satisfies $\|f(x) - g(x)\| \leq \varepsilon(x)$ for all $x \in E$.

The results of Section 2 are of independent interest, as they generalize important theorems on diffeomorphic *extractions* of some kind of sets. Although it is well known (see [20, 51, 30, 31] and the references therein) that every two separable, homotopy equivalent, infinite-dimensional Hilbert manifolds M, N are in fact diffeomorphic, a diffeomorphism $h : M \to N$ provided by this deep result has not been (and, in general, cannot be) shown to be limited by an arbitrary open cover $\mathcal{G}$ of M, a property that is essential in Step 3 above. The finest result we know of which provides a diffeomorphism $h : E \to E \setminus X$ limited by a given open cover $\mathcal{G}$ of E, where E is a separable infinite-dimensional Hilbert space E and X is a closed subset of E, is a theorem of J.E. West [67] in which X is assumed to be locally compact. However, in the proof of Theorem 1.2, we do not work necessarily with Hilbert spaces and we need to diffeomorphically extract a closed set X which is not necessarily locally compact but merely locally contained in the graph of a continuous mapping defined on a complemented subspace of infinite codimension in E and taking values in its linear complement (for a precise explanation of this terminology, see the statement of Theorem 1.4 below). In Section 2, we construct diffeomorphisms h which extract such closed sets X. Observe also that we do not even assume separability of the Banach space, in contrast with all the results about diffeomorphic extraction of closed sets that one can find in the literature.

The main result of Section 2 is the following.

Theorem 1.3. *Let E be an infinite-dimensional Hilbert space, X a closed subset of E which is locally contained in the graph of a continuous function defined on a subspace of infinite codimension in E and taking values in its orthogonal complement, $\mathcal{G}$ an open cover of E, and U an open subset of E. Then, there exists a C^∞ diffeomorphism h of $E \setminus X$ onto $E \setminus (X \setminus U)$ which is the identity on $(E \setminus U) \setminus X$ and is limited by $\mathcal{G}$.*

Recall that h is said to be limited by $\mathcal{G}$ provided that the set $\{\{x, h(x)\} : x \in E \setminus X\}$ refines $\mathcal{G}$; that is, for every $x \in E \setminus X$, we may find a $G_x \in \mathcal{G}$ such that both x and $h(x)$ are in G_x.

Theorem 1.3 is a straightforward consequence of the following much more general result, which is true for many Banach spaces not necessarily Hilbertian.

Theorem 1.4. *Let E be a Banach space, $p \in \mathbb{N} \cup \{\infty\}$, and $X \subset E$ be a closed set with the property that, for each $x \in X$, there exist a neighborhood U_x of x in E, Banach spaces $E_{(1,x)}$ and $E_{(2,x)}$, and a continuous mapping $f_x : C_x \to E_{(2,x)}$, where C_x is a closed subset of $E_{(1,x)}$, such that:*

1. *$E = E_{(1,x)} \oplus E_{(2,x)}$;*
2. *$E_{(1,x)}$ has C^p smooth partitions of unity;*
3. *$E_{(2,x)}$ is infinite-dimensional and has a (not necessarily equivalent) norm of class C^p;*
4. *$X \cap U_x \subset G(f_x)$, where*

$$G(f_x) = \{y = (y_1, y_2) \in E_{(1,x)} \oplus E_{(2,x)} \,:\, y_2 = f_x(y_1), y_1 \in C_x\}.$$

Then, for every open cover $\mathcal{G}$ of E and every open subset U of E, there exists a C^p diffeomorphism h from $E \setminus X$ onto $E \setminus (X \setminus U)$ which is the identity on $(E \setminus U) \setminus X$ and is limited by $\mathcal{G}$. Moreover, the same conclusion is true if we replace E with an open subset of E.

The proof of Theorem 1.4 combines ideas and techniques from Peter Renz's Ph.D. thesis [58], James West's paper [67], and some previous work of the first and second-named authors [4, 5]; see Section 2 for more information. It should be noted that Theorem 1.3 generalizes West's theorem [67], because a closed locally compact subset of an infinite-dimensional Hilbert space E, locally, can be regarded as the graph of a continuous mapping defined on a closed subset of an infinite-codimensional subspace of E; see, for instance, [58].
The proof of Theorem 1.1 will show that E and F can be replaced with open subsets U and V of E and F respectively. Then, by combining such an equivalent statement of Theorem 1.1 with the well known result [30, 46] stating that every separable infinite-dimensional Hilbert manifold is diffeomorphic to an open subset of ℓ_2, one may easily deduce the following.

Theorem 1.5. *Let M, N be separable infinite-dimensional Hilbert manifolds. For every continuous mapping $f : M \to N$ and every open cover $\mathcal{U}$ of N, there exists a C^∞ mapping $g : M \to N$ such that g has no critical point and $\{\{f(x), g(x)\} : x \in M\}$ refines $\mathcal{U}$.*

Alternatively, one can also adjust the proof of Theorem 1.1 to obtain a direct proof of Theorem 1.5.

It is worth noting that Theorems 1.1 and 1.2 are immediate consequences of the following more general (but also more technical) results. For spaces E which are reflexive and have a certain "composite" structure, we have the following.

Theorem 1.6. *Let E be a separable reflexive Banach space of infinite dimension, and F be a Banach space. In the case that F is infinite-dimensional, let us assume furthermore that:*

1. *E is isomorphic to $E \oplus E$.*
2. *There exists a linear bounded operator from E onto F (equivalently, F is a quotient space of E).*

Then, for every continuous mapping $f : E \to F$ and every continuous function $\varepsilon : E \to (0, \infty)$ there exists a C^1 mapping $g : E \to F$ such that $\|f(x) - g(x)\| \leq \varepsilon(x)$ and $Dg(x) : E \to F$ is a surjective linear operator for every $x \in E$.

Note that there exists separable, reflexive Banach spaces E such that E is not isomorphic to $E \oplus E$. The first example of such a space was given by Figiel in 1972 [34].
See also Theorem 5.1 and Theorem 5.2 below for more general variants of this result.
For spaces which are not necessarily reflexive but have an appropriate Schauder basis we have the following.

Theorem 1.7. *Let E be an infinite-dimensional Banach space, and F be a Banach space such that:*

1. *E has an equivalent locally uniformly convex norm $\|\cdot\|$ which is C^1 smooth.*

(2) *$E = (E, \|\cdot\|)$ has a (normalized) Schauder basis $\{e_n\}_{n\in\mathbb{N}}$ such that for every $x = \sum_{j=1}^{\infty} x_j e_j$ and every $j_0 \in \mathbb{N}$ we have that*

$$\left\| \sum_{j\in\mathbb{N},\, j\neq j_0} x_j e_j \right\| \leq \left\| \sum_{j\in\mathbb{N}} x_j e_j \right\|.$$

(3) *In the case that F is infinite-dimensional, there exists a subset $\mathbb{P}$ of $\mathbb{N}$ such that both $\mathbb{P}$ and $\mathbb{N}\setminus\mathbb{P}$ are infinite and, for every infinite subset J of $\mathbb{P}$, there exists a linear bounded operator from $\overline{span}\{e_j : j\in J\}$ onto F (equivalently, F is a quotient space of $\overline{span}\{e_j : j \in J\}$).*

Then, for every continuous mapping $f : E \to F$ and every continuous function $\varepsilon : E \to (0,\infty)$ there exists a C^1 mapping $g : E \to F$ such that $\|f(x) - g(x)\| \leq \varepsilon(x)$ and $Dg(x) : E \to F$ is a surjective linear operator for every $x \in E$.

Recall that a norm $\|\cdot\|$ in a Banach space E is said to be locally uniformly convex (LUC) (or locally uniformly rotund (LUR)) provided that, for every sequence $(x_n) \subset E$ and every point x_0 in E, we have that

$$\lim_{n\to\infty} 2\left(\|x_0\|^2 + \|x_n\|^2\right) - \|x_0 + x_n\|^2 = 0 \implies \lim_{n\to\infty} \|x_n - x_0\| = 0.$$

Condition (2) is equivalent to the fact that, for every (equivalently, finite) set $A \subset \mathbb{N}$, $\|P_A\| \leq 1$, where P_A stands for the projection $P_A(x) = \sum_{j\in A} x_j e_j$. This, in particular, implies that $\{e_n\}_{n\in N}$ is an unconditional basis; for more details see [2, p. 53] or [1].

The proofs of these theorems will be provided in Sections 3 and 4. These results combine to yield Theorem 1.2 for $k = 1$ (see also Remark 5.5 in Section 5 for an explanation of why the space c_0 satisfies the assumptions of Theorem 1.7). In order to deduce Theorem 1.2 in the cases of higher order smoothness, we just have to use Nicole Moulis's results on C^1 fine approximation in Banach spaces [52] or the more general results of [38, Corollary 7.96], together with the following fact.

Proposition 1.8. *Assume that the Banach spaces E, F satisfy the following properties:*

(1) *For every continuous mapping $f : E \to F$ and every continuous function $\delta : E \to (0,\infty)$ there exists a C^1 smooth mapping $\varphi : E \to F$ such that $\|f(x) - \varphi(x)\| \leq \delta(x)$ and $D\varphi(x) : E \to F$ is surjective for all $x \in E$.*
(2) *For every C^1 mapping $\varphi : E \to F$ and every continuous function $\eta : E \to (0,\infty)$ there exists a C^k mapping $g : E \to F$ such that $\|f(x) - \varphi(x)\| \leq \eta(x)$ and $\|D\varphi(x) - Dg(x)\| \leq \eta(x)$ for all $x \in E$.*

Then, for every continuous mapping $f : E \to F$ and every continuous function $\varepsilon : E \to (0,\infty)$ there exists a C^k smooth mapping $g : E \to F$ such that $\|f(x) - g(x)\| \leq \varepsilon(x)$ and $Dg(x) : E \to F$ is surjective for every $x \in E$.

Nevertheless, it should be noted that our proof of Theorem 1.6 directly provides C^∞ approximations without critical points in the case that E is a separable Hilbert space; see Remark 5.3 in Section 5 below. An easy proof of Proposition 1.8, together with some examples, remarks and more technical variants of our results, is given in Section 5.
Finally, let us mention that as a straightforward application of Theorem 1.2, we obtain that, for all Banach spaces E and F appearing in Theorem 1.2, every continuous mapping $f : E \to F$ can be uniformly approximated by *open* mappings of class C^k. For a more general statement, see Remark 5.9. Obviously, the latter result is false in the case that E is finite-dimensional.

2. Extracting closed sets which are locally contained in graphs of infinite codimension

In this section we will combine ideas and tools of [58, 67, 5] in order to prove Theorem 1.4. We will split the proof into four subsections. First, in Section 2.1, we will see that each piece of X contained in the graph $G(f_x)$ as provided by condition (4) of the statement can be flattened by means of homeomorphisms $h_x, \varphi_x : E \to E$ which are sufficiently close to each other, and whose restrictions to $E \setminus G(f_x)$ and $E \setminus (G(f_x) \setminus U)$ are diffeomorphisms, respectively. Next, in Sections 2.2 and 2.3, we will show that there exists a diffeomorphism $g_x : E \setminus (C_x \times \{0\}) \to E \setminus ((C_x \times \{0\}) \setminus h_x(U))$ which is the identity on $(E \setminus h_x(U)) \setminus (C_x \times \{0\})$ and moves no point more than a fixed small number ε_x. Then, the composition $\varphi_x^{-1} \circ g_x \circ h_x$ will extract the local chunk of graph $U_x \cap G(f_x)$ and will move no point too much. Finally, in Section 2.4, we will see how one can patch a collection of diffeomorphisms extracting pieces of X into a diffeomorphism h which extracts X and is limited by $\mathcal{G}$.
In Section 2.1, we will closely follow Peter Renz's results from [58, 59]. In Sections 2.2 and 2.3, we will combine ideas and techniques from [58, 4, 5]. Finally, in Section 2.4, we will borrow a technique of James West's [67, p. 288-290].

2.1. **Flattening graphs.** Here we will prove the following.

Theorem 2.1. *Let E_1 be a Banach space with C^p smooth partitions of unity and E_2 be a Banach space which admits a (not necessarily equivalent) C^p norm. Let $(E = E_1 \times E_2, \|\cdot\|)$ and $\pi_1 : E \to E_1$ be the natural projection, i.e., $\pi_1(x_1, x_2) = x_1$, $(x_1, x_2) \in E$. Let $X_1 \subset E_1$ be a closed set, $f : X_1 \to E_2$ a continuous mapping, $U \subset E$ an open set, and $\varepsilon > 0$. Write $G(f) = \{(x_1, x_2) \in E : x_2 = f(x_1),\ x_1 \in X_1\}$. Then there exist a couple of homeomorphisms $h, \varphi : E \to E$ such that:*

1. *$h(G(f)) \subset E_1 \times \{0\}$ and $\varphi(G(f) \setminus U) \subset (E_1 \times \{0\}) \setminus h(U)$;*
2. *$h = \varphi$ off of U;*
3. *$\pi_1 \circ h = \pi_1 = \pi_1 \circ \varphi$;*
4. *h restricted to $E \setminus G(f)$ is a C^p diffeomorphism of $E \setminus G(f)$ onto $E \setminus (X_1 \times \{0\})$;*
5. *φ restricted to $E \setminus (G(f) \setminus U)$ is a C^p diffeomorphism of $E \setminus (G(f) \setminus U)$ onto $E \setminus ((X_1 \times \{0\}) \setminus h(U))$.*
6. *$\|h^{-1}(x) - \varphi^{-1}(x)\| \leq \varepsilon$ for every $x \in E$.*
7. *$h^{-1}(x_1, x_2)$ is uniformly continuous with respect to the second coordinate x_2 (meaning that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|x_2 - x_2'\| < \delta$ then $\|h^{-1}(x_1, x_2) - h^{-1}(x_1, x_2')\| < \varepsilon$ for all x_1).*

We will assume without loss of generality that $\varepsilon \leq 1$.
In what follows, slightly abusing notation, we will indistinctly use the symbol $\|\cdot\|$ to denote the norms $\|\cdot\|_{E_1}$, $\|\cdot\|_{E_2}$, and $\|\cdot\|$ with which the Banach spaces E_1, E_2 or $E_1 \times E_2$ are endowed. We may and do assume that $\|x_1\|_{E_1} = \|(x_1, 0)\|$ and $\|x_2\|_{E_2} = \|(0, x_2)\|$ for all $(x_1, x_2) \in E_1 \times E_2$.
Now, we state and prove a sequence of lemmas that will be employed in proving the above theorem. The most important are Lemmas 2.3 and 2.5. Basically, we follow the ideas of Renz's paper [59] and Ph.D. thesis [58], with some minor but very important changes.
The proof of our first lemma is a consequence of the existence of C^p smooth partition of unity on E_1.

Lemma 2.2. *The function $f : X_1 \to E_2$ extends to a continuous function $\bar{f} : E_1 \to E_2$ such that $\bar{f}|E_1 \setminus X_1$ is C^p smooth.*

Lemma 2.3. *Let E_1 be a Banach space with C^p smooth partitions of unity, E_2 be a Banach space which admits a (not necessarily equivalent) C^p norm, X_1 be a closed subset of E_1, and $f : X_1 \to E_2$ be a continuous mapping. For every $n \in \mathbb{N}$, write*

$$W_n = \left\{ x_1 \in E_1 : \operatorname{dist}(x_1, X_1) \leq \frac{1}{n} \right\}.$$

Assume $\bar{f} : E_1 \to E_2$ is a continuous extension of f such that $f|E_1 \setminus X_1$ is C^p smooth. Then, there is a continuous mapping

$$F : \mathbb{R} \times E_1 \to E_2$$

such that

(1) $F(r, x_1) = \bar{f}(x_1)$ *for all* $(r, x_1) \in (r_n, \infty) \times E_1 \setminus W_n$ *and some* $0 < r_n < 1$*; in particular,* $F(r, x_1) = \bar{f}(x_1)$ *for all* (r, x_1) *in some neighborhood of the set* $\{1\} \times (E_1 \setminus X_1)$ *in* $\mathbb{R} \times (E_1 \setminus X_1)$*;*
(2) $F|\mathbb{R} \times (E_1 \setminus X_1) \cup (-\infty, 1) \times E_1$ *is* C^p *smooth;*
(3) $F(r, x_1) = f(x_1)$ *for* $r \geq 1$ *and* $x_1 \in X_1$*;*
(4) $\|D_1 F(r, x_1)\| \leq \frac{1}{2}$ *for all* $r \in \mathbb{R}$, $x_1 \in X_1$.

Proof. For every $n \in \mathbb{N}$ we can find a sequence of C^p functions $\bar{f}_n : E_1 \to E_2$ such that

$$||\bar{f}(x_1) - \bar{f}_n(x_1)|| \leq 2^{-2n-4}$$

for every $x_1 \in E_1$. The existence of such a sequence is again guaranteed by the existence of C^p partitions of unity in E_1 (see, for instance, [26, Theorem 8.2]). We will now improve the sequence $\{\bar{f}_n\}_{n\geq 1}$ to $\{f_n\}_{n\geq 1}$ so that the sequence $\{f_n|E_1 \setminus X_1\}$ locally stabilizes with respect to n. To achieve this, we use the existence of C^p partitions of unity to find a C^p function $\lambda_n : E_1 \to [0, 1]$ which is 1 on $E \setminus W_n$ and 0 on W_{n+1}. Define

$$f_n(x_1) = \lambda_n(x_1)\bar{f}(x_1) + (1 - \lambda_n(x_1))\bar{f}_n(x_1)$$

for all $x_1 \in E_1$. It follows that $\|f_n(x_1) - f_{n+1}(x_1)\| \leq 2^{-2n-3}$ for all $x_1 \in E_1$.
For $n \in \mathbb{R}$, pick a nondecreasing C^∞ function $h_n : \mathbb{R} \to [0, 1]$ such that $h_n(r) = 0$ for $r \leq 1 - 2^{1-n}$, $h_n(r) = 1$ for $r \geq 1 - 2^{-n}$, and $h_n'(r) \leq 2^{n+1}$. One can check that

$$F(r, x_1) = f_1(x_1) + \sum_{n=1}^{\infty} h_{n+1}(r)(f_{n+1}(x_1) - f_n(x_1))$$

defines a required mapping. □

Observe that in fact $F(r, x_1)$ is Lipschitz with constant 1 with respect to the first variable $r \in \mathbb{R}$. That is,

$$||F(r, x_1) - F(r', x_1)|| \leq \sum_{n=1}^{\infty} |h_{n+1}(r) - h_{n+1}(r')|||f_{n+1}(x_1) - f_n(x_1)|| \leq$$
$$\leq \sum_{n=1}^{\infty} 2^{n+1}|r - r'|2^{-2n-3} \leq |r - r'|$$

for every $x_1 \in E_1$.
We will write

$$U_0 = \pi_1(G(\bar{f}) \cap U),$$

which is an open set in E_1, and also

$$Y_1 = X_1 \setminus U_0 = X_1 \setminus \pi_1(G(\bar{f}) \cap U) = \pi_1(G(f) \setminus U),$$

which is a closed subset of E_1. By replacing U with $U \cap \pi_1^{-1}(U_0)$, we can assume that

$$U_0 = \pi_1(U).$$

Lemma 2.4. *With the above notation, take a decreasing sequence of positive numbers* $\{\delta_n\}_{n\geq 1}$ *converging to zero. Then there exists an increasing sequence of open subsets in* E_1

$$V_1 \subseteq \overline{V_1} \subseteq V_2 \subseteq \overline{V_2} \subseteq \cdots \subseteq \overline{V_n} \subseteq V_{n+1} \subseteq \cdots \subseteq U_0$$

such that $\bigcup_{n=1}^{\infty} V_n = U_0$ *and the sets*

$$U_n := \{(x_1, x_2) \in E : \, \|x_2 - \bar{f}(x_1)\| < \delta_n, \, x_1 \in V_n\}$$

are contained in U*.*

Proof. To be able to get the required inclusions between the sets V_n, we first take an auxiliary sequence of open sets W_n in U_0 such that $\overline{W_n} \subseteq W_{n+1}$ for every $n \in N$ and $\bigcup_{n=1}^{\infty} W_n = U_0$.
Then we define

$$V_n' = \left\{x_1 \in U_0 : \, \{(x_1, x_2) \in E : \|x_2 - \bar{f}(x_1)\| < \delta_n\} \subseteq U\right\}$$

for every $n \in \mathbb{N}$. Observe that we have $V_n' \subseteq V_{n+1}'$ and $\bigcup_{n=1}^{\infty} V_n' = U_0$, but we cannot assure that $\overline{V_n'} \subseteq V_{n+1}'$ for every $n \in N$. So now we mix these sets with the previous W_n, that is, we let $V_n = W_n \cap V_n'$. Obviously, by definition, for every $n \in \mathbb{N}$ the set $U_n = \{(x_1, x_2) \in E : \, \|x_2 - \bar{f}(x_1)\| < \delta_n, \, x_1 \in V_n\}$ is contained in U. Now, we have that $\overline{V_n} \subseteq V_{n+1}$ for every $n \in N$; also $\bigcup_{n=1}^{\infty} V_n = U_0$. □

The following lemma resembles [59, Lemma 2.2] and [58, Lemma 2] (in which only one function ϕ is considered). However, Theorem 2.1 requires constructing two homeomorphisms h and φ which are identical outside U. The building block in constructing those homeomorphisms are two functions ϕ and $\tilde{\phi}$ whose existence is claimed in the lemma below. The existence of $\tilde{\phi}$ is crucial. Incidentally, let us note that Renz's proof of [58, Theorem 4] is flawed (and this is the reason why we must deal with two functions ϕ and $\tilde{\phi}$ instead of just the function ϕ), but can be corrected by using Theorem 2.1.

Lemma 2.5. *Let* $\bar{f} : E_1 \to E_2$ *be the uniform limit of* C^p *functions. Then there are two continuous functions* ϕ, $\tilde{\phi} : E \to [0, 1]$ *such that*

(1) $\phi^{-1}(1) = G(\bar{f})$ *and* $\tilde{\phi}^{-1}(1) = G(\bar{f}) \setminus U$*;*
(2) $\phi|E \setminus G(\bar{f})$ *and* $\tilde{\phi}|E \setminus (G(\bar{f}) \setminus U)$ *are* C^p *smooth;*
(3) $\|D_2\phi(x_1, x_2)\| \leq \frac{1}{2}$ *for all* $(x_1, x_2) \in E \setminus G(\bar{f})$*, and* $\|D_2\tilde{\phi}(x_1, x_2)\| \leq \frac{1}{2}$ *for all* $(x_1, x_2) \in E \setminus (G(\bar{f}) \setminus U)$*;*
(4) $\phi = \tilde{\phi}$ *outside* U*.*

Proof. To construct ϕ we will follow [59, Lemma 2.2]. A similar argument will be used to construct $\tilde{\phi}$; however, we have to make sure that $\tilde{\phi}|G(\bar{f}) \cap U < 1$.
For $n \in \mathbb{N}$, let $a_n, b_n, c_n, d_n, \varepsilon_n$ be positive numbers with the following properties:

(1) they tend to zero as n tends to infinity;
(2) $a_n < b_n$ for all n;
(3) $\varepsilon_{n+1} + b_{n+1} < a_n - \varepsilon_n$ for all n;
(4) $\sum_{n=1}^{\infty} c_n \leq \frac{\epsilon}{2} \leq 1$;
(5) $\sum_{n=1}^{\infty} d_n \leq \frac{1}{2}$

(for instance, let us set $a_n = \epsilon 2^{-2n}$, $b_n = \epsilon 2 a_n$, $c_n = \epsilon 2^{-4n}$, $d_n = \epsilon 2^{-2n}$ and $\varepsilon_n = \epsilon 2^{-4(n+1)}$). Let h_n be a nonincreasing C^p function from $\mathbb{R}$ to $\mathbb{R}$ satisfying

$$\begin{aligned} c_n = h_n(r) = h_n(0) > 0 \qquad & \text{whenever } r \leq a_n, \\ h_n(r) = 0 \qquad & \text{whenever } r \geq b_n, \\ |h_n'(r)| \leq d_n \qquad & \text{for all } r \text{ in } \mathbb{R}, \end{aligned}$$

and $g_n : E_1 \to E_2$ be a C^p mapping such that $\|g_n(x_1) - \bar{f}(x_1)\| \leq \varepsilon_n$ for every $x_1 \in E_1$. Then

$$\psi_n(x_1, x_2) = h_n\left(\|x_2 - g_n(x_1)\|\right)$$

defines a nonnegative C^p function on $E_1 \times E_2 = E$ satisfying

$$\begin{aligned} &c_n = \psi_n(x_1,x_2) = h_n(0) > 0 && \text{if } \ \|x_2 - \bar f(x_1)\| \le a_n - \varepsilon_n, \\ &\psi_n(x_1,x_2) = 0 && \text{if } \ \|x_2 - \bar f(x_1)\| \ge b_n + \varepsilon_n, \\ &\|D_2\psi_n(x_1,x_2)\| \le d_n && \text{for all } \ (x_1,x_2) \in E_1 \times E_2. \end{aligned}$$

The nonnegativity and first two properties of ψ_n are evident, and it is easy to see that ψ_n is C^p on E. The bound on the norm of the derivative $D_2\psi_n$ is established by using the chain rule and the fact that the operator norm of the derivative of the norm of any Banach space is less than or equal to one. Define

$$\psi(x_1,x_2) = \sum_{n=1}^{\infty} \psi_n(x_1,x_2) \tag{2.1}$$

for all $(x_1,x_2) \in E$.
Similarly, define

$$\tilde\psi(x_1,x_2) = \sum_{n=1}^{\infty} \lambda_n(x_1)\psi_n(x_1,x_2) \tag{2.2}$$

for all $(x_1,x_2) \in E$, where $\lambda_n : E_1 \to [0,1]$ is a C^p smooth function such that $\lambda_n(x_1) = 1$ if $x_1 \notin V_n$ and $\lambda_n(x_1) = 0$ if $x \in V_{n-1}$. Here, the sets V_n are provided by Lemma 2.4 for the sequence $\delta_n := \varepsilon_n + b_n$ (let $V_0 = \emptyset$ and assume $V_1 \neq \emptyset$). In particular, observe that since $\bigcup_{n=1}^{\infty} V_n = U_0$ then $\lambda_n(x_1) = 1$ for every $x_1 \notin U_0$; hence, $\psi(x_1,x_2) = \tilde\psi(x_1,x_2)$ for $x_1 \notin U_0$.
Since the functions ψ and $\tilde\psi$ are defined via absolutely and uniformly convergent series of continuous functions, they are continuous.
If $(x_1,x_2) \in E \setminus G(\bar f)$, then $\|x_2 - \bar f(x_1)\| > b_n + \varepsilon_n$ for some $n \in \mathbb{N}$. By continuity, the inequality holds in a neighborhood of (x_1,x_2) so ψ_k vanishes for $k \ge n$. Hence, ψ is locally a finite sum of C^p functions, and in particular is C^p on $E \setminus G(\bar f)$.
Also, if $(x_1,x_2) \in G(\bar f) \cap U$, then $x_1 \in U_0$ and $x_1 \in V_n$ for some $n \in \mathbb{N}$. So $\lambda_k(x_1) = 0$ for all $k \ge n+1$. This means that $\tilde\psi$ is locally a finite sum of C^p functions and, thus, is of class C^p on $E \setminus (G(\bar f) \setminus U)$.
The derived series for $D_2\psi$ and $D_2\tilde\psi$ are absolutely and uniformly convergent in view of the bounds on $\|D_2\psi_n\|$ and the fact that $D_2\lambda_n = 0$. Then differentiation term by term is justified and $\|D_2\psi(x_1,x_2)\| \le \frac{1}{2}$ for all $(x_1,x_2) \in E \setminus G(\bar f)$ and $\|D_2\tilde\psi(x_1,x_2)\| \le \frac{1}{2}$ for all $(x_1,x_2) \in E \setminus (G(\bar f) \setminus U)$.
Each point $(x_1,x_2) \in G(\bar f)$ satisfies $0 = \|x_2 - \bar f(x_1)\| < a_n - \varepsilon_n$ for all $n \in \mathbb{N}$, consequently ψ equals the constant

$$d^* = \sum_{n=1}^{\infty} \psi_n(x_1, \bar f(x_1)) = \sum_{n=1}^{\infty} h_n(0) = \sum_{n=1}^{\infty} c_n \le 1.$$

On the other hand, if $(x_1,x_2) \in G(\bar f) \setminus U$, then $0 = \|x_2 - \bar f(x_1)\| < a_n + \varepsilon_n$ and $\lambda_n(x_1) = 1$ for all $n \in \mathbb{N}$, so $\tilde\psi$ equals again the constant d^*. In fact d^* is the supremum of ψ and of $\tilde\psi$, and is easily seen to be attained in $G(\bar f)$ and $G(\bar f) \setminus U$, respectively.
To show that ψ and $\tilde\psi$ are equal outside U take $(x_1,x_2) \in E$. By a remark after the definition of ψ and $\tilde\psi$, we can assume $x_1 \in U_0$.

Claim 2.6. *For every $n \in \mathbb{N}$, if $x_1 \in V_n \setminus V_{n-1}$ and if $x_2 \in E_2$ is such that $\|x_2 - \bar f(x_1)\| \ge b_n + \varepsilon_n$, then $\psi(x_1,x_2) = \tilde\psi(x_1,x_2)$.*

Proof of Claim. If $\|x_2 - \bar f(x_1)\| \ge b_n + \varepsilon_n$ we have that $\psi_k(x_1,x_2) = 0$ for all $k \ge n$. So we have to see that $\lambda_k(x_1) = 1$ for $k = 1,\dots,n-1$. But this is clear since $x \notin V_{n-1}$ and hence $x \notin V_k$ for any $k = 1,\dots,n-1$. □

Now, we can conclude that for each $n \in \mathbb{N}$, $\psi = \tilde{\psi}$ on the set

$$((V_n \setminus V_{n-1}) \times E_2) \setminus U_n \supseteq ((V_n \setminus V_{n-1}) \times E_2) \setminus U.$$

Since $\bigcup_{n=1}^{\infty} V_n \setminus V_{n-1} = U_0$ and $\bigcup_{n=1}^{\infty} U_n \subseteq U$, it follows that ψ is equal to $\tilde{\psi}$ outside U.
Finally, to obtain functions ϕ and $\tilde{\phi}$ with the desired properties it is sufficient to set

$$\phi(x_1, x_2) = \psi(x_1, x_2) + 1 - d^*$$
$$\tilde{\phi}(x_1, x_2) = \tilde{\psi}(x_1, x_2) + 1 - d^*$$

for all $(x_1, x_2) \in E$. This ensures that the supremum, which is attained precisely on $G(\bar{f})$ for ϕ and precisely on $G(\bar{f}) \setminus U$ for $\tilde{\phi}$, is equal to 1.

□

In the proof of Theorem 2.1, we will employ a well known fact stating that the identity mapping perturbed by a contracting mapping is a homeomorphism (even a diffeomorphism provided that the contracting mapping is smooth). This fact is stated and proved in Lemma 3 of Renz's Ph.D. thesis [58].

Lemma 2.7. *Let E_1 be a normed linear space and E_2 be a Banach space. Let $E = E_1 \times E_2$ and let $d : E \to E_2$ be a continuous mapping satisfying the following condition*

$$\|d(x_1, x_2) - d(x_1, x_2')\| \leq \frac{1}{2}\|x_2 - x_2'\|$$

for all $x_1 \in E_1$ and $x_2, x_2' \in E_2$. Then the mapping defined by $h(x_1, x_2) = (x_1, x_2 - d(x_1, x_2))$ is a homeomorphism of E onto itself. Moreover, h is a C^p diffeomorphism when restricted to any open set (onto its image) on which d is C^p smooth.

Let us now present the proof of Theorem 2.1.

Proof of Theorem 2.1. Basically, we will follow the proof of Theorem 1 of Renz's Ph.D. thesis [58]. First, we apply Lemma 2.2 to $f : X_1 \to E_2$ to obtain a continuous mapping $\bar{f} : E_1 \to E_2$ such that $\bar{f}|X_1 = f$ and $\bar{f}|E_1 \setminus X_1$ is C^p smooth. Then, we apply Lemma 2.3 to the mapping $\bar{f}$ to obtain a mapping $F : \mathbb{R} \times E_1 \to E_2$ satisfying conditions (1)–(4) of Lemma 2.3. Next, we apply Lemma 2.5 to $\bar{f}$ to obtain functions ϕ and $\tilde{\phi}$ satisfying conditions (1)–(4) of Lemma 2.5. Now, we define

$$d(x_1, x_2) = F(\phi(x_1, x_2), x_1)$$
$$\tilde{d}(x_1, x_2) = F(\tilde{\phi}(x_1, x_2), x_1).$$

Let us check that Lemma 2.7 is applicable to d and $\tilde{d}$ so that

$$h(x_1, x_2) = (x_1, x_2 - d(x_1, x_2))$$

and

$$\varphi(x_1, x_2) = (x_1, x_2 - \tilde{d}(x_1, x_2))$$

are homeomorphisms (which, additionally, will satisfy the conditions enumerated in Theorem 2.1).
Both functions d and $\tilde{d}$ are continuous as compositions of continuous functions. We compute $D_2 d$ and $D_2 \tilde{d}$ to obtain

$$D_2 d(x_1, x_2) = D_1 F(\phi(x_1, x_2), x_1) \circ D_2 \phi(x_1, x_2)$$
$$D_2 \tilde{d}(x_1, x_2) = D_1 F(\tilde{\phi}(x_1, x_2), x_1) \circ D_2 \tilde{\phi}(x_1, x_2).$$

The estimates of the norms of D_1F, $D_2\phi$, and $D_2\tilde{\phi}$ yields $\|D_2d(x_1,x_2)\|$, $\|D_2\tilde{d}(x_1,x_2)\| \le \frac{1}{4}$ when $(x_1,x_2)\notin G(\bar{f})$. Since d and $\tilde{d}$ are continuous and $E\setminus G(\bar{f})$ is dense in E, by the mean value theorem, we can write

$$\|d(x_1,x_2)-d(x_1,x_2')\| \le \frac{1}{4}\|x_2-x_2'\|$$
$$\|\tilde{d}(x_1,x_2)-\tilde{d}(x_1,x_2')\| \le \frac{1}{4}\|x_2-x_2'\|$$

for all $x_1\in E_1$ and all x_2, $x_2'\in E_2$. Hence, Lemma 2.7 applies and yields that h and φ are homeomorphisms.

Let us show conditions (1)–(7) of Theorem 2.1.

First, we will verify condition (1). If $(x_1,x_2)\in G(f)$, then $d(x_1,x_2)=F(1,x_1)=f(x_1)=x_2$ and

$$h(x_1,x_2)=(x_1,x_2-d(x_1,x_2))=(x_1,0). \tag{2.3}$$

If $(x_1,x_2)\in G(f)\setminus U$, then $\tilde{d}(x_1,x_2)=F(1,x_1)=x_2$ and

$$\varphi(x_1,x_2)=(x_1,x_2-\tilde{d}(x_1,x_2))=(x_1,0). \tag{2.4}$$

Condition (3) is obvious. Since ϕ and $\tilde{\phi}$ are equal outside U we obtain (2).

Let us see that $d|E\setminus G(f)$ and $\tilde{d}|E\setminus(G(f)\setminus U)$ are C^p diffeomorphisms. If $(x_1,x_2)\notin G(\bar{f})$ then ϕ and $\tilde{\phi}$ are C^p smooth and $\phi(x_1,x_2)$, $\tilde{\phi}(x_1,x_2)<1$ by condition (2) and (1) of Lemma 2.5. It follows that $F(\phi(x_1,x_2),x_1)$ and $F(\tilde{\phi}(x_1,x_2),x_1)$ are C^p smooth in a neighborhood of (x_1,x_2). Thus h and φ are C^p on $E\setminus G(\bar{f})$.

On the other hand, we have $\phi|G(\bar{f})\setminus G(f)=1$. Then, by continuity of ϕ and condition (1) of Lemma 2.3, we infer that $F(\phi(x_1,x_2),x_1)=\bar{f}(x_1)$ and, consequently, $h(x_1,x_2)=(x_1,x_2-\bar{f}(x_1))$ in a neighborhood of $G(\bar{f})\setminus G(f)$. We have proved that d is C^p smooth on $E\setminus G(f)$.

It remains to show that $\varphi|U$ is C^p smooth. By condition (1) of Lemma 2.5, we have $\tilde{\phi}|U<1$; by condition (2) of Lemma 2.3, $\tilde{d}|U$ is C^p smooth. The proof that $\tilde{d}$ and, therefore, φ restricted to $E\setminus(G(f)\setminus U)$ is C^p smooth is complete.

Now, Lemma 2.7 tells us that h and φ are C^p diffeomorphisms of $E\setminus G(f)$ onto $h(E\setminus G(f))$ and $E\setminus(G(f)\setminus U)$ onto $\varphi(E\setminus(G(f)\setminus U))$. So to get (4) and (5) it is sufficient to show that $h(G(f))=X_1\times\{0\}$ and $\varphi(G(f)\setminus U)=(X_1\times\{0\})\setminus h(U)$. The first equality is clear from equation (2.3). For the second one, observe that (2.4) tells us that $\varphi(G(f)\setminus U)=(X_1\setminus U_0)\times\{0\}=Y_1\times\{0\}$. So, we must check that

$$(X_1\times\{0\})\setminus h(U)=(X_1\setminus U_0)\times\{0\},$$

or, what is the same, that $\pi_1(h(U))=U_0$. The latter follows from condition (3) and the fact that $\pi_1(U)=U_0$.

Let us finish the proof by showing (6) and (7). Firstly let us check that $||h^{-1}(x_1,x_2)-\varphi^{-1}(x_1,x_2)||\le\varepsilon$ for every $(x_1,x_2)\in E_1\times E_2$. Since h^{-1} preserves the first coordinate, we can write $h^{-1}(x_1,x_2)=(x_1,y_2)$ and $\varphi^{-1}(x_1,x_2)=(x_1,z_2)$ where $y_2,z_2\in E_2$ are such that

$$y_2-d(x_1,y_2)=x_2$$
$$z_2-\tilde{d}(x_1,z_2)=x_2.$$

We then have that $||h^{-1}(x_1,x_2)-\varphi^{-1}(x_1,x_2)||\le\varepsilon$ if and only if $||y_2-z_2||\le\varepsilon$ and if and only if $||d(x_1,y_2)-\tilde{d}(x_1,z_2)||\le\varepsilon$. Since $r\mapsto F(r,x_1)$ is 1-Lipschitz, this is true if $|\phi(x_1,y_2)-\tilde{\phi}(x_1,z_2)|\le\epsilon$, or what is the same if $|\psi(x_1, y_2) - \tilde{\psi}(x_1, z_2)| \leq \epsilon$. And this is the case because

$$|\psi(x_1, y_2) - \tilde{\psi}(x_1, z_2)| \leq |\psi(x_1, y_2)| + |\tilde{\psi}(x_1, z_2)| = |\sum_{n=1}^{\infty} \psi_n(x_1, y_2)| + |\sum_{n=1}^{\infty} \lambda_n(x_1)\psi_n(x_1, z_2)| \leq$$

$$\leq \sum_{n=1}^{\infty} c_n + \sum_{n=1}^{\infty} c_n \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Secondly, let us see that for every $\eta > 0$ there exists $\delta > 0$ such that if $||(x_1, x_2) - (x_1, x_2')|| = ||x_2 - x_2'|| \leq \delta$ then $||h^{-1}(x_1, x_2) - h^{-1}(x_1, x_2')|| \leq \eta$. It will be enough to set $\delta = \frac{\eta}{2}$. Indeed, take $(x_1, x_2), (x_1, x_2') \in E_1 \times E_2$ such that $||(x_1, x_2) - (x_1, x_2')|| = ||x_2 - x_2'|| \leq \frac{\eta}{2}$. Write $h^{-1}(x_1, x_2) = (x_1, y_2)$ and $h^{-1}(x_1, x_2') = (x_1, y_2')$ where $y_2, y_2' \in E_2$ are such that

$$y_2 - d(x_1, y_2) = x_2$$
$$y_2' - \tilde{d}(x_1, y_2') = x_2'.$$

Then we have that

$$||h^{-1}(x_1, x_2) - h^{-1}(x_1, x_2')|| = ||y_2 - y_2'|| \leq ||x_2 - x_2'|| + ||d(x_1, y_2) - d(x_1, y_2')|| \leq$$

$$\leq ||x_2 - x_2'|| + \frac{1}{2}||y_2 - y_2'||,$$

which implies that $||y_2 - y_2'|| \leq 2||x_2 - x_2'|| = \eta$, and the proof is complete. □

2.2. **An extracting scheme tailored for closed subsets of a subspace of infinite codimension.** In this subsection we will establish the following.

Theorem 2.8. *Let E_1 and E_2 be Banach spaces such that E_2 is infinite-dimensional and admits a (not necessarily equivalent) C^p smooth norm, where $p \in \mathbb{N} \cup \{\infty\}$. Define $E = E_1 \times E_2$ and, for $i = 1, 2$, write $\pi_i : E \to E_i$ for the natural projections, that is, $\pi_i(x_1, x_2) = x_i$ for $(x_1, x_2) \in E$. Let W_1 be an open subset of E_1, and $\psi : W_1 \to [0, \infty)$ be a continuous function such that ψ is of class C^p on $\psi^{-1}(0, \infty)$. Denote $K = \psi^{-1}(0) \times \{0\}$. Then, there exists a C^p diffeomorphism h from $(W_1 \times E_2) \setminus K$ onto $W_1 \times E_2$ which satisfies $\pi_1 \circ h = h$, is the identity off of a certain open subset U of $W_1 \times E_2$. Specifically, the set U is defined as follows*

$$U := \{x = (x_1, x_2) \in W_1 \times E_2 \, : \, S(\psi(x_1), \omega(x_2)) < 1\},$$

where S is a certain C^∞ norm on $\mathbb{R}^2$ and $\omega : E_2 \to [0, \infty)$ is a certain (not necessarily symmetric) subadditive and positive-homogeneous functional of class C^p on $E_2 \setminus \{0\}$; see Lemmas 2.10 and 2.13 for precise definitions.

By a C^p smooth norm on E_2 we mean a (possibly nonequivalent) norm on E_2 which is of class C^p on $E_2 \setminus \{0\}$.
We will need to use the following three auxiliary results from [4, 5].

Lemma 2.9. *Let $F : (0, \infty) \longrightarrow [0, \infty)$ be a continuous function such that, for every $\beta \geq \alpha > 0$,*

$$F(\beta) - F(\alpha) \leq \frac{1}{2}(\beta - \alpha), \quad \text{and} \quad \limsup_{t \to 0^+} F(t) > 0.$$

Then there exists a unique $\alpha > 0$ such that $F(\alpha) = \alpha$.

Proof. See [4, Lemma 2]. □

It is not known whether every infinite-dimensional Banach space with a C^1 equivalent norm possesses a C^1 smooth non-complete norm.[1] The following lemma shows that for every Banach space with a C^p smooth norm there exists a kind of C^p *asymmetric non-complete subadditive functional* which successfully replaces the smooth non-complete norm in Bessaga's technique [14] for extracting points.

Lemma 2.10. *Let $(E_2, \|\cdot\|)$ be an infinite-dimensional Banach space which admits a (not necessarily equivalent) C^p smooth norm, where $p \in \mathbb{N} \cup \{\infty\}$. Then there exists a continuous function $\omega : E_2 \longrightarrow [0, \infty)$ which is C^p smooth on $E_2 \setminus \{0\}$ and satisfies the following properties:*

(1) *$\omega(x + y) \leq \omega(x) + \omega(y)$, and, consequently, $\omega(x) - \omega(y) \leq \omega(x - y)$, for every $x, y \in E_2$;*
(2) *$\omega(rx) = r\omega(x)$ for every $x \in E_2$, and $r \geq 0$;*
(3) *$\omega(x) = 0$ if and only if $x = 0$;*
(4) *$\omega(\sum_{k=1}^{\infty} z_k) \leq \sum_{k=1}^{\infty} \omega(z_k)$ for every convergent series $\sum_{k=1}^{\infty} z_k$ in $(E_2, \|\cdot\|)$; and*
(5) *for every $\varepsilon > 0$, there exists a sequence of vectors $(y_k) \subset E_2$ such that*
$$\omega(y_k) \leq \frac{\varepsilon}{4^{k+1}}, \quad ;$$
for every $k \in \mathbb{N}$, and
$$\liminf_{n\to\infty} \omega(y - \sum_{j=1}^{n} y_j) > 0$$
for every $y \in E_2$.

Notice that ω need not be a norm in E_2, as in general we have $\omega(x) \neq \omega(-x)$.

Proof. See [5, Lemma 2.3] □

Using the properties of the functional ω we can construct an *extracting curve* as follows.

Lemma 2.11. *Let $(E_2, \|\cdot\|)$ be a Banach space, and let ω be a functional satisfying conditions (1), (2), and (5) of Lemma 2.10. Then there exists a C^∞ curve $\gamma : (0, \infty) \longrightarrow E_2$ such that*

(1) *$\omega(\gamma(\alpha) - \gamma(\beta)) \leq \frac{1}{2}(\beta - \alpha)$ if $\beta \geq \alpha > 0$;*
(2) *$\limsup_{t\to 0^+} \omega(y - \gamma(t)) > 0$ for every $y \in E_2$; and*
(3) *$\gamma(t) = 0$ if $t \geq 1$.*
(4) *$\omega(\gamma(t)) \leq \varepsilon$ for every $t \in (0, +\infty)$.*

Proof. Let $\theta : [0, \infty) \longrightarrow [0, 1]$ be a non-increasing C^∞ function such that $\theta = 1$ on $[0, 1/2]$, $\theta = 0$ on $[1, \infty)$ and $\sup\{|\theta'(t)| : t \in [0, \infty)\} \leq 4$. Let us choose a sequence of vectors $(y_k) \subset E_2$ which satisfies condition (5) of Lemma 2.10, and define $\gamma : (0, \infty) \longrightarrow E_2$ by the following formula
$$\gamma(t) = \sum_{k=1}^{\infty} \theta(2^{k-1}t) y_k.$$
Observe for example that $\omega(\gamma(t)) \leq \sum_{k=1}^{\infty} \omega(y_k) \leq \varepsilon$. It is not difficult to check that this curve satisfies the properties of the statement. See [5, Lemma 2.5] for details. □

We will also need a technical tool (see, for instance, [8, Lemmas 2.27 and 2.28]) that allows us to obtain, on the product space $E_1 \times E_2$, a norm which preserves the smoothness properties that the corresponding norms of the factors may have. Notice that the natural formula $\left(\|x_1\|_1^2 + \|x_2\|_2^2\right)^{1/2}$ defines a C^1 norm in $E_1 \times E_2 \setminus \{0\}$ but, in general, this norm will not be C^2 on this set, even if $\|\cdot\|_1$ and $\|\cdot\|_2$ are C^∞ on $E_1 \setminus \{0\}$ and $E_2 \setminus \{0\}$, respectively, because the function $x_2 \mapsto \|x_2\|_2^2$ may not be C^2 smooth on all of E_2 even though it is C^∞ smooth on $E_2 \setminus \{0\}$. As a matter of fact, it is not

[1] For C^k with $k \geq 2$ in place of C^1, the answer to this question is positive; see [23].

difficult to show that, for every Banach space $(E, \|\cdot\|)$, if $\|\cdot\|^2$ is twice Fréchet differentiable at 0, then E is isomorphic to a Hilbert space; see, for instance [33, Exercise 10.4, pp. 475-476].

Definition 2.12. *We will say that a subset $\mathcal{S}$ of the plane $\mathbb{R}^2$ is a smooth square provided that:*

(i) *$\mathcal{S} \subset \mathbb{R}^2$ is a bounded, symmetric convex body with $0 \in int(\mathcal{S})$, and whose boundary $\partial\mathcal{S}$ is C^∞ smooth.*
(ii) *$(x, y) \in \partial\mathcal{S} \Leftrightarrow (\epsilon_1 x, \epsilon_2 y) \in \partial\mathcal{S}$ for each couple $(\epsilon_1, \epsilon_2) \in \{-1, 1\}^2$ (that is, $\mathcal{S}$ is symmetric about the cordinate axes).*
(iii) *$[-\frac{1}{2}, \frac{1}{2}] \times \{-1, 1\} \cup \{-1, 1\} \times [-\frac{1}{2}, \frac{1}{2}] \subset \partial\mathcal{S}$.*
(iv) *$\mathcal{S} \subset [-1, 1] \times [-1, 1]$.*

Of course, it is elementary to produce smooth squares in $\mathbb{R}^2$.
The following lemma enumerates the essential properties of a smooth square. Recall that the Minkowski functional of a convex body $\mathcal{A}$ such that $0 \in \text{int}(\mathcal{A})$ is defined by

$$\mu_{\mathcal{A}}(x) = \inf\{t > 0 : \frac{1}{t}x \in \mathcal{A}\}.$$

Lemma 2.13. *Let $\mathcal{S} \subset \mathbb{R}^2$ be a smooth square. Then its Minkowski functional $\mu_{\mathcal{S}} : \mathbb{R}^2 \to \mathbb{R}$ is a C^∞ smooth norm on $\mathbb{R}^2$ such that, for every $(x, y) \in \mathbb{R}^2$, we have*

(1) $\mu_{\mathcal{S}}(x, y) \leq |x| + |y| \leq 2\mu_{\mathcal{S}}(x, y)$;
(2) $\max(|x|, |y|) \leq \mu_{\mathcal{S}}(x, y) \leq 2\max(|x|, |y|)$;
(3) $\mu_{\mathcal{S}}(0, y) = |y|, \quad \mu_{\mathcal{S}}(x, 0) = |x|$;
(4) *For every $(x_0, y_0) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, there exists $\sigma > 0$ so that*

$$\mu_{\mathcal{S}}(x, y) = |x| \quad \text{if} \quad \max(|x - x_0|, |y|) \leq \sigma \quad \text{and} \quad \mu_{\mathcal{S}}(x, y) = |y| \quad \text{if} \quad \max(|x|, |y - y_0|) \leq \sigma.$$

(5) *The functions $(0, \infty) \ni t \to \mu_{\mathcal{S}}(x, ty)$ and $(0, \infty) \ni t \to \mu_{\mathcal{S}}(tx, y)$ are both nondecreasing.*

Note that property (4) (which is related to properties (iii) and (iv) of Definition 2.12) means that every sphere of $(\mathbb{R}^2, \mu_{\mathcal{S}})$ centered at the origin, which coincides with $\lambda(\partial\mathcal{S})$ for some $\lambda > 0$, is orthogonal to the coordinate axes and is locally flat on a neighborhood of the intersection of $\lambda(\partial\mathcal{S})$ with the lines $\{x = 0\} \cup \{y = 0\}$. By using this property it is easy to show that, for any couple of Banach spaces $(E_1, \|\cdot\|_1)$ and $(E_2, \|\cdot\|_2)$ with C^p smooth norms, the expression

$$\mu_{\mathcal{S}}\left(\|x_1\|_1, \|x_2\|_2\right)$$

defines an equivalent norm of class C^p in $E_1 \times E_2$.

Proof. Properties $(1) - (3)$ and (5) are easy to show. Let us prove (4). Assume for instance that $x_0 \neq 0$, and set $\sigma = |x_0|/4$. If $\max(|x - x_0|, |y|) \leq \sigma$, then we have

$$\frac{|y|}{|x|} \leq \frac{\sigma}{|x_0| - \sigma} = \frac{|x_0|/4}{|x_0| - |x_0|/4} = \frac{1}{3} < \frac{1}{2},$$

hence

$$\left(\frac{x}{|x|}, \frac{y}{|x|}\right) \in \{-1, 1\} \times \left[-\frac{1}{2}, \frac{1}{2}\right] \subset \partial\mathcal{S},$$

and it follows that $\mu_{\mathcal{S}}(x, y) = |x|$. $\square$

The lemma below shows how, with the help of a smooth square, we can combine the given C^p smooth function $\psi : W_1 \to [0, \infty)$ together with the C^p smooth functional $\omega : E_2 \to [0, \infty)$ obtained in Lemma 2.10, in order to obtain a C^p smooth function on $W_1 \times E_2$ which behaves more or less like $\psi(x_1) + \omega(x_2)$ (or, equivalently, like $\left(\psi(x_1)^2 + \omega(x_2)^2\right)^{1/2}$).

Lemma 2.14. *Let $E = E_1 \times E_2$ be a Banach space, $\rho_1 : E_1 \to [0, \infty)$ and $\rho_2 : E_2 \to [0, \infty)$ continuous functions which are of class C^p on $E_1 \setminus \rho_1^{-1}(0)$ and $E_2 \setminus \rho_1^{-1}(0)$, respectively. Then, for any smooth square $\mathcal{S}$ of $\mathbb{R}^2$, the function $\rho : E_1 \times E_2 \to [0, \infty)$ defined by*

$$\rho(x) = \rho(x_1, x_2) = \mu_{\mathcal{S}}(\rho_1(x_1), \rho_2(x_2)), \quad x = (x_1, x_2) \in E_1 \times E_2,$$

is continuous on E and of class C^p on $E \setminus \big(\rho_1^{-1}(0) \times \rho_2^{-1}(0)\big)$.
The same is true if we replace E_1 with an open subset W_1 of E_1.

Proof. It is clear that ρ is continuous on E, and that it is C^p smooth on $\{(x_1, x_2) \in E : \rho_1(x_1) \neq 0 \neq \rho_2(x_2)\}$. Let us see that ρ is also C^p smooth on a neighborhood of the set

$$(\{(x_1, x_2) \in E : \rho_1(x_1) = 0\} \cup \{(x_1, x_2) \in E : \rho_2(x_2) = 0\}) \setminus \big(\rho_1^{-1}(0) \times \rho_2^{-1}(0)\big)\,.$$

Suppose for instance that $\rho_1(x_1) \neq 0 = \rho_2(x_2)$. Then, by continuity of ρ_1, ρ_2 and by property (4) of Lemma 2.13, there exist a neighborhood U of the point (x_1, x_2) such that $U \subset \{(y_1, y_2) \in E : \rho_1(y_1) \neq 0\}$ and

$$\rho(y_1, y_2) = \rho_1(y_1)$$

for all $(y_1, y_2) \in U$. It follows that ρ is of class C^p on U. The case $\rho_1(x_1) = 0 \neq \rho_2(x_2)$ can be treated similarly. □

Now we are ready to prove Theorem 2.8.

Proof of Theorem 2.8. From now on we will fix a smooth square $\mathcal{S}$ on $\mathbb{R}^2$, and we will denote

$$S = \mu_{\mathcal{S}}.$$

Thus, by Lemma 2.14 applied to $\rho_1 = \psi$ and $\rho_2 = \omega$ (recall that ω was constructed in Lemma 2.10), the function

$$\rho(x_1, x_2) := S(\psi(x_1), \omega(x_2))$$

is continuous on $W_1 \times E_2$ and of class C^p on $(W_1 \times E_2) \setminus \big(\psi^{-1}(0) \times \{0\}\big)$.
Let us define $h : (W_1 \times E_2) \setminus K \to E$ by

$$h(x_1, x_2) = (x_1,\, x_2 + \gamma \circ \rho(x_1, x_2)) = (x_1,\, x_2 + \gamma\,(S(\psi(x_1), \omega(x_2))))\,, \quad (x_1, x_2) \in (W_1 \times E_2) \setminus K,$$

where γ is provided by Lemma 2.11. Note that

$$K = \psi^{-1}(0) \times \{0\} = \rho^{-1}(0).$$

Let (y_1, y_2) be an arbitrary point of $W_1 \times E_2$, and let $F_{y_1,y_2} : (0, \infty) \longrightarrow [0, \infty)$ be defined by

$$F_{y_1,y_2}(\alpha) = \rho(y_1, y_2 - \gamma(\alpha)) = S\,(\psi(y_1), \omega(y_2 - \gamma(\alpha))) \tag{2.5}$$

for $\alpha > 0$. Let us see that $F_{y_1,y_2}(\alpha)$ satisfies the conditions of Lemma 2.9. As for the first condition, we consider two cases: if $\omega(y_2 - \gamma(\beta)) \leq \omega(y_2 - \gamma(\alpha))$ then, since the function $(0, \infty) \ni t \mapsto S(\psi(y_1), t)$ is increasing (see condition (5) of Lemma 2.13), we have that

$$S(\psi(y_1), \omega(y_2 - \gamma(\beta))) \leq S(\psi(y_1), \omega(y_2 - \gamma(\alpha))),$$

and therefore

$$F_{y_1,y_2}(\beta) - F_{y_1,y_2}(\alpha) \leq 0 \leq \frac{1}{2}(\beta - \alpha)$$

trivially for all $\beta \geq \alpha > 0$. Otherwise, we have $\omega(y_2 - \gamma(\beta)) - \omega(y_2 - \gamma(\alpha)) > 0$, and therefore, using the fact that S is a norm in $\mathbb{R}^2$, condition (3) of Lemma 2.13, the properties of the functional ω, and condition (1) of Lemma 2.11, we obtain

$$S(\psi(y_1), \omega(y_2 - \gamma(\beta))) - S(\psi(y_1), \omega(y_2 - \gamma(\alpha))) \leq S\,(0, \omega(y_2 - \gamma(\beta)) - \omega(y_2 - \gamma(\alpha))) =$$
$$\omega(y_2 - \gamma(\beta)) - \omega(y_2 - \gamma(\alpha)) \leq \omega\big(y_2 - \gamma(\beta) - (y_2 - \gamma(\alpha))\big) = \omega(\gamma(\alpha) - \gamma(\beta)) \leq \frac{1}{2}(\beta - \alpha)$$

for every $\beta \geq \alpha > 0$. In either case we have that

$$F_{y_1,y_2}(\beta) - F_{y_1,y_2}(\alpha) \leq \frac{1}{2}(\beta - \alpha) \tag{2.6}$$

for all $\beta \geq \alpha > 0$.
On the other hand, by condition (2) of Lemma 2.11 we know that

$$\limsup_{\alpha \to 0^+} \omega(y_2 - \gamma(\alpha)) > 0,$$

and therefore, by condition (2) of Lemma 2.13, we have

$$\limsup_{\alpha \to 0^+} F_{y_1,y_2}(\alpha) = \limsup_{\alpha \to 0^+} S(\psi(y_1), \omega(y_2 - \gamma(\alpha))) \geq \limsup_{\alpha \to 0^+} \omega(y_2 - \gamma(\alpha)) > 0,$$

so that F_{y_1,y_2} also satisfies the second condition of Lemma 2.9.
Then, applying Lemma 2.9, we deduce that the equation $F_{y_1,y_2}(\alpha) = \alpha$ has a unique solution. This means that, for each $(y_1, y_2) \in W_1 \times E_2$, a number $\alpha(y_1, y_2) > 0$ with the property

$$S\left(\psi(y_1), \omega\big(y_2 - \gamma(\alpha(y_1, y_2))\big)\right) = \rho\left(y_1, y_2 - \gamma(\alpha(y_1, y_2))\right) = \alpha(y_1, y_2), \tag{2.7}$$

is uniquely determined.
Let us see why these facts imply that h is a C^p diffeomorphism from $W_1 \times E_2 \setminus K$ onto $W_1 \times E_2$. Assume first that $h(x_1, x_2) = (y_1, y_2) = h(z_1, z_2)$, that is to say $x_1 = y_1 = z_1$, and

$$x_2 + \gamma\left(\rho(y_1, x_2)\right) = y_2 = z_2 + \gamma\left(\rho(y_1, z_2)\right), \tag{2.8}$$

or equivalently

$$(y_1, x_2) = (y_1, y_2 - \gamma(\rho(y_1, x_2)) \quad \text{and} \quad (y_1, z_2) = (y_1, y_2 - \gamma(\rho(y_1, z_2)) .$$

Applying ρ to all sides of the above equations and using (2.5), we obtain

$$\rho(y_1, x_2) = \rho\left(y_1, y_2 - \gamma(\rho(y_1, x_2))\right) = F(\rho(y_1, x_2)) \text{ and } \rho(y_1, z_2) = \rho\left(y_1, y_2 - \gamma(\rho(y_1, z_2))\right) = F(\rho(y_1, z_2)).$$

It follows that both $\rho(y_1, x_2)$ and $\rho(y_1, z_2)$ are fixed points of F_{y_1,y_2}. By the uniqueness of the fixed point, we conclude that

$$\alpha(y_1, y_2) = \rho(y_1, x_2) = \rho(y_1, z_2).$$

Now applying (2.8), we have

$$x_2 = z_2 \quad \text{and} \quad x_2 = y_2 - \gamma(\alpha(y_1, y_2)).$$

This shows that h is one to one, and also that, given $(y_1, y_2) \in W_1 \times E_2$ we have

$$h\left(y_1, y_2 - \gamma(\alpha(y_1, y_2))\right) = (y_1, y_2).$$

Hence h is also onto, and $h^{-1} : W_1 \times E_2 \to W_1 \times E_2 \setminus K$ is given by

$$h^{-1}(y_1, y_2) = \left(y_1, y_2 - \gamma(\alpha(y_1, y_2))\right).$$

It is clear that h is of class C^p. In order to see that h^{-1} is C^p as well, let us define $\Phi : W_1 \times E_2 \times (0, \infty) \longrightarrow \mathbb{R}$ by

$$\Phi(y_1, y_2, \alpha) = \alpha - S(\psi(y_1), \omega(y_2 - \gamma(\alpha))) = \alpha - \rho(y_1, y_2 - \gamma(\alpha)).$$

On the one hand, according to (2.7) and the fact that S is a norm in $\mathbb{R}^2$, we have

$$(\psi(y_1), \omega(y_2 - \gamma(\alpha(y_1, y_2))) \neq (0, 0)$$

for every $(y_1, y_2) \in W_1 \times E_2$. Since S is C^∞ smooth away from $(0, 0)$, this implies that Φ is C^p smooth on a neighborhood of every point $(y_1, y_2, \alpha(y_1, y_2))$ in $W_1 \times E_2 \times (0, \infty)$. On the other hand, we know that $F_{y_1,y_2}(\beta) - F_{y_1,y_2}(\alpha) \leq \frac{1}{2}(\beta - \alpha)$ for $\beta \geq \alpha > 0$, which implies that $F'_{y_1,y_2}(\alpha) \leq \frac{1}{2}$ for every α in a neighborhood of $\alpha(y_1, y_2)$, and therefore

$$\frac{\partial \Phi(y_1, y_2, \alpha)}{\partial \alpha} = 1 - F'_{y_1,y_2}(\alpha) \geq 1 - 1/2 > 0.$$

Hence, by the implicit function theorem, the mapping $(y_1, y_2) \to \alpha(y_1, y_2)$ is of class C^p on $W_1 \times E_2$, and, since γ is C^p smooth, so is h^{-1}.
Finally, it is obvious that $\pi_1 \circ h = \pi_1$, and the fact that $\gamma(t) = 0$ whenever $t \geq 1$ implies that h is the identity off of the set $\{x = (x_1, x_2) \in W_1 \times E_2 \,:\, S(\psi(x_1), \omega(x_2)) < 1\}$. □

2.3. **Extracting pieces of continuous graphs of infinite codimension.** Now we will prove the following extractibility result.

Theorem 2.15. *Let $E = E_1 \times E_2$ be a product of Banach spaces such that E_1 admits C^p smooth partitions of unity and E_2 admits a C^p (not necessarily equivalent) norm. Assume that X_1 is a closed subset of E_1, that $f : E_1 \to E_2$ is a continuous mapping, and that E_2 is infinite-dimensional. Define*

$$X = \{(x_1, x_2) \in E_1 \times E_2 \,:\, x_1 \in X_1, x_2 = f(x_1)\}.$$

Let U be an open subset of E and $\varepsilon > 0$. Then there exists a C^p diffeomorphism g from $E \setminus X$ onto $E \setminus (X \setminus U)$ such that g is the identity on $(E \setminus U) \setminus X$ and moves no point more than ε.

Proof. We may of course assume $U \cap X \neq \emptyset$ (as otherwise the result holds trivially with g equal to the identity map).

Claim 2.16. *It is sufficient to prove the result for $f = 0$ and such that the extracting diffeomorphism preserves the first coordinate, that is $g(x_1, x_2) = (x_1, \pi_2(g(x_1, x_2)))$.*

Proof. Let h and φ be homeomorphisms given by Theorem 2.1 such that $||\varphi^{-1}(x) - h^{-1}(x)|| \leq \frac{\varepsilon}{2}$ for every $x \in E$. By the uniform continuity of $h^{-1}(x_1, x_2)$ with respect to the second variable $x_2 \in E_2$, we may choose $\delta > 0$ such that if $||(x_1, x_2) - (x_1, x'_2)|| \leq \delta$ then

$$||h^{-1}(x_1, x_2) - h^{-1}(x_1, x'_2)|| \leq \frac{\varepsilon}{2}.$$

Assuming the result is true for $f = 0$ we can find a C^p diffeomorphism $g : E \setminus (X_1 \times \{0\}) \to E \setminus ((X_1 \times \{0\}) \setminus h(U))$ such that g is the identity on $(E \setminus h(U)) \setminus (X_1 \times \{0\})$, moves no point more than δ and preserves the first coordinate. Then the composition

$$\varphi^{-1} \circ g \circ h : E \setminus X \to (E \setminus (X \setminus U)$$

defines a C^p diffeomorphism with the required properties. Observe that

$$||g(h(x)) - h(x)|| = ||(x_1, \pi_2(g(h(x)))) - (x_1, \pi_2(h(x)))|| \leq \delta,$$

hence

$$||\varphi^{-1}(g(h(x))) - x|| \leq ||\varphi^{-1}(g(h(x))) - h^{-1}(g(h(x)))|| + ||h^{-1}(g(h(x))) - x|| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for every $x \in E \setminus X$. □

So it will be enough to see that if X_1 is a closed subset of E_1 and W is an open subset of E such that $W \cap X_1 \times \{0\} \neq \emptyset$ then there exists a C^p diffeomorphism g from $E \setminus (X_1 \times \{0\})$ onto $E \setminus ((X_1 \times \{0\}) \setminus W)$ such that g is the identity on $(E \setminus W) \setminus (X_1 \times \{0\})$ and moves no point more than δ.
To this end we next construct some auxiliary functions following Renz's strategy [58, pp. 54-59]. In what follows ω will denote the smooth asymmetric subadditive functional on E_2 given by Lemma 2.10.

Lemma 2.17. *There exists a continuous function $\varphi : E_1 \to [0, \frac{\delta}{2}]$ such that:*

(1) *φ is of class C^p on $E_1 \setminus \partial\varphi^{-1}(0)$.*
(2) *$W \cap (X_1 \times \{0\}) \subset \{(x_1, x_2) \in E : \|x_2\| < \varphi(x_1)\} \subset W$.*

Proof. Let $\pi_1 : E \to E_1$ denote the canonical projection defined by $\pi_1(x_1, x_2) = x_1$. The set

$$W_1 := \pi_1 (W \cap (E_1 \times \{0\}))$$

is open in E_1, and the function $G : E_1 \to [0, \infty)$ defined by

$$G(x_1) = \min\{\frac{\delta}{2}, \operatorname{dist}((x_1, 0), E \setminus W)\}$$

is continuous and satisfies that $G > 0$ on W_1 and $G = 0$ on $\pi_1 ((E_1 \times \{0\}) \setminus W)$. Since E_1 has C^p smooth partitions of unity and G is continuous and strictly positive on W_1, we can find a C^p smooth function F on W_1 such that

$$0 < \frac{1}{4}G(x_1) < F(x_1) < \frac{1}{2}G(x_1)$$

for every $x_1 \in W_1$. Now let us define $\varphi : E_1 \to [0, 1]$ by

$$\varphi(x_1) = \begin{cases} F(x_1) & \text{if } x_1 \in W_1 \\ 0 & \text{if } x_1 \in E_1 \setminus W_1. \end{cases}$$

It is immediately seen that φ is continuous, and of course φ is of class C^p on $E_1 \setminus \partial\varphi^{-1}(0) = W_1 \cup \operatorname{int}(E_1 \setminus W_1)$. Since $\varphi(x_1) = F(x_1) > 0$ for all $x_1 \in W_1$, it is obvious that

$$W \cap (X_1 \times \{0\}) \subset W_1 \times \{0\} \subset \{(x_1, x_2) \in E : \|x_2\| < \varphi(x_1)\}.$$

On the other hand, if $\|x_2\| < \varphi(x_1)$ then observe first that $x_1 \in W_1$ (because if $\varphi(x_1) = 0$ the inequality is impossible). We then must have $(x_1, x_2) \in W$, as otherwise we would get

$$\operatorname{dist}((x_1, 0), E \setminus W) \leq \operatorname{dist}((x_1, 0), (x_1, x_2)) + \operatorname{dist}((x_1, x_2), E \setminus W) =$$
$$\|x_2\| + 0 = \|x_2\| < \varphi(x_1) < \frac{1}{2}G(x_1) \leq \frac{1}{2}\operatorname{dist}((x_1, 0), E \setminus W),$$

which is absurd. This shows that $\{(x_1, x_2) \in E : \|x_2\| < \varphi(x_1)\} \subset W$ and concludes the proof of the lemma. □

We will also need to use a diffeomorphism h_2 of E_2 onto itself which carries the unit ball of E_2 onto the convex body $\{x_2 \in E_2 : \omega(x) \leq 1\}$ and such that $h_2(0) = 0$. The existence of h_2 is ensured by the following lemma. We say that a convex body U which contains 0 as an interior point is *radially bounded* provided that for every $x \in U$ the set $\{tx : t \in [0, \infty)\} \cap U$ is bounded.

Lemma 2.18. *Let X be a Banach space, and let U_1, U_2 be radially bounded, C^p smooth convex such that the origin is an interior point of both U_1 and U_2. Then there exists a C^p diffeomorphism $g : X \longrightarrow X$ such that $g(U_1) = U_2$, $g(0) = 0$, and $g(\partial U_1) = \partial U_2$.*

Proof. If U and V are C^p smooth, radially bounded convex bodies such that the origin is an interior point of both U and V, and we additionally assume that $U \subseteq V$, such a diffeomorphism can be constructed as follows: let $\theta(t)$ be a non-decreasing real function of class C^∞ defined for $t > 0$, such that $\theta(t) = 0$ for $t \leq 1/2$ and $\theta(t) = 1$ for $t \geq 1$, and define

$$g(x) = \left(\theta(\mu_U(x))\frac{\mu_U(x)}{\mu_V(x)} + 1 - \theta(\mu_U(x))\right)x$$

for $x \neq 0$, and $g(0) = 0$. Here μ_A denotes the Minkowski functional of A.
In the general case, let $U = \{x \in X : \mu_{U_1}(x) + \mu_{U_2}(x) \leq 1\}$, then $U \subseteq U_j$, for $j = 1, 2$, and there exist diffeomorphisms $g_1, g_2 : X \to X$ such that $g_j(U) = U_j$ and $g_j(\partial U) = \partial U_j$, $j = 1, 2$. Then $g = g_2 \circ g_1^{-1}$ does the job. See [29] for details. □

The following lemma is an immediate consequence of the existence of partitions of unity in E_1.

Lemma 2.19. *Suppose that E_1 is a Banach space with C^p smooth partitions of unity, and let X_1 be a closed subset of E_1. Then there exists a continuous function $\eta : E_1 \to [0, \infty)$ such that:*
(1) $X_1 = \eta^{-1}(0)$;
(2) *η is of class C^p on $E_1 \setminus X_1$.*

We are ready to proceed with the proof of Theorem 2.15. Let η be a function as in the statement of Lemma 2.19, and pick a diffeomorphism $h_2 : E_2 \to E_2$ such that $h_2(0) = 0$ and

$$h_2\left(\{x_2 \in E_2 : \|x_2\| \leq 1\}\right) = \{x_2 \in E_2 \, : \, \omega(x_2) \leq 1\}.$$

Let us define

$$A := \varphi^{-1}\left((0, 1]\right) \times E_2 = W_1 \times E_2,$$

and

$$\Phi(x_1, x_2) = \left(x_1, h_2\left(\frac{1}{\varphi(x_1)} x_2\right)\right), \;\; (x_1, x_2) \in A.$$

It is clear that $\Phi : A \to A$ is a C^p diffeomorphism, with inverse

$$\Phi^{-1}(y_1, y_2) = \left(y_1, \varphi(y_1) h_2^{-1}(y_2)\right),$$

and also that

$$\Phi\left((X_1 \times \{0\}) \cap A\right) = (X_1 \times \{0\}) \cap A.$$

Next let us define $\psi : \varphi^{-1}\left((0, 1]\right) = \pi_1(A) = W_1 \to [0, \infty)$ by

$$\psi(x_1) = \frac{\eta(x_1)}{\varphi(x_1)},$$

and notice that ψ is continuous, that

$$\psi^{-1}(0) = \pi_1\left((X_1 \times \{0\}) \cap A\right) = X_1 \cap W_1,$$

and that ψ is of class C^p outside $\psi^{-1}(0)$. By Theorem 2.8 we can find a C^p diffeomorphism H from $A \setminus (X_1 \times \{0\})$ onto A such that H is the identity outside $\{(x_1, x_2) \in A \setminus (X_1 \times \{0\}) \, : \, S(\psi(x_1), \omega(x_2)) < 1\}$, where S is a smooth square. Since $\Phi : A \to A$ is a C^p diffeomorphism which takes $(X_1 \times \{0\}) \cap A$ onto itself, we have that the composition $\Phi^{-1} \circ H \circ \Phi$ defines a C^p diffeomorphism from $A \setminus (X_1 \times \{0\})$ onto A. Now we extend this diffeomorphism outside $A \setminus (X_1 \times \{0\})$ by defining $g : E \setminus (X_1 \times \{0\}) \to E$ by

$$g(x) = \begin{cases} \Phi^{-1} \circ H \circ \Phi(x) & \text{if } x \in A \setminus (X_1 \times \{0\}) \\ x & \text{if } x \in E \setminus (A \cup (X_1 \times \{0\})). \end{cases}$$

This mapping is clearly a bijection. Thus, in order to see that g is a C^p diffeomorphism, it is enough to see that g is locally a C^p diffeomorphism. We already know this is so for all points of $E \setminus \big(\partial(A \setminus (X_1 \times \{0\})) \cup (X_1 \times \{0\})\big)$. Let us show that this is also true for every point (x_1, x_2) of $\partial(A \setminus (X_1 \times \{0\})) \setminus (X_1 \times \{0\})$. We have $\varphi(x_1) = 0$, and also either $\eta(x_1) > 0$ or $\|x_2\| > 0$. Then

$$\lim_{A \ni (y_1, y_2) \to (x_1, x_2)} \max\left\{\frac{\eta(y_1)}{\varphi(y_1)}, \frac{\|y_2\|}{\varphi(y_1)}\right\} = \infty,$$

hence there exists a neighborhood V of (x_1, x_2) in $E \setminus (X_1 \times \{0\})$ such that

$$\max\left\{\frac{\eta(y_1)}{\varphi(y_1)}, \frac{\|y_2\|}{\varphi(y_1)}\right\} > 1 \quad \text{for all } (y_1, y_2) \in V \cap A.$$

By Lemma 2.13(2) it follows that

$$S\left(\psi \circ \pi_1 \circ \Phi(y), \omega \circ \pi_2 \circ \Phi(y)\right) > 1 \quad \text{for all } y \in V \cap A,$$

hence that H is the identity on $\Phi(V\cap A)$, and consequently that g is the identity on V, and in particular a C^p diffeomorphism locally at (x_1,x_2). Thus $g : E\setminus(X_1\times\{0\})\to E$ is a C^p diffeomorphism.
Furthermore, if $\|x_2\|\geq\varphi(x_1)$, $(x_1,x_2)\in A\setminus(X_1\times\{0\})$, then we have $\omega(\pi_2\circ\Phi(x_1,x_2))\geq 1$. Hence, as above by Lemma 2.13(2), we conclude that $H(\Phi(x_1,x_2))=\Phi(x_1,x_2)$, and it follows that $g(x_1,x_2)=(x_1,x_2)$. Thus g is the identity off of the set $\{(x_1,x_2)\in E\setminus(X_1\times\{0\}) : \|x_2\|<\varphi(x_1)\}$. Since

$$\{(x_1,x_2)\in E\setminus(X_1\times\{0\}) : \|x_2\|<\varphi(x_1)\}\subset(E\setminus W)\setminus(X_1\times\{0\}),$$

g is the identity off of the set $(E\setminus W)\setminus(X_1\times\{0\})$ as well.
Finally let us check that g does not move any point more than δ. We know that if g moves a point $(x_1,x_2)\in E\setminus(X_1\times\{0\})$ then $||x_2||<\varphi(x_1)$, and also that g_2 only moves the second coordinate x_2, that is $g(x_1,x_2)=(x_1,\pi_2(g(x_1,x_2)))$. Hence

$$||g(x_1,x_2)-(x_1,x_2)||\leq||\pi_2(g(x_1,x_2))-x_2||\leq||\pi_2(g(x_1,x_2))||+||x_2||\leq\varphi(x_1)+\varphi(x_1)\leq\delta.$$

The proof of Theorem 2.15 is complete. □

2.4. **Patching local diffeomorphisms together.** Throughout this section E will be a Banach space. Our goal in this section is to complete the proof of Theorem 1.4.
First let us introduce the following definitions.

Definition 2.20. We will say that a subset X of E has the strong C^p extraction property with respect to an open set U if $X\subseteq U$, X is relatively closed in U, and for every open set $V\subseteq U$ and every subset $Y\subseteq X$ relatively closed in U there exists a C^p diffeomorphism φ from $U\setminus Y$ onto $U\setminus(Y\setminus V)$ which is the identity on $(U\setminus V)\setminus Y$. If in addition for any $\varepsilon>0$ we can require the diffeomorphism φ not to move any point more than ε, we will say that X has the ε-strong C^p extraction property with respect to U.
We will also say that such a closed set X has locally the strong (or the ε-strong) local C^p extraction property if for every point $x\in X$ there exists an open neighborhood U_x of x such that $X\cap\overline{U_x}$ has the strong (ε-strong, respectively) C^p extraction property with respect to every open set U with $X\cap\overline{U_x}\subseteq U$. (Equivalently, there exists an open neighborhood U_x of x such that $X\cap U_x$ has the strong (ε-strong respectively) C^p extraction property with respect to every open set U for which $X\cap U_x$ is a relatively closed subset of U.)

Remark 2.21. *Let (U,W), $W\subset U$, be a pair of open sets in a Banach space E. We say that (U,W) has the strong C^p expansion property if, for every open subsets V and U' of U, $W\subset U'$, there exists a C^p diffeomorphism $U'\cap V\to V$ which, by letting $\varphi(x)=x$ for $x\in U'\setminus V$, extends to C^p diffeomorphism $\varphi : U'\to U'\cup V$.*
In particular, letting $U'=W$, there exists a C^p diffeomorphism $W\cap V\to V$ which extends to a C^p diffeomorphism of W onto $W\cup V$ via the identity off $W\cap V$. Hence, W is smoothly expanded to $W\cup V$; this justifies the term of C^p expansion. Should this expansion be valid for all open sets U', $W\subset U'$, then we would have the strong C^p expansion property.
Notice that a relatively closed subset X has the strong C^p extraction property with respect to U if and only if $(U,W)=(U,U\setminus X)$ has the strong C^p expansion property.

We say that an open subset W of E has locally the strong C^p expansion property if every $x\in E\setminus W$ has an open neighborhood U_x such that $(U,U_x\cap W)$ has the strong expansion property for every open set $U\supset U_x\cap W$.
Notice that a closed set X has locally the strong C^p extraction property if and only if $W=E\setminus X$ has locally the strong C^p expansion property.

Some basic properties that can be derived from Definition 2.20 are listed in the following lemma.

Lemma 2.22. *Let us suppose that $X \subset E$ has the ε-strong C^p extraction property with respect to an open set U of E. Then:*

1. *For every closed set $Y \subseteq X$, Y has the ε-strong C^p extraction property with respect to U;*
2. *For every open subset $U' \subseteq U$, $X \cap U'$ has the ε-strong C^p extraction property with respect to U'.*
3. *If h is a C^p diffeomorphism defined on U and such that $h(U)$ is open, then $h(X)$ has the strong C^p extraction property with respect to $h(U)$.*

Proof.
(1) This follows directly from the definition.

(2) Take an open subset $V' \subseteq U'$, a subset $Y \subseteq X \cap U'$ relatively closed in U'. Since X has the strong C^p extraction property with respect to U there exists a C^p diffeomorphism φ from $U \setminus \overline{Y}$ onto $U \setminus (\overline{Y} \setminus V')$ which is the identity on $(U \setminus V') \setminus \overline{Y}$. When restricting φ to $U' \setminus Y$ we actually get a C^p diffeomorphism from $U' \setminus Y$ onto $U' \setminus (Y \setminus V')$ which is the identity on $(U' \setminus V') \setminus Y$.

(3) Take an open subset V of $h(U)$, a subset $Y \subseteq h(X)$ relatively closed in $h(U)$. Since X has the strong C^p extraction property with respect to U and $h^{-1}(Y)$ is relatively closed in U, there exists a C^p diffeomorphism φ from $U \setminus h^{-1}(Y)$ onto $U \setminus (h^{-1}(Y) \setminus h^{-1}(V))$ which is the identity on $(U \setminus h^{-1}(V)) \setminus h^{-1}(Y)$. Then the mapping

$$g := h \circ \varphi \circ h^{-1} : h(U) \setminus Y \longrightarrow h(U) \setminus (Y \setminus V)$$

is a surjective C^p diffeomorphism which restricts to the identity on $(h(U) \setminus V) \setminus Y$. □

Remark 2.23. In Lemma 2.22 (3) we do not have in general the ε-strong C^p extraction property of $h(X)$ with respect to $h(U)$, but we still have the following: suppose h does not move any points more than some $\varepsilon > 0$. For every $\eta > 0$ in the proof of Lemma 2.22 (3) we can assume that φ does not move any point more than η. Hence $g \circ h = (h \circ \varphi \circ h^{-1}) \circ h = h \circ \varphi$ does not move any point more than $\varepsilon + \eta$.

Let us state the main result of this section, which is crucial in the proof of Theorem 1.4 provided underneath.

Theorem 2.24. *Let E be a Banach space and X be a closed subset of E which has locally the ε-strong C^p extraction property. Let U be an open subset of E and $\mathcal{G} = \{G_r\}_{r \in \Omega}$ be an open cover of E. Then there exists a C^p diffeomorphism g from $E \setminus X$ onto $E \setminus (X \setminus U)$ which is the identity on $(E \setminus U) \setminus X$ and is limited by $\mathcal{G}$.*

Proof of Theorem 1.4. Let us show that X from the statement of Theorem 1.4 has locally the strong C^p extraction property.
To this end, fix $x \in X$ and choose a neighborhood U_x such that $X \cap U_x \subset G(f_x)$; we can assume that $U_x = \overline{U_x}$. Further, we can assume that f_x is defined and continuous on the whole $E_{(1,x)}$. We will show that $X' := X \cap \overline{U_x}$ has the ε-strong C^p extraction property with respect to every open set U with $X' \subseteq U$. Notice that $X' = G(f_x | X'_1)$ for a certain closed $X'_1 \subset E_{(1,x)}$. Furthermore, if $Y' \subset X'$ is relatively closed in U, then Y' is closed in X'. Hence, $Y' = G(f_x | Y'_1)$ for a certain closed $Y'_1 \subset X'_1 \subset E_{(1,x)}$. Let V be an open subset of U. Take now $\varepsilon > 0$ and apply Theorem 2.15 to $E_1 := E_{(1,x)}$, $E_2 := E_{(2,x)}$, $f := f_x$, $X := Y'$, and V (in place of U) to obtain a C^p diffeomorphism g from $E \setminus Y'$ onto $E \setminus (Y' \setminus V)$ such that g is the identity on $(E \setminus V) \setminus Y'$ and moves no point more than ε. Then $\varphi = g|U$ is as required in the definition of the ε-strong C^p extraction of X' with respect to U.
Now, an application of Theorem 2.24 concludes our proof. □

The remaining part of this section is devoted to proving Theorem 2.24. Firstly, the fact that we are working with a set X that has locally the strong C^p extraction property, and the requirement that our final C^p diffeomorphism must be limited by a given open cover $\mathcal{G}$ forces us to employ *good* refinements of covers of the Banach space E. In the separable case star-finite refinements provide an adequate tool to face the problem (see West [67]). Recall that a cover is said to be *star-finite* provided that each element of the cover intersects at most finitely many others. However, in the nonseparable case, getting a star-finite refinement of an open cover, in general, is not possible. We will use sigma-discrete refinements as shown in the following.

Lemma 2.25. *Let E be a Banach space and X be a closed subset of E which has locally the ε-strong C^p extraction property. Let $\mathcal{G} = \{G_r\}_{r\in\Omega}$ be an open cover of E, where the cardinality of the indexing set Ω is the density of E. Then there exist collections $\{X_i\}_{i\geq 1}$, $\{W_i\}_{i\geq 1}$, $\{V_i\}_{i\geq 1}$, such that:*

(1) *$X_i \subseteq W_i \subseteq \overline{W_i} \subseteq V_i$ for all $i \in \mathbb{N}$;*
(2) *$\{V_i\}_{i\geq 1}$ and $\{W_i\}_{i\geq 1}$ are star-finite open covers of E;*
(3) *$\{X_i\}_{i\geq 1}$ is a cover of X by closed subsets of X;*
(4) *Each W_i and V_i admits an open discrete cover $\{W_{i,r}\}_{r\in\Omega}$ and $\{V_{i,r}\}_{r\in\Omega}$, repectively; more precisely,*
$$W_i = \bigcup_{r\in\Omega} W_{i,r} \quad \textit{and} \quad V_i = \bigcup_{r\in\Omega} V_{i,r},$$
$$\overline{W_{i,r}} \subseteq V_{i,r} \quad \textit{for every} \quad r \in \Omega,$$
and
$$\operatorname{dist}(V_{i,r}, V_{i,r'}) \geq \frac{1}{2^{i+1}} \quad \textit{for every} \quad r, r' \in \Omega, r \neq r';$$
(5) *$\{W_{i,r}\}_{i\geq 1, r\in\Omega}$ and $\{V_{i,r}\}_{i\geq 1, r\in\Omega}$ are open refinements of $\mathcal{G}$;*
(6) *Each X_i can be written as $X_i = \bigcup_{r\in\Omega} X_{i,r}$, where $X_{i,r}$ is a closed subset of X satisfying the following requirements*
$$X_{i,r} \subseteq W_{i,r} \subseteq \overline{W_{i,r}} \subseteq V_{i,r}$$
and
$$X_{i,r} \quad \textit{has the } \varepsilon - \textit{strong} \quad C^p \quad \textit{extraction property with respect to} \quad V_{i,r}.$$

Proof. For each $x \in E$, let U_x be an open neighborhood of x such that $\overline{U_x} \subseteq G$ for some $G \in \mathcal{G}$ and also satisfying that

(1) if $x \in X$ then $X \cap \overline{U_x}$ has the ε-strong C^p extraction property with respect to every open set U with $X \cap \overline{U_x} \subseteq U$, and
(2) if $x \notin X$ then $X \cap \overline{U_x} = \emptyset$.

Since the cardinality of Ω is the density of E, we can extract a subcover $\mathcal{U} = \{U_r : r \in \Omega\}$ from $\{U_x : x \in E\}$. Now we use a result of Rudin [61] (see also [38, p. 390]) to obtain two open refinements $\{A_{j,r}\}_{j\geq 1, r\in\Omega}$ and $\{B_{j,r}\}_{j\geq 1, r\in\Omega}$ of $\mathcal{U}$ such that

(1) $A_{j,r} \subseteq B_{j,r} \subseteq U_r$ for all $j \in \mathbb{N}$ and $r \in \Omega$;
(2) $\operatorname{dist}(A_{j,r}, E \setminus B_{j,r}) \geq \frac{1}{2^j}$ for all $j \in \mathbb{N}$ and $r \in \Omega$;
(3) $\operatorname{dist}(B_{j,r}, B_{j,r'}) \geq \frac{1}{2^{j+1}}$ for all $j \in \mathbb{N}$ and $r, r' \in \Omega$, $r \neq r'$;
(4) Letting $A_j = \bigcup_{r\in\Omega} A_{j,r}$ and $B_j = \bigcup_{j\in\Omega} B_{j,r}$ each collection $\{A_j\}_{j\geq 1}$, $\{B_j\}_{j\geq 1}$ forms a locally finite open cover of E.

Observe that $\overline{A_j} \subseteq B_j$ for every $j \in \mathbb{N}$.
For every j, there exists a sequence of open sets B_j^n, $n \geq j$, so that
$$\overline{A_j} \subset B_j^j \subset \overline{B_j^j} \subset B_j^{j+1} \subset \overline{B_j^{j+1}} \subset \cdots \subset B_j^n \subset \overline{B_j^n} \subset B_j^{n+1} \subset \cdots \subset B_j.$$

For each j, write $\mathcal{B}_j = \left\{B_j^n : n \geq j\right\}$. Clearly, $\mathcal{B} = \bigcup_{j=1}^{\infty} \mathcal{B}_j$ is an open cover of E; likewise, the family $\left\{\overline{B} \cap X : B \in \mathcal{B}\right\}$ is a closed cover of X.
Defining for each $n \in \mathbb{N}$: $Y_n := \bigcup_{j=1}^{n} B_j^n$, $H_n := Y_n \setminus \overline{Y_{n-3}}$ and $K_n := \overline{Y_n} \setminus Y_{n-1}$ (let $Y_{-2} = Y_{-1} = Y_0 = \emptyset$), we have the following properties:

- $E = \bigcup_{n=1}^{\infty} Y_n$;
- $\overline{Y_n} \subseteq Y_{n+1}$ for all $n \in \mathbb{N}$;
- $K_n \subseteq H_{n+1}$ for all $n \in \mathbb{N}$;
- $E = \bigcup_{n=1}^{\infty} K_n$;
- $H_m \cap H_n = \emptyset$ for all m, n with $|m-n| \geq 3$.

Hence, the collection

$$\bigcup_{n=1}^{\infty} \{K_n \cap \overline{B_j^n} : j = 1, \ldots, n\}$$

is a closed cover of E and therefore

$$\bigcup_{n=1}^{\infty} \{H_{n+1} \cap B_j^{n+1} : j = 1, \ldots, n\}$$

is an open cover of E. Both covers are countable and star-finite, and they are refinements of $\{B_j\}_{j\geq 1}$. We call the first one $\{T_i\}_{i\geq 1}$ and the second one $\{V_i\}_{i\geq 1}$, that is, for every i there corresponds a unique pair (j, n), $n \geq j$, with $T_i = K_n \cap \overline{B_j^n}$ and $V_i = H_{n+1} \cap B_j^{n+1}$. Consequently, we have $T_i \subseteq V_i$ for every $i \in \mathbb{N}$.
Now for each $i \in \mathbb{N}$ we take $j = j(i) \in \mathbb{N}$ such that $T_i \subseteq V_i \subseteq B_j$. Let us assume without loss of generality that $j(i) \leq i$. We can write

$$T_i = \bigcup_{r\in\Omega} T_i \cap B_{j,r} \ \text{ and } \ V_i = \bigcup_{r\in\Omega} V_i \cap B_{j,r}$$

and we define $T_{i,r} = T_i \cap B_{j,r}$ and $V_{i,r} = V_i \cap B_{j,r}$ for every $i \in \mathbb{N}$ and $r \in \Omega$. Clearly we have that $T_{i,r} \subseteq V_{i,r}$ for all $i \in \mathbb{N}$ and $r \in \Omega$. Also $\operatorname{dist}(V_{i,r}, V_{i,r'}) \geq \frac{1}{2^{j+1}} \geq \frac{1}{2^{i+1}}$ for all $i \in \mathbb{N}$ and $r, r' \in \Omega$, $r \neq r'$.

Finally let us define $X_{i,r} = X \cap T_{i,r}$. Bearing in mind that $T_{i,r} \subset V_{i,r} \subset B_{j,r} \subset U_x$ for some $x \in X$ and that $T_{i,r}$ is closed, we obtain $X_{i,r} = X \cap T_{i,r} \subset X \cap \overline{U_x}$. Since $X \cap \overline{U_x}$ has the ε-strong C^p extraction property with respect to every open set U with $X \cap \overline{U_x} \subset U$, applying Lemma 2.22(1), we get that $X_{i,r}$ has the ε-strong C^p extraction property with respect to every such an open set U. Finally, applying Lemma 2.22(2), $X_{i,r}$ has the strong C^p extraction property with respect to every open set U' with $X_{i,r} \subset U'$. In particular, $X_{i,r}$ has the ε-strong C^p extraction property with respect to $V_{i,r}$.

Let us consider now for each $i \in \mathbb{N}$ and $r \in \Omega$ an open set $W_{i,r}$ with $T_{i,r} \subseteq W_{i,r} \subseteq \overline{W_{i,r}} \subseteq V_{i,r}$ and call $W_i = \bigcup_{r\in\Omega} W_{i,r}$. We still have that $\{W_{i,r}\}_{i\geq 1, r\in\Omega}$ is a refinement of $\mathcal{G}$, and that $\{W_i\}_{i\geq 1}$ is a star-finite open cover of E.
Then the collections $\{X_{i,r}\}_{i\geq 1, r\in\Omega}$, $\{W_{i,r}\}_{i\geq 1, r\in\Omega}$, $\{V_{i,r}\}_{i\geq 1, r\in\Omega}$ have the required properties. Note that each $X_{i,r}$ has the strong C^p extraction property with respect to every open set containing it. □

Lemma 2.26. *Let X_i, V_i, and $V_{i,r}$ be as in Lemma 2.25. Then, for every $i, j \in \mathbb{N}$, $X_i \cap V_j$, has the strong C^p extraction property with respect to V_j. Moreover, if $\varphi_{i,j}$ is a C^p diffeomorphism satisfying the definition of the ε-strong extraction property for these sets, then $\varphi(V_{i,r}) \subset V_{i,r}$ and $\varphi(V_{j,r}) \subset V_{j,r}$ for every $r \in \Omega$.*

Proof. Take $V \subseteq V_j$ an open set. Let $r \in \Omega$ and consider the open set $V_{j,r}$. For each $s \in \Omega$ the set $X_{i,s} \cap V_{j,r}$ has the ε-strong C^p extraction property with respect to the open set $V_{i,s} \cap V_{j,r}$. We have

$$X_{i,s} \cap V_{j,r} \subseteq W_{i,s} \cap V_{j,r} \subseteq \overline{W_{i,s}} \cap V_{j,r} \subseteq V_{i,s} \cap V_{j,r}.$$

There exists a C^p diffeomorphism $h_{r,s} : (V_{i,s} \cap V_{j,r}) \setminus X_{i,s} \to V_{i,s} \cap V_{j,r} \setminus (X_{i,s} \setminus (W_{i,s} \cap V))$ which is the identity on $((V_{i,s} \cap V_{j,r} \setminus (W_{i,s} \cap V)) \setminus X_{i,s}$. Outside $V_{i,s} \cap V_{j,r}$ we define $h_{r,s}$ to be the identity. Since $\overline{W_{i,s}} \subseteq V_{i,s}$ we have a well-defined C^p diffeomorphism

$$h_{r,s} : V_{j,r} \setminus X_{i,s} \to V_{j,r} \setminus (X_{i,s} \setminus (W_{i,s} \cap V))$$

which is the identity on $(V_{j,r} \setminus (W_{i,s} \cap V)) \setminus X_{i,s}$. In particular $h_{r,s}$ is the identity on $V_{i,s'} \cap V_{j,r}$ for every $s' \in \Omega, s \neq s'$.
Having defined $h_{r,s}$ for each $s \in \Omega$ in the way described above, we finally define

$$h_r = \bigcirc_{s \in \Omega} h_{r,s}$$

as an infinite composition of $h_{r,s}$, $s \in \Omega$. It is easy to see that h_r is well defined and provides a C^p diffeomorphism of $V_{j,r} \setminus X_i$ onto $V_{j,r} \setminus (X_i \setminus V)$ which is the identity on $(V_{j,r} \setminus V) \setminus X_i$.
By the discreteness of the family $\{V_{j,r}\}_{r \in \Omega}$, the formula $h(x) = h_r(x)$, $x \in V_{j,r} \setminus X_i$, defines a C^p diffeomorphism of $\bigcup_{r \in \Omega} V_{j,r} \setminus X_i = V_j \setminus X_i$ onto $\bigcup_{r \in \Omega} (V_{j,r} \setminus (X_i \setminus V)) = V_j \setminus (X_i \setminus V)$ which is the identity on $\bigcup_{r \in \Omega} ((V_{j,r} \setminus V) \setminus X_i) = (V_j \setminus V) \setminus X_i$.
To end the proof observe that each $h_{r,s}$ is the identity outside $V_{i,s} \cap V_{j,r}$, so h sends $V_{i,r}$ into $V_{i,r}$ and $V_{j,r}$ into $V_{j,r}$ for every $r \in \Omega$. □

Notice that in the previous two Lemmas 2.26 and 2.25 one can replace the ε-strong C^p extraction property with just the strong C^p extraction property.
The last tool that we need to introduce before going into the proof of Theorem 2.24 is the next lemma (see Statement A in West's paper [67, pp. 289]).

Lemma 2.27. *Let $V_0, \dots, V_n$ be open sets of E, and $X_0, \dots, X_n$ be subsets of $V_0, \dots, V_n$. Take also an open set U. Suppose each X_i is relatively closed in $V = \bigcup_{i=0}^n V_i$ and has the ε-strong C^p extraction property with respect to V. Then for every $\varepsilon > 0$ there exists a C^p diffeomorphism of $V \setminus X_0$ onto $V \setminus (X_0 \setminus U)$ which is the identity outside $V_0 \cap U$, carries $X_i \setminus X_0$ into V_i for each $i = 1, \dots, n$, and moves no point more than ε.*

Proof. We will divide the set X_0 that we want to extract as follows. For each $j = 0, \dots, n$, let

$$Q_j := \left\{ Z : \ Z = \bigcap_{i=0}^{n-j} X_{p(i)} \setminus \bigcup_{i=n-j+1}^{n} X_{p(i)} \text{ for some permutation } p \text{ of } \{0, \dots, n\} \text{ carrying } 0 \text{ to } 0 \right\};$$

these are families of subsets of X_0. Let $Q = \bigcup_{j=0}^n Q_j$. The family Q is a pairwise disjoint cover of X_0 with cardinality $\leq 2^n$. Order Q in such a manner that if $j < k$ then all elements of Q_j precede those of Q_k. We then will list the elements of Q as $Z_1, Z_2, \dots, Z_{2^n}$ (bearing in mind that some Z_m's may repeat in that listing); that is, $Q = \{Z_m\}_{m=1}^{2^n}$ and $\bigcup_{i=1}^{2^n} Z_i = X_0$. Note that $Q_0 = \{Z_1\}$ and $Q_n = \{Z_{2^n}\}$, where

$$Z_1 = \bigcap_{i=0}^{n} X_i \quad \text{and} \quad Z_{2^n} = X_0 \setminus \bigcup_{i=1}^{n} X_i.$$

Likewise, for each $m = 1, \dots, 2^n$, if $Z_m \in Q_j$ and p is a permutation for which $Z_m = \bigcap_{i=0}^{n-j} X_{p(i)} \setminus \bigcup_{i=n-j+1}^{n} X_{p(i)}$, we define

$$N_m = \bigcap_{i=0}^{n-j} V_{p(i)} \subset V_0.$$

The family $\{N_m\}_{m=1}^{2^n}$ is an open cover of V_0; we also have $N_1 = \bigcap_{i=0}^n V_i$ and $N_{2^n} = V_0$. Denote by Q_k^* the union of all the elements of Q_k, $k = 0, 1, \ldots, n$; notice that $Q_0^* = Z_1$ and $Q_n^* = Z_{2^n}$. For each $j > 0$, the elements of Q_j form a pairwise disjoint family of relatively closed subsets of the open set $V \setminus \bigcup_{k=0}^{j-1} Q_k^*$. Also each Z_m in Q_j lies in N_m. Therefore, for each $j > 0$, there exists a collection of pairwise disjoint open sets M_m in $V \setminus \bigcup_{k=0}^{j-1} Q_k^*$, one for each Z_m in Q_j (that is, if $Z_m = Z_{m'} \in Q_j$, $m \neq m'$, then $M_m = M_{m'}$), such that

$$Z_m \subseteq M_m \subseteq N_m \setminus \bigcup_{i=n-j+1}^{n} X_{p(i)},$$

where j is such that Z_m is in Q_j and p is a permutation defining Z_m as above.
The set $Z_1 = \bigcap_{i=0}^n X_i \subseteq N_1 = \bigcap_{i=0}^n V_i$ is relatively closed in V and has the ε-strong C^p extraction property with respect to V, so there is a C^p diffeomorphism

$$h_1 : V \setminus Z_1 \to V \setminus (Z_1 \setminus U)$$

which is the identity outside $(N_1 \cap U) \cup Z_1$ and moves no point more than $\frac{\varepsilon}{2^n}$; in particular, $h_1(Z_2) \setminus U = Z_2 \setminus U$ should $Z_2 \neq Z_1$.
We will apply induction to prove that for $1 < m \leq 2^n$ there exists a C^p diffeomorphism

$$h_m : (V \setminus (\bigcup_{i=1}^{m-1} Z_i \setminus U)) \setminus h_{m-1} \circ \cdots \circ h_1(Z_m) \to V \setminus (\bigcup_{i=1}^{m} Z_i \setminus U)$$

which is the identity outside $(h_{m-1} \circ \cdots \circ h_1(M_m) \cap N_m \cap U) \cup h_{m-1} \circ \cdots \circ h_1(Z_m)$, satisfies

$$h_m \circ h_{m-1} \circ \cdots \circ h_1(V \setminus \bigcup_{i=1}^{m} Z_i) = V \setminus (\bigcup_{i=1}^{m} Z_i \setminus U) \text{ and } h_m \circ h_{m-1} \circ \cdots \circ h_1(Z_{m+1}) \setminus U = Z_{m+1} \setminus U,$$

and such that $h_m \circ \cdots \circ h_1$ moves no point more than $\frac{m\varepsilon}{2^n}$.
Suppose this is true for every $1 \leq k \leq m-1$, and let us check so it is for m. Assume $Z_m \in Q_j$; additionally, we can assume that $Z_{m-1} \neq Z_m$, otherwise, the identity in place of h_m will do. By the definition of the ε-strong C^p extractibility and Lemma 2.22(2), Z_m has the ε-strong C^p extraction property with respect to $V \setminus \bigcup_{k=0}^{j-1} Q_k^*$. Furthermore, one more application of Lemma 2.22(2) yields that $Z_m \subset V \setminus \bigcup_{i=1}^{m-1} Z_i$ is relatively closed and has the ε-strong C^p extraction property with respect to $V \setminus \bigcup_{i=1}^{m-1} Z_i$. By Lemma 2.22 (3), $h_{m-1} \circ \cdots \circ h_1(Z_m) \subset h_{m-1} \circ \cdots \circ h_1(V \setminus \bigcup_{i=1}^{m-1} Z_i)$ is relatively closed and has the strong C^p extraction property with respect to the open set

$$h_{m-1} \circ \cdots \circ h_1(V \setminus \bigcup_{i=1}^{m-1} Z_i) = V \setminus (\bigcup_{i=1}^{m-1} Z_i \setminus U).$$

Considering the open set $h_{m-1} \circ \cdots \circ h_1(M_m) \cap N_m \cap U$, there exists a C^p diffeomorphism h_m from

$$(V \setminus (\bigcup_{i=1}^{m-1} Z_i \setminus U)) \setminus h_{m-1} \circ \cdots \circ h_1(Z_m)$$

onto

$$(V \setminus (\bigcup_{i=1}^{m-1} Z_i \setminus U)) \setminus (h_{m-1} \circ \cdots \circ h_1(Z_m) \setminus (h_{m-1} \circ \cdots \circ h_1(M_m) \cap N_m \cap U) \tag{2.9}$$

which is the identity outside $(h_{m-1}\circ\cdots\circ h_1(M_m)\cap N_m\cap U)\cup h_{m-1}\circ\cdots\circ h_1(Z_m)$. Furthermore by Remark 2.23 and the fact that $h_{m-1}\circ\cdots\circ h_1$ moves no point more than $\frac{(m-1)\varepsilon}{2^n}$ (induction hypothesis), we have that $h_m\circ(h_{m-1}\circ\cdots\circ h_1)$ moves no point more than $\frac{\varepsilon}{2^n}+\frac{(m-1)\varepsilon}{2^n}=\frac{m\varepsilon}{2^n}$.
Finally note that the expression (2.9) is equal to $V\setminus(\bigcup_{i=1}^m Z_i\setminus U)$ because $h_{m-1}\circ\cdots\circ h_1(Z_m)\subseteq h_{m-1}\circ\cdots\circ h_1(M_m)\cap N_m$ and the fact that $h_{m-1}\circ\cdots\circ h_1(Z_m)\setminus U=Z_m\setminus U$.

To conclude define

$$h=h_{2^n}\circ\cdots\circ h_1:V\setminus X_0\to V\setminus(X_0\setminus U).$$

This is a C^p diffeomorphism of $V\setminus X_0$ onto $V\setminus(X_0\setminus U)$ which is the identity outside $(V_0\cap U)\cup X_0$ and moves no point more than ε.
To end the proof we will show that h carries $X_i\setminus X_0$ into V_i for each $i=1,\dots,n$. Take $x\in X_i\setminus X_0$ for some $i=1,\dots,n$ and let us see that $h(x)\in V_i$. If $h_1(x)\neq x$ then $h_1(x)\in N_1$, so because $N_1\subseteq V_i$ we have that $h_1(x)\in V_i$. Suppose now that $h_{m-1}\circ\cdots\circ h_1(x)\in V_i$. If $h_m\circ\cdots\circ h_1(x)\neq h_{m-1}\circ\cdots\circ h_1(x)$ then we have that $x\in M_m$ and $h_m\circ\cdots\circ h_1(x)\in N_m$. We have that $x\in X_i\cap M_m$ and $M_m\subseteq N_m\setminus\bigcup_{i=n-j+1}^n X_{p(i)}$ where j is such that Z_m is in Q_j and p is a permutation that defines Z_m. Obviously we must have that $i=p(i_0)$ where $i_0=\{1,\dots,n-j\}$. Also $N_m=\bigcap_{k=0}^{n-j}V_{p(k)}\subseteq V_i$, hence $h_m\circ\cdots\circ h_1(x)$ must also lie in V_i. Applying induction we have that $h(x)\in V_i$.

□

Remark 2.28. Notice that if we only have the strong C^p extraction property of the sets X_i with respect to V, we get the same result except that for any $\varepsilon>0$ we can not assure that the final extracting diffeomorphism moves all points less than ε. In such a case we will say that we have a *weak version* of Lemma 2.27.
However, suppose we have that $X_0\subseteq V_0, X_1\subseteq V_1,\dots,X_n\subseteq V_n$ are of the form $g(X_i)\subseteq g(V_i)$, $i=0,1,\dots,n$, where g is a C^p diffeomorphism that moves points less than some $\delta_1>0$. In such a case, using Remark 2.23, for any $\delta_2>0$ we can make the final extracting diffeomorphism h of the proof of Lemma 2.27 satisfy that $h\circ g$ moves no point more than $\delta_1+\delta_2$. In particular if $g(x)=x$ then we have that $||h(g(x))-x||\leq\delta_2$.

Remark 2.29. Assume, additionally, that the set V of Lemma 2.27 is of the form $V=\bigcup_{r\in\Omega}V_r$, where Ω is a set of indexes, each V_r is an open set , and $V_r\cap V_{r'}=\emptyset$ for every $r,r'\in\Omega$, $r\neq r'$. Then we can also require that the extracting C^p diffeomorphism of $V\setminus X_0$ onto $V\setminus(X_0\setminus U)$ sends each set $V_r\setminus X_0$ into V_r for every $r\in\Omega$. To prove this, fix $r\in\Omega$ and replace the sequences $V_0,V_1,\dots,V_n$ and $X_0,X_1,\dots X_n$ with $V_0\cap V_r, V_1\cap V_r\dots,V_n\cap V_r$ and $X_0\cap V_r, X_1\cap V_r,\dots,V_n\cap V_r$, respectively. Further, observe that $\bigcup_{i=0}^n V_i\cap V_r=V_r$ and that each $X_i\cap V_r$ has the strong C^p extraction property with respect to V_r by Lemma 2.22 (2). According to the assertion of Lemma 2.27, we conclude that there exists a C^p diffeomorphism $h_r:V_r\setminus X_0\to V_r\setminus(X_0\setminus U)$ satisfying the required conditions. Finally, it is enough to set $h:V\setminus X_0\to V\setminus(X_0\setminus U)$ by letting $h(x)=h_r(x)$ for $x\in V_r\setminus X_0$.

The rest of the proof of Theorem 2.24 goes as in [67, Theorem 1], with some modifications due to the facts that we work here with an open set U not necessarily containing X, and that E is not necessarily separable.

Proof of Theorem 2.24. Apply Lemma 2.25 to the given cover $\mathcal{G}$ to find collections $\{X_i\}_{i\geq 1}$, $\{W_i\}_{i\geq 1}$ and $\{V_i\}_{i\geq 1}$ of subsets of E satisfying conditions (1)–(6) of that lemma. By Lemma 2.26, for all $i,j\in\mathbb{N}$, $X_i\cap V_j$ has the ε-strong C^p extraction property with respect to V_j. Moreover, if $\varphi_{i,j}$ is a C^p diffeomorphism with this property, then $\varphi(V_{i,r})\subset V_{i,r}$ and $\varphi(V_{j,r})\subset V_{j,r}$ for every $r\in\Omega$.
Let us now define the required C^p diffeomorphism $g:E\setminus X\to E\setminus(X\setminus U)$ which is the identity on $(E\setminus U)\setminus X$ and is limited by $\mathcal{G}$.

(1) For a given V_1, define $I_1 = \{1_1 = 1, 1_2, \dots, 1_{n(1)}\} \subset \mathbb{N}$ to be the finite set of positive integers such that $W_{1_1}, W_{1_2}, \dots, W_{1_{n(1)}}$ are the only $W_i's$ sets for which $V_1 \cap W_i \neq \emptyset$ (if there were infinitely many such $W_i's$, then we would have $V_1 \cap V_i's \neq \emptyset$ for infinitely many i's, which would contradict the star-finiteness of $\{V_i\}_{i\geq 1}$; obviously, we assume that $V_i \neq V_{i'}$ for $i \neq i'$). Since $\{W_i\}_{i\geq 1}$ is a cover we have $\bigcup_{i\in I_1} W_i \cap V_1 = V_1$. (A priori V_1 can be covered by a proper subfamily of $\{V_2, \dots, V_n\}$). Assuming that i_1 is the greatest number in I_1 (in particular $i_1 \geq 1$) we set
$$\varepsilon_1 = \frac{1}{2} \cdot \frac{1}{2^{i_1+1}} > 0.$$
We want to apply Lemma 2.27 for the sets
$$X_{1_1} = X_1 = X_1 \cap V_1, X_{1_2} \cap V_1, \dots, X_{1_{n(1)}} \cap V_1$$
which play the role of $X_0, \dots, X_n$ in the statement of the Lemma 2.27, and for the sets
$$W_{1_1} = W_1 = W_1 \cap V_1, W_{1_2} \cap V_1, \dots, W_{1_{n(1)}} \cap V_1,$$
which play the role of $V_0, \dots, V_n$ respectively, and for the positive number $\varepsilon_1 > 0$. Observe that each $X_i \cap V_1$, $i \in I_1$, has the strong C^p extraction property with respect to V_1. Hence, applying Lemma 2.27, we find a C^p diffeomorphism g_1 of $V_1 \setminus X_1$ onto $V_1 \setminus (X_1 \setminus U)$ which is the identity outside $(W_1 \cap U) \cup X_1$, carries $(X_l \setminus X_1) \cap V_1$ into $W_l \cap V_1$ for each $l > 1$ and moves no point more than ε_1.
By Remark 2.29 we may also assume that g_1 sends each set $V_{1,r} \setminus X_1$ into $V_{1,r}$. This means in particular that g_1 refines $\mathcal{G}$. Also, since g_1 moves no point more than ε_1 we cannot have that $x \in V_{i,r}$ and $g_1(x) \in V_{i,r'}$ for some $i \in I_1$ and different $r, r' \in \Omega$ (recall that $\text{dist}(V_{i,r}, V_{i,r'}) \geq \frac{1}{2^{i+1}} > \varepsilon_1$).
Since $W_1 \cap U \subseteq \overline{W_1} \cap U \subseteq V_1 \cap U$, by making g_1 be the identity outside $V_1 \setminus X_1$, there exists a well-defined natural extension of g_1 from $V_1 \setminus X_1$ to $E \setminus X_1$. Now we have a C^p diffeomorphism g_1 such that
  (a) g_1 acts from $E \setminus X_1$ onto $E \setminus (X_1 \setminus U)$.
  (b) g_1 is the identity on $E \setminus [(W_1 \cap U) \cup X_1]$. In particular g_1 is the identity on $(E \setminus U) \setminus X_1$.
  (c) g_1 carries $(X_l \setminus X_1) \cap V_1$ into $W_l \cap V_1$ for each $l > 1$.
  (d) We require that if $X_1 = \emptyset$ or $X_{1,r} = \emptyset$ then g_1 is the identity on V_1 or $V_{1,r}$, respectively.
  (e) g_1 moves no point more than $\varepsilon_1 = \frac{1}{2} \cdot \frac{1}{2^{i_1+1}}$.
  (f) g_1 sends each set $V_{1,r} \setminus X_1$ into $V_{1,r}$ for every $r \in \Omega$, so g_1 refines $\mathcal{G}$.

(2) Consider now the set V_2 and define $I_2 = \{2_1 = 2, 2_2, \dots, 2_{n(2)}\} \subset \mathbb{N}$ to be the finite set of natural numbers such that $W_{2_1}, W_{2_2}, \dots, W_{2_n}$ are the only $W_i's$ sets for which $V_2 \cap W_i \neq \emptyset$ (if there were infinitely many such $W_i's$ then $V_2 \cap V_i's \neq \emptyset$ for infinitely many i's, which would contradict the star-finiteness of $\{V_i\}_{i\geq 1}$). Assume that i_2 is the greatest number in I_2 (in particular $i_2 \geq 2$), and set
$$\varepsilon_2 = \frac{1}{2^2} \cdot \frac{1}{2^{i_2+1}} > 0.$$
Since $\{W_i\}_{i\geq 1}$ is a cover we have
$$\bigcup_{i\in I_2} g_1(V_2 \setminus X_1) \cap W_i \cap V_2 = g_1(V_2 \setminus X_1) \cap V_2.$$
Again, we want to apply Lemma 2.26 for the sets
$$\{g_1((X_i \setminus X_1) \cap V_2) \cap V_2 : \, i \in I_2\}$$

playing the role of $X_0, \dots, X_n$ in the statement of the lemma, for

$$\{g_1(V_2 \setminus X_1) \cap W_i \cap V_2 : i \in I_2\}$$

playing the role of the sets $V_0 \dots, V_n$ respectively. Here we should recall that $g_1(X_i \setminus X_1) \subseteq W_i$. Observe that by Lemma 2.22 (3), each $g_1((X_i \setminus X_1) \cap V_2) \cap V_2$ has the strong C^p extraction property with respect to the open set $g_1(V_2 \setminus X_1) \cap V_2$. Applying the weak version of Lemma 2.27 to these sets we get a C^p diffeomorphism g_2 of

$$[g_1(V_2 \setminus X_1) \cap V_2] \setminus [g_1(X_2 \setminus X_1) \cap V_2] = g_1(V_2 \setminus (X_1 \cup X_2)) \cap V_2$$

onto

$$[g_1(V_2 \setminus X_1) \cap V_2] \setminus [(g_1(X_2 \setminus X_1) \cap V_2) \setminus U]$$

which is the identity outside

$$(g_1(V_2 \setminus (X_1 \cup X_2)) \cap W_2 \cap V_2 \cap U) \cup (g_1(X_2 \setminus X_1) \cap V_2)$$

and carries

$$g_1(X_k \setminus (X_1 \cup X_2) \cap V_2) \cap V_2$$

into

$$g_1(V_2 \setminus X_1) \cap W_k \cap V_2$$

for each $k > 2$. Moreover using Remark 2.28 one can also assume that $g_2 \circ g_1$ moves no point more than $\varepsilon_1 + \varepsilon_2$.

Because $\overline{g_1(V_2 \setminus (X_1 \cup X_2)) \cap W_2} \cap U \subseteq g_1(V_2 \setminus (X_1 \cup X_2)) \cap V_2 \cap U$, by letting g_2 be the identity outside $g_1(V_2 \setminus (X_1 \cup X_2)) \cap V_2$ there exists a well-defined natural extension of g_2 to $g_1(E \setminus (X_1 \cup X_2))$. To sum up we have the following properties:

(a) g_2 acts from

$$g_1(E \setminus (X_1 \cup X_2))$$

onto

$$\begin{aligned} g_1(E \setminus X_1) \setminus [g_1(X_2 \setminus X_1) \setminus U] &= E \setminus (X_1 \setminus U) \setminus [(g_1((X_2 \setminus X_1) \cap U) \cup g_1((X_2 \setminus X_1)) \setminus U) \setminus U] = \\ &= E \setminus (X_1 \setminus U) \setminus [(X_2 \setminus X_1) \setminus U] = E \setminus ((X_1 \cup X_2) \setminus U). \end{aligned}$$

(Here we are using the fact that $g_1((X_2 \setminus X_1) \cap U) \subseteq U$ and that g_1 is the identity outside U).

(b) g_2 is the identity on $E \setminus [(g_1(W_2 \setminus (X_1 \cup X_2)) \cap W_2 \cap U) \cup (g_1(X_2 \setminus X_1) \cap V_2)]$. Since

$$(g_1(W_2 \setminus (X_1 \cup X_2)) \cap W_2 \cap U) \cup (g_1(X_2 \setminus X_1) \cap V_2) \subseteq U \cup X_1 \cup X_2,$$

in particular g_2 is the identity on $E \setminus (U \cup (X_1 \cup X_2))$. Because g_1 is the identity outside U then g_2 is the identity on $g_1(E \setminus (U \cup (X_1 \cup X_2)) = g_1((E \setminus U) \setminus (X_1 \cup X_2))$.

(c) g_2 carries

$$g_1(X_l \setminus (X_1 \cup X_2)) \cap V_2$$

into

$$W_l \cap V_2$$

for each $l > 2$.

(d) If $X_{2,r} = \emptyset$ then we require that g_2 is the identity on $g_1(V_{2,r} \setminus X_1) \cap V_2$ and in $g_1(V_2 \setminus X_1) \cap V_{2,r}$.

(e) We have

$$\begin{cases} ||g_2(g_1(x)) - x|| \leq \varepsilon_1 + \varepsilon_2 = \frac{1}{2} \cdot \frac{1}{2^{i_1+1}} + \frac{1}{4} \cdot \frac{1}{2^{i_2+1}} & \text{for every } x \in E \setminus (X_1 \cup X_2) \\ ||g_2(g_1(x)) - x|| < \frac{1}{2^{2+1}} & \text{for every } x \in V_2 \setminus (X_1 \cup X_2). \end{cases} \tag{2.10}$$

The first inequality is clear. For the second one, when $x \notin V_2 \setminus V_1$ we have $g_1(x) = x$, hence

$$||g_2(g_1(x)) - x|| = ||g_2(x) - x|| \leq \varepsilon_2 = \frac{1}{4} \cdot \frac{1}{2^{i_2+1}} < \frac{1}{2^{2+1}},$$

and if $x \in V_1 \cap V_2$ then $V_1 \cap V_2 \neq \emptyset$, so $i_1 \geq 2$ and

$$||g_2(g_1(x)) - x|| \leq \frac{1}{2} \cdot \frac{1}{2^{i_1+1}} + \frac{1}{4} \cdot \frac{1}{2^{i_2+1}} < \left(\frac{1}{2} + \frac{1}{4}\right) \cdot \frac{1}{2^{2+1}}.$$

(f) The composition $g_2 \circ g_1$ is a mapping from $E \setminus (X_1 \cup X_2)$ onto $E \setminus ((X_1 \cup X_2) \setminus U)$ and it is the identity outside $(E \setminus U) \setminus (X_1 \cup X_2)$. Let us check that it refines $\mathcal{G}$. Take $x \in E \setminus (X_1 \cup X_2)$. If $g_2(g_1(x)) = g_1(x)$, since g_1 refines $\mathcal{G}$ we are done. Otherwise we have $g_2(g_1(x)) \neq g_1(x)$, and then $g_1(x), g_2(g_1(x)) \in g_1(V_2 \setminus (X_1 \cup X_2)) \cap V_2$. We have that $x, g_2(g_1(x)) \in V_2$. We must have $x, g_2(g_1(x)) \in V_{2,r}$ for some $r \in \Omega$ (as otherwise $x \in V_{2,r}$ and $g_2(g_1(x)) \in V_{2,r'}$ a contradiction with (2.10) since $\text{dist}(V_{2,r}, V_{2,r'}) \geq \frac{1}{2^{2+1}}$).

(3) We go on doing this process by successive applications of Lemma 2.27. We want to apply induction to prove that for each $j \geq 3$ we can find a C^p diffeomorphism g_j such that

(a) g_j acts from

$$g_{j-1} \circ \cdots \circ g_1 \left(E \setminus \bigcup_{k \leq j} X_k \right)$$

onto

$$E \setminus \left(\bigcup_{k \leq j} X_k \setminus U \right).$$

(b) g_j is the identity on $g_{j-1} \circ \cdots \circ g_1 \left((E \setminus U) \setminus (\bigcup_{k \leq j} X_k) \right)$.

(c) g_j carries

$$g_{j-1} \circ \cdots \circ g_1(X_l \setminus \bigcup_{k \leq j} X_k) \cap V_j$$

into

$$W_l \cap V_j$$

for each $l > j$.

(d) If $X_{j,r} = \emptyset$ then g_j is the identity on $g_{j-1} \circ \cdots \circ g_1(V_{j,r} \setminus \bigcup_{k<j} X_k) \cap V_j$ and on $g_{j-1} \circ \cdots \circ g_1(V_j \setminus \bigcup_{k<j} X_k) \cap V_{j,r}$.

(e) We have

$$\begin{cases} ||g_j \circ \cdots \circ g_1(x) - x|| \leq \sum_{k=1}^{j} \frac{1}{2^k} \cdot \frac{1}{2^{i_k+1}} & \text{for every } x \in E \setminus (\bigcup_{k \leq j} X_k) \\ ||g_j \circ \cdots \circ g_1(x) - x|| < \frac{1}{2^{j+1}} & \text{for every } x \in V_j \setminus (\bigcup_{k \leq j} X_k), \end{cases} \tag{2.11}$$

where i_k is the greatest number such that $V_{i_k} \cap V_k \neq \emptyset$. Let us call for every $k = 1, \ldots, j$,

$$\varepsilon_k = \frac{1}{2^k} \cdot \frac{1}{2^{i_k+1}} > 0.$$

(f) $g_j \circ \cdots \circ g_1$ refines $\mathcal{G}$.
Suppose this is true for $j-1$ and let us check this is so for j.

The idea is the same as in steps (1) and (2). We first find the set $I_j = \{j_1 = j, j_2, \dots, j_{n(j)}\} \subset \mathbb{N}$ such that $W_{j_1}, W_{j_2}, \dots, W_{j_{n(j)}}$ are the only $W_i's$ sets for which $V_j \cap W_i \neq \emptyset$. Assume i_j is the greatest number in I_j (in particular $i_j \geq j$) and set

$$\varepsilon_j = \frac{1}{2^j} \cdot \frac{1}{2^{i_j+1}} > 0.$$

We have that

$$\bigcup_{i \in I_j} g_{j-1} \circ \cdots \circ g_1(V_j \setminus \bigcup_{k<j} X_k) \cap W_i \cap V_j = g_{j-1} \circ \cdots \circ g_1(V_j \setminus \bigcup_{k<j} X_k) \cap V_j.$$

We want to apply Lemma 2.26 for the sets

$$\{g_{j-1} \circ \cdots \circ g_1((X_i \setminus \bigcup_{k<j} X_k) \cap V_j) \cap V_j \, : \, i \in I_j\}$$

playing the role of $X_0, \dots, X_n$ in the statement of the lemma, for

$$\{g_{j-1} \circ \cdots \circ g_1(V_j \setminus \bigcup_{k<j} X_k) \cap W_i \cap V_j \, : \, i \in I_j\}$$

playing the role of the sets $V_0 \dots, V_n$ respectively.
Observe that by Lemma 2.22 (3), each $g_{j-1} \circ \cdots \circ g_1((X_i \setminus \bigcup_{k<j} X_k) \cap V_j) \cap V_j$ has the strong C^p extraction property with respect to the open set $g_{j-1} \circ \cdots \circ g_1(V_j \setminus \bigcup_{k<j} X_k) \cap V_j$. Applying the weak version of Lemma 2.27 to these sets we get a C^p diffeomorphism g_j from

$$\left[g_{i-1} \circ \cdots \circ g_1(V_i \setminus \bigcup_{j<i} X_j) \cap V_i \right] \setminus \left[g_{j-1} \circ \cdots \circ g_1(X_j \setminus \bigcup_{k<j} X_k) \cap V_j \right] = g_{j-1} \circ \cdots \circ g_1(V_j \setminus \bigcup_{k \leq j} X_k) \cap V_j$$

onto

$$\left[g_{j-1} \circ \cdots \circ g_1(V_j \setminus \bigcup_{k<j} X_k) \cap V_j \right] \setminus \left[g_{j-1} \circ \cdots \circ g_1(X_j \setminus \bigcup_{k<j} X_k) \cap V_j \setminus U \right]$$

which is the identity outside

$$\left((g_{j-1} \circ \cdots \circ g_1(V_j \setminus \bigcup_{k \leq j} X_k) \cap W_j \cap U \right) \cup \left(g_{j-1} \circ \cdots \circ g_1(X_j \setminus \bigcup_{k<j} X_k) \cap V_j \right)$$

and carries

$$g_{j-1} \circ \cdots \circ g_1(X_l \setminus \bigcup_{k \leq j} X_k) \cap V_j$$

into

$$W_l \cap V_j$$

for each $l > j$. This last property establishes (c).
Define g_j to be the natural extension of $g_{j-1} \circ \cdots \circ g_1(V_j \setminus \bigcup_{k \leq j} X_k) \cap V_j$ to $g_{j-1} \circ \cdots \circ g_1(E \setminus$

$\bigcup_{k\leq j} X_k)$, so now g_j is defined from

$$g_{j-1}\circ\cdots\circ g_1\left(E\setminus\bigcup_{k\leq j} X_k)\right)=$$

onto

$$\left[g_{j-1}\circ\cdots\circ g_1(E\setminus\bigcup_{k<j} X_k)\right]\setminus\left[g_{j-1}\circ\cdots\circ g_1(X_j\setminus\bigcup_{k<j} X_k)\setminus U\right]=$$

$$E\setminus(\bigcup_{k<j} X_j\setminus U)\setminus\left[\left(g_{j-1}\circ\cdots\circ g_1((X_j\setminus\bigcup_{k<j} X_k)\cap U)\cup g_{j-1}\circ\cdots\circ g_1((X_j\setminus\bigcup_{k<j} X_k)\setminus U)\right)\setminus U\right]=$$

$$=E\setminus(\bigcup_{k<j} X_k\setminus U)\setminus\left[(X_j\setminus\bigcup_{k<j} X_k)\setminus U\right]=E\setminus\left(\bigcup_{k\leq j} X_k\setminus U\right).$$

Here we are using that $g_{j-1}\circ\cdots\circ g_1((X_j\setminus\bigcup_{k<j} X_k)\cap U\subseteq U$ and also (a) from the induction hypothesis. This establishes (a).

We also have that g_j is the identity on $(E\setminus U)\setminus(\bigcup_{k\leq j} X_k)=g_{j-1}\circ\cdots\circ g_1\left((E\setminus U)\setminus(\bigcup_{k\leq j} X_k)\right)$, which establishes (b).

If $X_{j,r}=\emptyset$ then we let g_j be the identity on $g_{j-1}\circ\cdots\circ g_1(V_{j,r}\setminus\bigcup_{k<j} X_k)\cap V_j$ and on $g_{j-1}\circ\cdots\circ g_1(V_j\setminus\bigcup_{k<j} X_k)\cap V_{j,r}$. This last property implies (d).

For the property (e), using Remark 2.28 one can assume that $g_j\circ\cdots\circ g_1$ moves no point more than $\sum_{k=1}^{j}\varepsilon_k$, because by the induction hypothesis $g_{j-1}\circ\cdots g_1$ moves no point more than $\sum_{k=1}^{j-1}\varepsilon_k$. Let us check then that the second property in (2.11) is satisfied. Take $x\in V_j\setminus(\bigcup_{k\leq j} X_j)$ and observe that if $g_k\circ\cdots\circ g_1(x)\neq g_{k-1}\circ\cdots\circ g_1(x)$ for some $k=1,\cdots,j$ means that $x\in V_k$, hence $k\in I_j$ and $i_k\geq j$. We can write

$$||g_j\circ\cdots\circ g_1(x)-x||\leq\sum_{k\in I_j,k\leq j}\left(\frac{1}{2^k}\cdot\frac{1}{2^{i_k+1}}\right)\leq\sum_{k=1}^{j}\frac{1}{2^k}\cdot\frac{1}{2^{j+1}}<\frac{1}{2^{j+1}}.$$

Now, using our induction hypothesis (f) that $g_{j-1}\circ\cdots\circ g_1$ refines $\mathcal{G}$, we can prove that $g_j\circ\cdots\circ g_1$ still refines $\mathcal{G}$. If we take an $x\in E\setminus(\bigcup_{k\leq j} X_k)$ and $g_j\circ\cdots\circ g_1(x)=g_{j-1}\circ\cdots\circ g_1(x)$, since $g_{j-1}\circ\cdots g_1$ refines $\mathcal{G}$ we are done. Otherwise $g_{j-1}\circ\cdots g_1(x), g_j\circ\cdots g_1(x)\in g_{j-1}\circ\cdots\circ g_1(V_j\setminus\bigcup_{k<j} X_k)\cap V_j$, so $x, g_j\circ\cdots\circ g_1(x)\in V_j$. We must have $x, g_j\circ\cdots\circ g_1(x)\in V_{j,r}$ for some $r\in\Omega$. Indeed, otherwise $x\in V_{j,r}$ and $g_j\circ\cdots\circ g_1(x)\in V_{j,r'}$ for different $r,r'\in\Omega$, a contradiction with (2.11) since $\operatorname{dist}(V_{j,r},V_{j,r'})\geq\frac{1}{2^{j+1}}$.

Hence we have finished our induction process.

To conclude, note that $g_j\circ\cdots\circ g_1(x)\neq g_{j-1}\circ\cdots\circ g_1(x)$ implies $x\in V_j$. This fact ensures the existence of a well-defined C^p diffeomorphism $g(x)=\lim_{j\to\infty} g_j\circ\cdots\circ g_1(x)$ from $E\setminus X$ onto $E\setminus(X\setminus U)$. The mapping g is the identity on $(E\setminus U)\setminus X$ and because $\{V_{j,r}\}_{j\geq 1,r\in\Omega}$ is a refinement of $\mathcal{G}$, g is limited by $\mathcal{G}$. □

3. Proof of Theorem 1.6

First of all notice that since the result we want to establish is invariant by diffeomorphisms, it is enough to prove it for a C^1 manifold M diffeomorphic to E in place of E. It will be very convenient for us to do so with $M = S^+$, the upper sphere of $E \times \mathbb{R}$.

Let $\|\cdot\|$ denote an equivalent norm in E which is LUR and C^1 (we will also denote by $\|\cdot\|$ the norm of F; this will do not do any harm because there will be no risk of confusion). Since E^* is separable there always exists such a norm; see [26, Corollary II.4.3] for instance.

Let us define $Y = E \times \mathbb{R}$, with norm

$$|(u,t)| = \left(\|u\|^2 + t^2\right)^{1/2},$$

and let us denote the upper sphere of Y by

$$S^+ := \{(u,t) : u \in E, t > 0, \|u\|^2 + t^2 = 1\}.$$

Observe that S^+ is the graph of

$$s(u) = \sqrt{1 - \|u\|^2},$$

which is a C^1 function[2] defined on the open unit ball B_E of E; hence,

$$d(u) = (u, s(u)) = (u, \sqrt{1 - \|u\|^2}),$$

$u \in B_E$, defines a C^1 diffeomorphism of B_E onto S^+. Since B_E is obviously C^1 diffeomorphic with E, the upper sphere S^+, which is a C^1 submanifold of codimension 1 of Y, is also diffeomorphic to E.

Therefore it will be enough to prove that every continuous mapping $f : S^+ \to F$ can be ε-approximated by a mapping $\varphi : S^+ \to F$ which is of class C^1 and has no critical points. As explained in the introduction, this will be done in three steps, the first of which consists in finding a smooth approximation of f whose critical set is a set that we can extract with the help of Theorem 1.4.

In order to find such a smooth approximation, as in [7] we will have to use a partition of unity $\{\psi_n : n \in \mathbb{N}\}$ in S^+ made out of slices of the unit ball of Y by linear functionals $g_k \in Y^*$, so that the derivative at $y \in S^+$ of a local sum $\sum \psi_k$ will belong to the span of the restrictions to $T_y S^+$ (the tangent space to S^+ at y) of a finite collection of g_k. However, the construction of the partition of unity, the technical properties that we will require, and the use that we will make of it, will be much simpler than in that paper.

To construct our partition of unity $\{\psi_n\}_{n\in\mathbb{N}}$ we next translate an old standard argument (going back to Eells in the Hilbert space case, and probably first appearing in [48, p. 28-30], later generalized by Bonic and Frampton [15] for separable Banach spaces with smooth bump functions; we follow [33, Theorem 8.25]) to the upper sphere, as in [65, 36, 7].

Let us denote $S := S_{|\cdot|}$, the unit sphere of $(Y, |\cdot|)$, and $S^* := S_{|\cdot|^*}$, the unit sphere of $(Y^*, |\cdot|^*)$. The duality mapping of the norm $|\cdot|$, defined as

$$D : S \longrightarrow S^*$$
$$D(x) = |\cdot|'(x),$$

is $|\cdot| - |\cdot|^*$ continuous since the norm $|\cdot|$ is of class C^1.

Since the norm $|\cdot|$ is locally uniformly convex we can find, for every $x \in S^+$, open slices $R_x = \{y \in S : \ g_x(y) > \delta_x\} \subset S^+$ and $P_x = \{y \in S : \ g_x(y) > \delta_x^2\} \subset S^+$, where $g_x = D(x) \in Y^*$, $0 < \delta_x < 1$, and $|g_x|^* = 1 = g_x(x)$, so that the oscillation of the functions f and ε on every P_x is less than $\varepsilon(x)/16$. We also assume, with no loss of generality, that $\operatorname{dist}(P_x,\, E \times \{0\}\,) > 0$.

[2] Smoothness of $s(u)$ at $u = 0$ is a consequence of the facts that $\|\cdot\|^2$ is trivially differentiable at 0, and that an everywhere differentiable convex function is always of class C^1; see for instance [16, Corollary 4.2.12].

Since Y is separable we can select a countable subfamily of $\{R_x\}_{x\in S^+}$ which covers S^+. Let us denote this countable subfamily by $\{R_n\}_n$, where $R_n = R_{x_n} = \{y \in S : g_n(y) > \delta_n\}$ and $g_n(x_n) = 1$. Recall that the oscillation of the functions f and ε on every $P_n = P_{x_n} = \{y \in S : g_n(y) > \delta_n^2\}$ is less than $\varepsilon(x_n)/16$; this implies that

$$\frac{15}{16}\varepsilon(x_n) \le \varepsilon(x) \le \frac{17}{16}\varepsilon(x_n) \text{ and } \|f(x) - f(y)\| \le \frac{\varepsilon(x_n)}{16}$$

for every $x, y \in P_n$. Note also that $\{P_n\}_{n\in\mathbb{N}}$ is an open cover of S^+.
For each $k \in \mathbb{N}$, let $\theta_k : \mathbb{R} \to [0,1]$ be a C^∞ function such that $\theta_k(t) = 1$ if and only if $t \ge \delta_k$, and $\theta(t) = 0$ if and only if $t \le \delta_k^2$. Next, for each $k \in \mathbb{N}$ we define $\varphi_k : S^+ \to [0,1]$ by

$$\varphi_k(x) = \theta_k(g_k(x)),$$

and note that the interior of the support of φ_k, which coincides with $\varphi_k^{-1}((0,1])$, is the open slice $P_k = \{y \in S : g_k(x) > \delta_k^2\}$.
Now, for $k = 1$, define $h_1 : S^+ \to \mathbb{R}$ by

$$h_1(x) = \varphi_1(x).$$

Notice that the interior of the support of h_1 is the open set $U_1 := P_1$.
For $k \ge 2$ let us define $h_k : S^+ \to \mathbb{R}$ by

$$h_k(x) = \varphi_k(x) \prod_{j<k} (1 - \varphi_j(x)),$$

and notice that the interior of the support of h_k is the set

$$U_k := \{y \in S^+ : g_k(y) > \delta_k^2 \text{ and } g_j(y) < \delta_j \text{ for all } j < k\}.$$

Claim 3.1. *The family $\{U_k\}_{k\in\mathbb{N}}$ is a locally finite open covering of S^+ that refines $\{P_k\}_{k\in\mathbb{N}}$. Therefore the functions*

$$\psi_n := \frac{h_n}{\sum_{k=1}^\infty h_k}, \quad n \in \mathbb{N},$$

define a C^1 partition of unity in S^+ subordinate to $\{P_k\}_{k\in\mathbb{N}}$.

Proof. Given $j \in \mathbb{N}$, if $x, y \in \{y : g_j(y) > \delta_j\}$ and $k > j$ then $\varphi_j(y) = 1$, hence $h_k(y) = 0$. Since $\{y : g_j(y) > \delta_j\}$ is a neighborhood of x in S^+ this implies that the family of supports of the h_k is locally finite. On the other hand, if $h_k(x) > 0$ for some x, k then $\varphi_k(x) > 0$, hence $x \in P_k$, and this shows that the family of the open supports of the functions h_k, which coincides with $\{U_k\}_{k\in\mathbb{N}}$, refines $\{P_k\}_{k\in\mathbb{N}}$. It only remains to prove that the family of the open supports of the functions h_k is indeed a cover, that is, for every $x \in S^+$ there exists some n_x such that $h_{n_x}(x) > 0$. We argue by contradiction: assume we had $h_k(x) = 0$ for all $k \in \mathbb{N}$, then we can show by induction that $\varphi_n(x) = 0$ for all $n \in \mathbb{N}$, which implies that $x \notin P_n$ for all n and contradicts the fact that $\{P_n\}_{n\in\mathbb{N}}$ covers S^+. Indeed, for $n = 1$ we have $0 = h_1(x) = \varphi_1(x)$. Now suppose that we have $\varphi_1(x) = \varphi_2(x) = ... = \varphi_n(x) = 0$. Then

$$0 = h_{n+1}(x) = \varphi_{n+1}(x) \prod_{j<n+1} (1 - \varphi_j(x)) = \varphi_{n+1}(x) = 0,$$

so it is also true that $\varphi_{n+1}(x) = 0$. □

We will employ the following remarkable fact.

Claim 3.2. *For every $k \in \mathbb{N}$ and every $y \in S^+$, we have*

$$h_k(y) = 0 \Longrightarrow Dh_k(y) = 0.$$

Proof. Let us assume that $h_k(y)=0$. Suppose first that $\varphi_k(y)=0$. Computing the derivative of h_k at y we get that

$$Dh_k(y)=D\varphi_k(y)\prod_{j<k}(1-\varphi_j(y))+\varphi_k(y)D\left(\prod_{j<k}(1-\varphi_j(y))\right)=D\varphi_k(y)\prod_{j<k}(1-\varphi_j(y)).$$

But $\varphi_k(y)=\theta_k(g_k(y))=0$ implies that $D\varphi_k(y)=0$, hence $Dh_k(y)=0$.
If $\varphi_k(y)\neq 0$ we must have $\varphi_j(y)=1$ for some $j<k$. But again it follows that $D\varphi_j(y)=0$, so one can check that also

$$Dh_k(y)=\varphi_k(y)D\left(\prod_{j<k}(1-\varphi_j(y))\right)=0.$$

□

Notice that, according to Claim 3.1, for every $x\in S^+$ there exist a number $n=n_x$ and an open neighborhood V_x of x in S^+ such that $\sum_{j\leq n}h_j(y)>0$ for every $y\in V_x$, and $h_k(y)=0$ for every $k>n$. This means that $\psi_k(y)=0$ for every $k>n$ and every $y\in V_x$. More precisely, n_x can be chosen as the first such j so that $x\in\{y\in S^+ : g_j(y)>\delta_j\}=R_j$ and $V_x=R_j$.
Let us also call $m=m_y$ the largest j such that $h_j(y)\neq 0$. Note that m_y is also the largest j for which $\psi_j(y)\neq 0$. Thus, for every $y\in V_x$, we have

$$m_y=\max\{j : y\in\psi_j^{-1}((0,1])\}\leq n_x.$$

In order to calculate the derivatives of this partition of unity in this neighborhood V_x of x, let us introduce the functions

$$H_k(y)=\frac{h_k(y)}{\sum_{j=1}^n h_j(y)},\qquad y\in V_x,\ \ k=1,...,n,$$

which are well defined on V_x and in fact can be extended as C^1 smooth functions to an open subset $\mathcal{S}_n$ of Y containing V_x. Specifically, noting that each h_k is well-defined and C^∞ smooth on the whole Y, one can choose $\mathcal{S}_n=\bigcup_{i=1}^n h_i^{-1}((0,1])$. Slightly abusing notation, we will keep denoting these extensions by H_k, and we will also think of the functions h_j, ψ_j, g_j, $j\leq n$, as being C^1 smooth functions defined on this open set $\mathcal{S}_n$. Therefore, to calculate the derivative of ψ_k on V_x for $k=1,...,n$, we only have to calculate the derivative of H_k at each $y\in V_x$ for $k=1,...,n$ and then restrict it to the tangent spaces T_yS^+. The exact expression for the derivative of the functions $H_1,...,H_n$ on V_x will not be particularly interesting or useful to us. The only thing we need to know is that there are C^1 smooth functions $\sigma_{k,j}$, $1\leq k,j\leq n$, (actually, $\sigma_{k,j}$ will be C^∞ smooth) defined on $\mathcal{S}_n$ so that

$$DH_k(y)=\sum_{j=1}^n\sigma_{k,j}(y)\,g_j$$

for $k=1,...,n$, $y\in V_x$, and that, in fact, as an immediate consequence of Claim 3.2 and the definition of m_y we have

$$DH_k(y)=\sum_{j=1}^{m_y}\sigma_{k,j}(y)\,g_j.$$

for $k=1,\ldots,m_y$, $y\in V_x$.
Note that even though each H_k is C^∞ smooth on an open subset $\mathcal{S}_n$ of Y, we cannot say that ψ_k is C^∞ smooth too, because S^+ has not a C^∞ smooth submanifold structure. The functions ψ_k are just C^1 because S^+ is just a C^1 manifold modeled on E. Now, if we want to know what the derivative of

the functions ψ_k, $k=1,...,n$, on V_x looks like, because $H_k=\psi_k$ on V_x and V_x is open in S^+, we only have to restrict DH_k to the tangent spaces T_yS^+ for each $y\in V_x$. Hence we have

$$D\psi_k(y)(v)=\sum_{j=1}^{n}\sigma_{k,j}(y)\,g_j(v)=\sum_{j=1}^{m_y}\sigma_{k,j}(y)\,g_j(v)$$

for each $v\in T_yS^+$.
This expression can be somewhat misleading at first sight, because one might think that, for $k=1,...,n$, $D\psi_k(y)$ is just a linear combination of the functionals $g_1,...,g_n$, and this is not exactly so. It is a linear combination of the restrictions of $g_1,...,g_n$ to T_yS^+, and therefore for every $y\in V_x$ it is a different linear combination of different linear functionals $g_1|_{T_yS^+}$, ..., $g_1|_{T_yS^+}$, each of them defined on a space depending on y.
In order to fully clarify this important point, let us calculate the tangent space T_yS^+ at $y=(u_y,t_y)$. Since S^+ is the graph of the function $s(u)=\sqrt{1-\|u\|^2}$, the most natural representation of T_yS^+ is given by

$$T_yS^+=\{(u,t)\in Y=E\times\mathbb{R}\,:\,t=L_y(u),u\in E\}=\{(u,L_y(u)):u\in E\}\subset Y, \tag{3.1}$$

where L_y is the derivative $Ds(u_y)$ of the function s evaluated at the point $u_y=d^{-1}(y)\in B_E$ (recall that $d(u)=(u,s(u))$); in other words, if $y\neq(0,1)$,

$$L_y(w)=-\frac{\|u_y\|}{\sqrt{1-\|u_y\|^2}}D\|\cdot\|(u_y)(w)=-\frac{\sqrt{1-t_y^2}}{t_y}D\|\cdot\|(u_y)(w)$$

for each $w\in E$. Of course we have $L_{(0,1)}(w)=Ds(0)(w)=0$ for each $w\in E$, and $T_{(0,1)}S^+=E\times\{0\}$.
This is the vectorial tangent hyperplane to S^+ at y, as opposed to the affine tangent hyperplane to S^+, which is just $y+T_yS^+$. Since derivatives of mappings act on vectorial tangent hyperplanes we may forget the affine hyperplanes $y+T_yS^+$ in what follows.
Now, recall that $g_j\in Y^*$ and therefore these functionals are of the form

$$g_j(u,t)=g_j^1(u)+g_j^2t,\qquad (u,t)\in E\times\mathbb{R}=Y,$$

where $g_j^1\in E^*$ and $g_j^2\in\mathbb{R}$.
Therefore the derivative of $g_j|_{S^+}$ at $y\in S^+$ is given by

$$Dg_j(y)(u,L_y(u))=g_j(u,L_y(u))=g_j^1(u)+g_j^2L_y(u) \tag{3.2}$$

for every $v=(u,L_y(u))\in T_yS^+$. Thus, for every $y\in V_x$ and every $k=1,...,n$, we have

$$D\psi_k(y)(v)=D\psi_k(y)(u,L_y(u))=\sum_{j=1}^{n}\sigma_{k,j}(y)\,(g_j^1(u)+g_j^2L_y(u))=\sum_{j=1}^{m_y}\sigma_{k,j}(y)\,(g_j^1(u)+g_j^2L_y(u))$$

for every $v=(u,L_y(u))\in T_yS^+$.
Finally, let us note that the points $x_n\in R_n$ satisfy that

$$\frac{15}{16}\,\varepsilon(x_n)\le\varepsilon(y)\le\frac{17}{16}\,\varepsilon(x_n)\quad\text{for every } y\in P_n,\quad\text{and}\quad\sup_{x,y\in P_n}\|f(x)-f(y)\|\le\frac{\varepsilon(x_n)}{16}.$$

The following lemma summarizes the properties of the partition of unity $\{\psi_n\}_{n\in\mathbb{N}}$ which will be most useful to us.

Lemma 3.3. *Given two continuous functions $f:S^+\to F$ and $\varepsilon:S^+\to(0,\infty)$, there exists a collection of norm-one linear functionals $\{g_k\}_{k\in\mathbb{N}}\subset Y^*$, an open covering $\{P_n\}_{n\in\mathbb{N}}$ of S^+, and a C^1 partition of unity $\{\psi_n\}_{n\in\mathbb{N}}$ on S^+ such that:*

(1) *$\{\psi_n\}_{n\in\mathbb{N}}$ is subordinate to $\{P_n\}_{n\in\mathbb{N}}$.*
(2) *For every $x \in S^+$ there exist a neighborhood V_x of x in S^+ and a number $n = n_x \in \mathbb{N}$ such that $\psi_m = 0$ on V_x for all $m > n$, and the derivatives of the functions $\psi_1, ..., \psi_n$ on V_x are of the form*

$$D\psi_k(y)(v) = \sum_{j=1}^{n} \sigma_{k,j}(y)\, g_j(v) = \sum_{j=1}^{m_y} \sigma_{k,j}(y)\, g_j(v),$$

for $v \in T_yS^+$, the tangent hyperplane to S^+ at $y \in S^+ \cap V_x$, and where $m_y \leq n$ is the largest number such that $\psi_{m_y}(y) \neq 0$. More precisely, if L_y denotes the derivative of the function $s(u) = \sqrt{1 - \|u\|^2}$ evaluated at the point $d^{-1}(y)$, where $d(u) = (u, s(u))$, we have

$$D\psi_k(y)(v) = D\psi_k(y)(u, L_y(u)) = \sum_{j=1}^{n} \sigma_{k,j}(y)\,(g_j^1(u) + g_j^2 L_y(u)) = \sum_{j=1}^{m_y} \sigma_{k,j}(y)\,(g_j^1(u) + g_j^2 L_y(u))$$

for every $k = 1, ..., n$, and for every $v = (u, L_y(u)) \in T_yS^+$, where the functions $\sigma_{k,j} : V_x \to \mathbb{R}$ are of class C^1, and $g_j^1 \in E^*$, $g_j^2 \in \mathbb{R}$, $j = 1, ..., n$.
(3) *For every $n \in \mathbb{N}$ there exist a point $y_n := x_n \in P_n$ such that*

$$\frac{15}{16}\,\varepsilon(y_n) \leq \varepsilon(y) \leq \frac{17}{16}\,\varepsilon(y_n) \quad \text{for every } y \in P_n, \quad \text{and} \quad \sup_{x,y\in P_n} \|f(x) - f(y)\| \leq \frac{\varepsilon(y_n)}{16}.$$

Now we are ready to start the construction of our approximating function $\varphi : S^+ \to F$, which will be of the form

$$\varphi(x) = \sum_{n=1}^{\infty} (f(y_n) + T_n(x))\, \psi_n(x),$$

where the y_n are the points given by condition (4) of the preceding lemma, and the operators $T_n : Y \to F$ will be carefully defined below.

Case 1: Assume that F is infinite-dimensional.
We will have to make repeated use of the following fact, whose proof is elementary and can be left to the interested reader.

Lemma 3.4. *If E is a Banach space which is isomorphic to $E \oplus E$ then for every finite-codimensional closed subspace V of E, there exists a decomposition*

$$E = E_1 \oplus E_2 \oplus E_3,$$

with factors E_1, E_2, E_3 isomorphic to E, such that $E_1 \oplus E_2 \subset V$.

We start considering a decomposition

$$E = E_{1,1} \oplus E_{1,2},$$

with infinite-dimensional factors isomorphic to E, and we define a continuous linear surjection $S_1 : E \to F$ such that $S_1 = 0$ on $E_{1,2}$. This can be done by taking a continuous linear surjection $R_1 : E_{1,1} \to F$ (which exists because by assumption there exists such an operator from E onto F and $E_{1,1}$ is isomorphic to E), and setting $S_1 = R_1 \circ P_{1,1}$, where $P_{1,1} : E_{1,1} \oplus E_{1,2} \to E_{1,1}$ is the projection onto the first factor associated to this decomposition of E. Next, recall that the linear functionals $g_n \in Y^* = (E \times \mathbb{R})^*$ are of the form

$$g_j(u, t) = g_j^1(u) + g_j^2 t,$$

where $g_j^1 \in E^*$ and $g_j^2 \in \mathbb{R}$. Of course, $\bigcap_{j=1}^2 \operatorname{Ker} g_j^1$ is a finite-codimensional subspace of E, so by using Lemma 3.4 (2) with $V = E_{1,2} \cap \bigcap_{j=1}^2 \operatorname{Ker}(g_j^1)$, and with $E_{1,2}$ in place of E, we may find a decomposition of the second factor $E_{1,2}$,

$$E_{1,2} = E_{2,1} \oplus E_{2,2} \oplus E_{2,3},$$

with factors $E_{2,1}$, $E_{2,2}$ and $E_{2,3}$ isomorphic to E, and

$$E_{2,1} \oplus E_{2,2} \subset \bigcap_{j=1}^{2} \operatorname{Ker} g_j^1,$$

and we may easily define a bounded linear operator S_2 from E onto F such that $S_2 = 0$ on $E_{1,1} \oplus E_{2,2} \oplus E_{2,3}$ (this can be done by taking a surjective operator $R_2 : E_{2,1} \to F$ and defining $S_2 = R_2 \circ P_{2,1} \circ P_{1,2}$, where $P_{1,2} : E_{1,1} \oplus E_{1,2} \to E_{1,2}$ and $P_{2,1} : E_{2,1} \oplus E_{2,2} \oplus E_{2,3} \to E_{2,1}$ are the projections associated to the corresponding decompositions).

We continue this process by induction: assuming that we have already defined decompositions $E = E_{1,1} \oplus E_{1,2}$, $E_{1,2} = E_{2,1} \oplus E_{2,2} \oplus E_{2,3}$, $E_{2,3} = E_{3,1} \oplus E_{3,2} \oplus E_{3,3}$, ..., $E_{n-1,3} = E_{n,1} \oplus E_{n,2} \oplus E_{n,3}$, and surjective operators

$$S_k : E = E_{1,1} \oplus (E_{2,1} \oplus E_{2,2}) \oplus ... \oplus (E_{k-1,1} \oplus E_{k-1,2}) \oplus (E_{k,1} \oplus E_{k,2} \oplus E_{k,3}) \to F, \;\; k = 2, ..., n,$$

so that S_k is zero on all the factors of this decomposition except $E_{k,1}$, and

$$E_{k,1} \oplus E_{k,2} \subset \bigcap_{j=1}^{k} \operatorname{Ker}(g_j^1),$$

we again apply Lemma 3.4 (2) to write

$$E_{n,3} = E_{n+1,1} \oplus E_{n+1,2} \oplus E_{n+1,3},$$

with factors isomorphic to E and

$$E_{n+1,1} \oplus E_{n+1,2} \subset \bigcap_{j=1}^{n+1} \operatorname{Ker}(g_j^1),$$

and we define a continuous linear surjection

$$S_{n+1} : E = E_{1,1} \oplus (E_{2,1} \oplus E_{2,2}) \oplus ... \oplus (E_{n,1} \oplus E_{n,2}) \oplus (E_{n+1,1} \oplus E_{n+1,2} \oplus E_{n+1,3}) \to F,$$

by setting it equal to 0 on all the factors of this decomposition except $E_{n+1,1}$, which is mapped onto F.

Having this collection of surjective operators $S_n : E \to F$ at our disposal, we finally define $T_n : Y \to F$ by

$$T_n(u,t) = \frac{\varepsilon(y_n)}{4\|S_n\|} S_n(u),$$

and $\varphi : S^+ \to F$ by

$$\varphi(x) = \sum_{n=1}^{\infty} \left(f(y_n) + T_n(x)\right) \psi_n(x).$$

It is clear that φ is well defined and of class C^1.

In the rest of the proof we will check that this mapping ε-approximates f on S^+, and that the set of critical points of φ is a set which can be diffeomorphically extracted by using Theorem 1.4. Then the proof will be completed by setting $g = \varphi \circ h$, where $h : S^+ \to S^+ \setminus C_\varphi$ is a C^1 diffeomorphism which is close enough to the identity.

Claim 3.5. *The mapping φ approximates f.*

Proof. By condition (3) of Lemma 3.3, we know that the oscillation of f in P_n is less that $\varepsilon(y_n)/16$, and by definition of T_n, we have $\|T_n\| \leq \varepsilon(y_n)/4$, hence $\|T_n(x)\| \leq \varepsilon(y_n)/4$ for all $x \in S^+$ too, because $\|x\| = 1$. Now, if $\psi_n(x) \neq 0$, then $x \in P_n$ and

$$\begin{aligned} \|f(y_n) + T_n(x) - f(x)\| &\leq \|f(y_n) - f(x)\| + \|T_n(x)\| \\ &\leq \varepsilon(y_n)/16 + \varepsilon(y_n)/4 = \frac{5}{16}\varepsilon(y_n) < \frac{15}{32}\varepsilon(y_n) \leq \varepsilon(x)/2. \end{aligned} \tag{3.3}$$

Therefore

$$\|\varphi(x) - f(x)\| = \Big\|\sum_{n=1}^{\infty}(f(y_n) + T_n(x) - f(x))\,\psi_n(x)\Big\| \leq \sum_{n=1}^{\infty}\|f(y_n) + T_n(x) - f(x)\|\,\psi_n(x) \leq \varepsilon(x)/2.$$

□

Let us now consider the question as to how big the critical set C_φ can be. We need to calculate the derivative of our function φ. To this end we first have to examine the expressions for the derivatives of the operators T_n restricted to S^+. These are simpler than those of the g_n's on S^+, because of the way the operators $T_n : Y \to F$ have been defined. Indeed, for every $v = (u, L_y(u)) \in T_yS^+$ we have that

$$T_n(v) = T_n(u, L_y(u)) = \frac{\varepsilon(y_n)}{4\|S_n\|}\,S_n(u), \tag{3.4}$$

and we have that $D(T_{n|_{S^+}})(y)$ is the restriction of $DT_n(y) = T_n$ to T_yS^+, that is to say, if $v = (u, L_y(u)) \in T_yS^+$ then

$$DT_n(y)(u, L_y(u)) = \frac{\varepsilon(y_n)}{4\|S_n\|}\,S_n(u). \tag{3.5}$$

Now, recall that, by condition (2) of Lemma 3.3, for every $x \in S^+$ there is a neighborhood V_x of x in S^+ and a number $n = n_x \in \mathbb{N}$ such that

$$\varphi(y) = \sum_{j=1}^{n}(f(y_j) + T_j(y))\,\psi_j(y)$$

for every $y \in V_x$. Fix $y \in V_x$ and recall that for $m = m_y$, the largest number j for which $h_j(y) \neq 0$ (or $\psi_j(y) \neq 0$), we have

$$\varphi(y) = \sum_{j=1}^{m}(f(y_j) + T_j(y))\,\psi_j(y).$$

By using (3.5) and the expression for $D\psi_j(y)$ given in Lemma 3.3, we easily see that

$$\begin{aligned} D\varphi(y)(u, L_y(u)) &= \sum_{j=1}^{n}\psi_j(y)\,\frac{\varepsilon(y_j)}{4\|S_j\|}\,S_j(u) + \sum_{j=1}^{n}\left(g_j^1(u) + g_j^2 L_y(u)\right)\alpha_{n,j}(y) = \\ &= \sum_{j=1}^{m}\psi_j(y)\,\frac{\varepsilon(y_j)}{4\|S_j\|}\,S_j(u) + \sum_{j=1}^{m}\left(g_j^1(u) + g_j^2 L_y(u)\right)\alpha_{m,j}(y) \end{aligned} \tag{3.6}$$

for every $(u, L_y(u)) \in T_yS^+$, $y \in V_x$, where the functions $\alpha_{n,j} : V_x \subset S^+ \to F$ are of class C^1 because $\alpha_{n,j}(y) = \sum_{i=1}^{n}(f(y_i) + T_i(y))\sigma_{i,j}(y)$.

Let us now show that the critical set of φ is relatively small.

When $n_x = 1$ we have $\varphi(y) = f(y_1) + T_1(y)$ on V_x, so $D\varphi(y)$ is the restriction of T_1 to T_yS^+, and equation (3.4) for $n = 1$ implies that this restriction is a surjective operator. So it is clear that φ has no critical point on V_x.

Claim 3.6. *If $n_x \geq 2$ then $C_\varphi \cap V_x$ is contained in the set*

$$A_x := \left\{ y \in S^+ : E_{n,2} \subset Ker\, L_y \right\}.$$

Recall that $L_y = Ds(u_y)$, where $s(u) = \sqrt{1 - \|u\|^2}$, $d(u) = (u, s(u))$, and $u_y = d^{-1}(y)$.

Proof. Let us see that, if $y \in V_x \setminus A_x$ then $D\varphi(y) : T_yS^+ \to F$ is surjective, that is, for every $w \in F$ there exists $v \in T_yS^+$ such that $D\varphi(y)(v) = w$. Let $m = m_y$ be the largest number such that $\psi_m(y) \neq 0$. Recall that $m \leq n$. Since the operator

$$S_m : E = E_{1,1}(\oplus E_{2,1} \oplus E_{2,2}) \oplus ... \oplus (E_{m-1,1} \oplus E_{m-1,2}) \oplus (E_{m,1} \oplus E_{m,2} \oplus E_{m,3}) \to F$$

is surjective and equal to zero on all the factors of this decomposition except $E_{m,1}$ (which is mapped onto F), we may find $u_{m,1} \in E_{m,1}$ so that

$$S_m(u_{m,1}) = 4\varepsilon(y_m)^{-1}\psi_m(y)^{-1}\|S_m\|\, w.$$

Now, since $y \notin A_x$ there exists some $e_{n,2} \in E_{n,2} \setminus \text{Ker } L_y$, that is to say, $L_y(e_{n,2}) \neq 0$, and this implies that, if we put

$$t_0 := -\frac{L_y(u_{m,1})}{L_y(e_{n,2})},$$

then the vector

$$u := u_{m,1} + t_0 e_{n,2},$$

satisfies that

$$L_y(u) = 0.$$

But recall that, for every $k \leq n$, we have $E_{k,1} \oplus E_{k,2} \subset \bigcap_{j=1}^k \text{Ker}(g_j^1)$; in particular, $E_{m,1} \oplus E_{m,2} \subset \bigcap_{j=1}^m \text{Ker}(g_j^1)$ because $m \leq n$. Hence, $g_j^1(u) = 0$ for every $1 \leq j \leq m$. It follows that

$$\sum_{j=1}^{m} \left(g_j^1(u) + g_j^2 L_y(u)\right) \alpha_{m,j}(y) = 0.$$

The rest of the operators $S_1, ..., S_{m-1}$ are zero on $E_{m,1} \oplus E_{n,2}$, so we have

$$S_j(u) = 0 \text{ for every } j = 1, ..., m-1,$$

and since S_m is zero on $E_{n,2} \subset E_{m,3}$ we also have $S_m(t_0 e_{n,2}) = 0$. Therefore, by combining these equalities with equation 3.6, we obtain that

$$D\varphi(y)(u, L_y(u)) = w,$$

and the proof of the claim is complete. □

Lemma 3.7. *For $x \in S^+$ with $n := n_x \geq 2$, the set A_x of Claim 3.6 is of the form*

$$A_x = d(G(f_x) \cap B_E),$$

where $G(f_x)$ is the graph of a continuous mapping $f_x : E_{1,1} \oplus (E_{2,1} \oplus E_{2,2}) \oplus \cdots \oplus (E_{n,1} \oplus E_{n,3}) \to E_{n,2}$.

Proof. Note that, by Claim 3.6,

$$A_x = \{d(u) = \left(u, \sqrt{1 - \|u\|^2}\right) : \|u\| < 1, u \in \mathcal{A}_x\},$$

where

$$\mathcal{A}_x := \bigcap_{e \in E_{n,2}} \{u \in E \setminus \{0\} \, : \, \langle D\| \cdot \|(u), e\rangle = 0\} \cup \{0\}.$$

Let us denote $E'_{n,2} = E_{1,1} \oplus (E_{2,1} \oplus E_{2,2}) \oplus \cdots \oplus (E_{n,1} \oplus E_{n,3})$, and let us see that there exists a mapping $f_x : E'_{n,2} \to E_{n,2}$ such that $\mathcal{A}_x = G(f_x) = \{w + f_x(w) : w \in E'_{n,2}\}$.
Pick a point $w \in E'_{n,2}$. Note that the function $E_{n,2} \ni v \mapsto \psi_w(v) := \|w+v\|^2$ is convex and continuous, and satisfies $\lim_{\|v\| \to \infty} \psi_w(v) = \infty$, hence, since $E_{n,2}$ is reflexive, ψ_w attains a minimum at some point $v_w \in E_{n,2}$; in fact this minimum point v_w is unique because the norm $\| \cdot \|$ is strictly convex. Let us denote

$$f_x(w) := v_w.$$

Note that the critical points of ψ_w, with $w \neq 0$, are exactly the points $v \in E_{n,2}$ such that

$$\frac{d}{dt}\|w + v + te\|^2 \, |_{t=0} = 0 \text{ for every } e \in E_{n,2}$$

or equivalently

$$\|w + v\| \, \langle D\| \cdot \|(w + v), e\rangle = 0 \text{ for every } e \in E_{n,2},$$

which in turn is equivalent to saying that $w + v \in \mathcal{A}_x$; we let $f_x(0) = v_0 = 0$.
Therefore the unique point $v \in E_{n,2}$ so that $w + v \in \mathcal{A}_x$ is the point $v = f_x(w)$. This shows that $\mathcal{A}_x$ is the graph of the function f_x.
Now let us see that the function $f_x : E'_{n,2} \to E_{n,2}$ is continuous. Suppose f_x is discontinuous at w_0 and let $v_0 := f_x(w_0)$. Then there exist sequences $w_k \to w_0$ in $E'_{n,2}$ and $v_k := f_x(w_k)$ in $E_{n,2}$ and a number $\varepsilon_0 > 0$ so that

$$\|v_k - v_0\| \geq \varepsilon_0 \text{ for all } k \in \mathbb{N}. \tag{3.7}$$

From the previous argument we know that the point v_k is characterized as being the unique point $v_k \in E_{n,2}$ for which we have

$$\|w_k + v_k\| \leq \|w_k + v_k + e\| \text{ for all } e \in E_{n,2}, \tag{3.8}$$

and similarly v_0 is the unique point $v_0 \in E_{n,2}$ for which

$$\|w_0 + v_0\| \leq \|w_0 + v_0 + e\| \text{ for all } e \in E_{n,2}. \tag{3.9}$$

By taking $e = -v_k$ in (3.8) we learn that

$$\|v_k\| - \|w_k\| \leq \|w_k + v_k\| \leq \|w_k\|,$$

hence $\|v_k\| \leq 2\|w_k\|$, and because $\|w_k\|$ converges to $\|w_0\|$ we deduce that (v_k) is bounded. Since $E_{n,2}$ is reflexive, this implies that (v_k) has a subsequence that weakly converges to a point $\xi_0 \in E_{n,2}$. We keep denoting this subsequence by (v_k).
Now, if we take $e = -v_k + e'$ in (3.8), with $e' \in E_{n,2}$, we obtain

$$\|w_k + v_k\| \leq \|w_k + e'\| \text{ for all } e' \in E_{n,2}.$$

This implies (using the facts that $v_k \rightharpoonup \xi_0$ and $w_k \to w_0$, and the weak lower semicontinuity of the norm) that

$$\|w_0 + \xi_0\| \leq \liminf_{k \to \infty} \|w_k + v_k\| \leq \liminf_{k \to \infty} \|w_k + e'\| = \|w_0 + e'\| \text{ for all } e' \in E_{n,2}. \tag{3.10}$$

That is, we have shown that

$$\|w_0 + \xi_0\| \leq \|w_0 + e'\| \text{ for all } e' \in E_{n,2}. \tag{3.11}$$

By taking $e' = \xi_0 + \xi$ with $\xi \in E_{n,2}$ we conclude that

$$\|w_0 + \xi_0\| \leq \|w_0 + \xi_0 + \xi\| \text{ for all } \xi \in E_{n,2}.$$

According to (3.9), v_0 is the only point which can satisfy this inequality. Hence $\xi_0 = v_0$.
But (3.10) tells us even more: by taking $e' = \xi_0$ we also learn that there exists a subsequence (w_{k_j}) of (w_k) such that

$$\|w_{k_j} + v_{k_j}\| \to \|w_0 + \xi_0\|.$$

Since we also know that $w_{k_j} + v_{k_j}$ converges to $w_0 + \xi_0$ weakly and the norm $\|\cdot\|$ is locally uniformly convex (hence $\|\cdot\|$ has the Kadec-Klee property), this implies that $w_{k_j} + v_{k_j}$ converges to $w_0 + \xi_0 = w_0 + v_0$ in the norm topology as well. As we also have $\lim_{j\to\infty} w_{k_j} = w_0$ in norm, we deduce that $\lim_{j\to\infty} \|v_{k_j} - v_0\| = 0$, which contradicts (3.7). □

Now we can easily finish the proof of Theorem 1.6. By Claim 3.6 and Lemma 3.7, we see that C_φ is a diffeomorphic image in S^+ of a relatively closed set Z of the open unit ball B_E of E which has the property of being locally contained in the graph of a continuous function defined on a complemented subspace of infinite codimension in E. Indeed, let $Z := d^{-1}(C_\varphi) \subset B_E$. Since C_φ is closed in S^+, Z is relatively closed in B_E. Also if we take $z \in Z$ then, according to Lemma 3.7 applied to $x = d(z) \in S^+$, $n = n_x$, and V_x, for a neighborhood $U_z := d^{-1}(V_x)$ of z, we have $Z \cap U_z \subseteq G(f_x)$, where

$$G(f_x) = \{u = (w, v) \in E'_{n,2} \oplus E_{n,2} = E : \, v = f_x(w)\}.$$

Observe that E has C^1 smooth partitions of unity since E has a separable dual. Therefore we may apply Theorem 1.4 to find a C^1 diffeomorphism which extracts C_φ from S^+; more precisely, there exists a diffeomorphism $h : S^+ \to S^+ \setminus C_\varphi$ which, in addition, is limited by the open cover $\mathcal{G}$ that we next define. Recall that we have

$$\|\varphi(x) - f(x)\| \leq \varepsilon(x)/2 \tag{3.12}$$

for all $x \in S^+$. Since φ and ε are continuous, for every $z \in S^+$ there exists $\delta_z > 0$ so that if $x, y \in B(z, \delta_z)$ then $\|\varphi(y) - \varphi(x)\| \leq \varepsilon(z)/4 \leq \varepsilon(x)/2$. We set $\mathcal{G} = \{B(x, \delta_x) \, : \, x \in S^+\}$.
Finally, let us define

$$g = \varphi \circ h.$$

Since h is limited by $\mathcal{G}$ we have that, for any given $x \in S^+$, there exists $z \in S^+$ such that $x, h(x) \in B(z, \delta_z)$, and therefore $\|\varphi(h(x)) - \varphi(x)\| \leq \varepsilon(z)/4$, that is, we have that

$$\|g(x) - \varphi(x)\| \leq \varepsilon(z)/4 \leq \varepsilon(x)/2.$$

By combining this inequality with (3.12), we obtain that

$$\|g(x) - f(x)\| \leq \varepsilon(x)$$

for all $x \in S^+$. Besides, it is clear that g does not have any critical point: since $h(x) \notin C_\varphi$, we have that the linear map $D\varphi(h(x)) : T_{h(x)}S^+ \to F$ is surjective, and $Dh(x) : T_xS^+ \to T_{h(x)}S^+$ is a linear isomorphism, so $Dg(x) = D\varphi(h(x)) \circ Dh(x)$ is a linear surjection from T_xS^+ onto F for every $x \in S^+$.

Case 2: Assume that $F = \mathbb{R}^m$. The main idea of the proof is very similar to that of Case 1. The fact that F is finite dimensional will allow us dispense with the hypothesis that $E = E \oplus E$. We will use the same partition of unity $\{\psi_n\}_{n\in\mathbb{N}}$ provided by Lemma 3.3. We will decompose E inductively as follows. Since $Kerg_1^1$ has infinite dimension we can write

$$E = E_1 \oplus G_1,$$

where $E_1 = \mathbb{R}^m$ and $G_1 \subseteq \text{Ker}g_1^1$. Then $G_1 \cap \bigcap_{j=1}^2 \text{Ker}g_j^1$ has codimension 0 or 1 in G_1, which is infinite-dimensional, and we can write

$$E = E_1 \oplus (E_{2,1} \oplus E_{2,2} \oplus G_2),$$

where $E_{2,1} = \mathbb{R}^m$, $E_{2,2} = \{0\}$ or $E_{2,2} = \mathbb{R}$, $G_1 = E_{2,1} \oplus E_{2,2} \oplus G_2$ for some G_2 with $\dim G_2 = \infty$, and

$$E_{2,1} \oplus G_2 = \bigcap_{j=1}^{2} \text{Ker}g_j^1 \cap G_1 \subseteq \bigcap_{j=1}^{2} \text{Ker}g_j^1.$$

Inductively, we can write

$$E = E_1 \oplus (E_{2,1} \oplus E_{2,2}) \oplus \cdots \oplus (E_{n-1,1} \oplus E_{n-1,2}) \oplus (E_{n,1} \oplus E_{n,2} \oplus G_n), \tag{3.13}$$

where $E_1, E_{2,1} \ldots, E_{n,1} = \mathbb{R}^m$, $E_{2,2}, \ldots, E_{n,2}$ are subspaces of dimension 0 or 1,

$$E_{n,2} \oplus G_n \subseteq \bigcap_{j=1}^{n} \text{Ker}g_j^1,$$

and

$$G_k = (E_{k+1,1} \oplus E_{k+1,2} \oplus G_{k+1})$$

for every $k = 1, \ldots, n$.

Now, for each $n \in \mathbb{N}$, we define a continuous linear surjection $S_n : E \to F$ by setting it to be 0 on all the factors of the decomposition (3.13) except on $E_{n,1}$, which is mapped onto $F = \mathbb{R}^m$, and we construct our approximating function φ exactly as in the proof of Theorem 1.6. At this point, we only need to show the following variant of Claim 3.6 (in which G_n replaces the subspace $E_{n,2}$ of the previous proof).

Claim 3.8. *If $n_x \geq 2$ then $C_\varphi \cap V_x$ is contained in the set*

$$A_x := \left\{ y \in S^+ : G_n \subset Ker\, L_y \right\}.$$

Recall that $L_y = Ds(u_y)$, where $s(u) = \sqrt{1 - \|u\|^2}$, $d(u) = (u, s(u))$, and $u_y = d^{-1}(y)$.

Proof. Let us see that, if $y \in V_x \setminus A_x$ then $D\varphi(y) : T_y S^+ \to F$ is surjective, that is, for every $w \in F$ there exists $v \in T_y S^+$ such that $D\varphi(y)(v) = w$. Let $m = m_y$ be the largest number such that $\psi_m(y) \neq 0$. Recall that $m \leq n$. Since the operator

$$S_m : E = E_{1,1}(\oplus E_{2,1} \oplus E_{2,2}) \oplus ... \oplus (E_{m-1,1} \oplus E_{m-1,2}) \oplus (E_{m,1} \oplus E_{m,2} \oplus G_m) \to F$$

is surjective and equal to zero on all the factors of the decomposition (3.13) except on $E_{m,1}$ (which is mapped onto F), we may find $u_{m,1} \in E_{m,1}$ so that

$$S_m(u_{m,1}) = 4\varepsilon(y_m)^{-1}\psi_m(y)^{-1}\|S_m\|\, w.$$

Now, since $y \notin A_x$ there exists $e_n \in G_n \setminus \text{Ker } L_y$. If we set

$$t_0 := -\frac{L_y(u_{m,1})}{L_y(e_n)},$$

then the vector

$$u := u_{m,1} + t_0 e_n,$$

satisfies that

$$L_y(u) = 0.$$

But recall that, for every $k \leq n$, we have $E_{k,1} \oplus G_k \subset \bigcap_{j=1}^{k} \text{Ker}(g_j^1)$; in particular, $E_{m,1} \oplus G_m \subset \bigcap_{j=1}^{m} \text{Ker}(g_j^1)$ because $m \leq n$. Hence, $g_j^1(u) = 0$ for every $1 \leq j \leq m$. It follows that

$$\sum_{j=1}^{m} \left(g_j^1(u) + g_j^2 L_y(u)\right) \alpha_{m,j}(y) = 0.$$

The rest of the operators $S_1, ..., S_{m-1}$ are zero on $E_{m,1} \oplus G_n$, so we have

$$S_j(u) = 0 \text{ for every } j = 1, ..., m-1,$$

and since S_m is zero on $G_n \subset G_m$ we also have $S_m(t_0 e_n) = 0$. Therefore, by combining these equalities with equation 3.6, we obtain that

$$D\varphi(y)(u, L_y(u)) = w,$$

and the proof of the claim is complete. □

Then we also have the following.

Lemma 3.9. *For $x \in S^+$ with $n := n_x \geq 2$, the set A_x of Claim 3.8 is of the form*

$$A_x = d(G(f_x) \cap B_E),$$

where $G(f_x)$ is the graph of a continuous mapping $f_x : E_{1,1} \oplus (E_{2,1} \oplus E_{2,2}) \oplus \cdots \oplus (E_{n,1} \oplus E_{n,2}) \to G_n$.

Proof. Repeat the proof Lemma 3.7, just replacing $E_{n,2}$ with G_n. □

Let $Z := d^{-1}(C_\varphi) \subset B_E$. According to Lemma 3.9 applied to $x = d(z) \in S^+$, $n = n_x$, and V_x, for a neighborhood $U_z := d^{-1}(V_x)$ of z, we have $Z \cap U_z \subseteq G(f_x)$, where

$$G(f_x) = \{u = (w, v) \in G'_n \oplus G_n = E : \, v = f_x(w)\},$$

with G'_n denoting $E_1 \oplus (E_{2,1} \oplus E_{2,2}) \oplus \cdots \oplus (E_{n-1,1} \oplus E_{n-1,2}) \oplus (E_{n,1} \oplus E_{n,2})$. Since G_n is infinite-dimensional, we may use Theorem 1.4, and the rest of the proof goes exactly as in Case 1. □

Remark 3.10. Observe that in the infinite-dimensional case we could have asked A_x to be

$$A_x := \{y \in S^+ : \, E_{n,3} \subset Ker L_y\},$$

using $E_{n,3}$ instead of $E_{n,2}$ and requiring that $E_{n,1} \oplus E_{n,3} \subseteq \bigcap_{j=1}^{n} Ker g_j^1$.

4. Proof of Theorem 1.7

First of all let us note that since E admits a C^1 equivalent norm, E cannot contain a closed subspace isomorphic to ℓ_1. Furthermore, as noted following the statement of Theorem 1.7, condition (2) implies that the basis $\{e_n^*\}_{n\in\mathbb{N}}$ is unconditional, and therefore by [50, Theorem 1.c.9], is *shrinking*, that is, we have that $E^* = \overline{\text{span}}\{e_n^* : n \in \mathbb{N}\}$, where $\{e_n^*\}_{n\in\mathbb{N}}$ are the biorthogonal functionals associated to $\{e_n\}_{n\in\mathbb{N}}$ (in particular, E^* is separable).
We keep using the notations $Y = E \times \mathbb{R}$ and S^+ from the proof of Theorem 1.6. As in the case of Theorem 1.6, it will be enough to prove Theorem 1.7 with S^+ in place of E. We define $e_0 = (0, 1) \in Y$, $e_0^* : Y = E \times \mathbb{R} \to \mathbb{R}$ by

$$e_0^*(u, t) = t,$$

and by slightly abusing notation we identify $e_n \in E$ to $(e_n, 0) \in Y$ and also extend the $e_n^* \in E^*$ to $e_n^* : Y = E \times \mathbb{R} \to \mathbb{R}$ by

$$e_n^*(u, t) = e_n^*(u) = u_n \quad \text{for all} \quad u = \sum_{j=1}^{\infty} u_j e_j, \in E, \, t \in \mathbb{R}.$$

Then we may we consider $\{e_n\}_{n\in\mathbb{N}}\cup\{e_0\}$ as a basis of $E\times\mathbb{R}=Y$ with associated coordinate functionals $\{e_n^*\}_{n\in\mathbb{N}}\cup\{e_0^*\}$, and we have that this basis is also shrinking.
Next we are going to construct a partition of unity in S^+, quite similar but not identical to that of the proof of Theorem 1.6
Since the norm $|\cdot|$ is locally uniformly convex we can find, for every $x\in S^+$, open slices $R_x=\{y\in S:\ f_x(y)>\delta_x\}\subset S^+$ and $P_x=\{y\in S:\ f_x(y)>\delta_x^4\}\subset S^+$, where $f_x\in Y^*$, $0<\delta_x<1$, and $|f_x|^*=1=f_x(x)$, so that the oscillation of the functions f and ε on every P_x is less than $\varepsilon(x)/16$. We also assume, with no loss of generality, that $\operatorname{dist}(P_x,\, E\times\{0\}\,)>0$.
Since Y is separable we can select a countable subfamily of $\{R_x\}_{x\in S^+}$, which covers S^+. Let us denote this countable subfamily by $\{R_n\}_n$, where $R_n=R_{x_n}=\{y\in S: f_n(y)>\delta_n\}$ and $f_n(x_n)=1$. Recall that the oscillation of the functions f and ε on every $P_n=P_{x_n}=\{y\in S: f_n(y)>\delta_n^4\}$ is less than $\varepsilon(x_n)/16$, and this implies that

$$\frac{15}{16}\,\varepsilon(x_n)\leq\varepsilon(x)\leq\frac{17}{16}\,\varepsilon(x_n)\ \text{ and }\ \|f(x)-f(y)\|\leq\frac{\varepsilon(x_n)}{16}$$

for every $x,y\in P_n$. Note that $\{P_n\}_{n\in\mathbb{N}}$ is an open cover of S^+.
• **For $\mathbf{k=1}$**, since $\operatorname{span}\{e_n^*: n\in\mathbb{N}\}$ is dense in E^*, we may find numbers $N_1\in\mathbb{N}$, $\epsilon_1,\gamma_1\in(0,1)$ with $\epsilon_1>\gamma_1$, and $\beta_{1,0},...,\beta_{1,N_1}\in\mathbb{R}$ with $\beta_{1,0}>0$ so that the functional g_1 defined by

$$g_1:=\sum_{j=0}^{N_1}\beta_{1,j}e_j^*$$

has norm 1 and satisfies

$$\{x\in S: f_1(x)>\delta_1^2\}\subset\{x\in S: g_1(x)>\epsilon_1\}\subset\{x\in S: g_1(x)>\gamma_1\}\subset\{x\in S: f_1(x)>\delta_1^3\}.$$

Let us define

$$h_1:\ S^+\longrightarrow\mathbb{R}$$
$$h_1=\theta_1(g_1),$$

where $\theta_1:\mathbb{R}\to[0,1]$ is a C^∞ function satisfying

$$\theta_1(t)=0\ \text{ if and only if } t\leq\gamma_1$$
$$\theta_1(t)=1\ \text{ if and only if } t\geq\epsilon_1.$$

Note that the interior of the support of h_1 is the open set $U_1:=\{x\in S^+: g_1(x)>\gamma_1\}$.
• **For $\mathbf{k=2}$**. We may again use the density of $\operatorname{span}\{e_n^*: n\in\mathbb{N}\}$ in E^*, in order to find numbers $N_2\in\mathbb{N}$, $\gamma_2,\epsilon_2\in(0,1)$ with $\gamma_2<\epsilon_2$, and $\beta_{2,0},...,\beta_{2,N_2}\in\mathbb{R}$ so that the linear functional

$$g_2:=\sum_{j=0}^{N_2}\beta_{2,j}e_j^*$$

has norm 1 and satisfies

$$\{x\in S: f_2(x)>\delta_2^2\}\subset\{x\in S: g_2(x)>\epsilon_2\}\subset\{x\in S: g_2(x)>\gamma_2\}\subset\{x\in S: f_2(x)>\delta_2^3\}.$$

We may assume without loss of generality that $N_1\leq N_2$ (otherwise we may set $\beta_{2,j}=0$ for $N_2<j\leq N_1$ and take a new N_2 equal to N_1).
Now we define

$$h_2:\ S^+\longrightarrow\mathbb{R}$$
$$h_2=\theta_2(g_2)\,(1-\theta_1(g_1))\,,$$

where $\theta_2 : \mathbb{R} \to [0,1]$ is a C^∞ function satisfying:

$$\theta_2(t) = 0 \text{ if and only if } t \le \gamma_2$$
$$\theta_2(t) = 1 \text{ if and only if } t \ge \epsilon_2.$$

Notice that the interior of the support of h_2 is the open set

$$U_2 = \{x \in S^+ : g_1(x) < \epsilon_1 \,,\ g_2(x) > \gamma_2\}.$$

• **For k = 3,** By density of span$\{e_n^* : n \in \mathbb{N}\}$ in E^* we may pick numbers $N_3 \in \mathbb{N}$, $\gamma_3, \epsilon_3 \in (0,1)$ with $\epsilon_3 > \gamma_3$, and $\beta_{3,0}, ..., \beta_{3,N_3} \in \mathbb{R}$ so that, for

$$g_3 := \sum_{j=0}^{N_3} \beta_{3,j} e_j^*$$

we have that $g_3 \in S^*$ and

$$\{x \in S : \ f_3(x) > \delta_3^2\} \subset \{x \in S : \ g_3(x) > \epsilon_3\} \subset \{x \in S : \ g_3(x) > \gamma_3\} \subset \{x \in S : f_3(x) > \delta_3^3\}$$

Again we may assume without loss of generality that $N_2 \le N_3$.
We define

$$h_3 : \ S^+ \longrightarrow \mathbb{R}$$
$$h_3 = \theta_3(g_3) \prod_{j=1}^{2} (1 - \theta_j(g_j)) \,,$$

where $\theta_3 : \mathbb{R} \to [0,1]$ is a C^∞ function satisfying

$$\theta_3(t) = 0 \quad \text{if and only if } t \le \gamma_3$$
$$\theta_3(1) = 1 \quad \text{if and only if } t \ge \epsilon_3.$$

Clearly the interior of the support of h_3 is the set

$$U_3 = \{x \in S^+ : \ g_1(x) < \epsilon_1 \,,\ g_2(x) < \epsilon_2 \ \text{ and } \ g_3(x) > \gamma_3\}.$$

We continue this process by induction.
• Assume that, in the steps $j = 2, ..., k$, with $k \ge 2$, we have selected points $y_j \in S^+$, positive integers $N_1 \le N_2 \le ... \le N_k$, and constants $\gamma_j, \epsilon_j \in (0,1)$, $\beta_{j,i} \in \mathbb{R}$ so that the functionals

$$g_j := \sum_{i=0}^{N_j} \beta_{j,i} e_i^*$$

belong to S^* and satisfy

$$\{x \in S : \ f_j(x) > \delta_j^2\} \subset \{x \in S : \ g_j(x) > \epsilon_j\} \subset \{x \in S : \ g_j(x) > \gamma_j\} \subset \{x \in S : \ f_j(x) > \delta_j^4\}, \tag{4.1}$$

for all $j = 2, ..., k$. Assume also that we have defined numbers γ_j and functions

$$h_j = \theta_j(g_j) \prod_{i<j} (1 - \theta_i(g_i)) \,,$$

where $\theta_j : \mathbb{R} \to [0,1]$ are C^∞ functions satisfying

$$\theta_j(t) = 0 \quad \text{if and only } t \le \gamma_j$$
$$\theta_j(t) = 1 \quad \text{if and only } t \ge \epsilon_j.$$

The interior of the support of h_j is the set

$$U_j = \{x \in S^+ : \ g_1(x) < \epsilon_1 \,, ..., \, g_{j-1}(x) < \epsilon_{j-1} \ \text{ and } \ g_j(x) > \gamma_j\}.$$

Then we may again use the density of $\operatorname{span}\{e_n^* : n \in \mathbb{N}\}$ in E^*, in order to find a positive integer $N_{k+1} \geq N_k$, and constants $\gamma_{k+1}, \epsilon_{k+1} \in (0,1)$, and $\beta_{k+1,0}, ..., \beta_{k+1,N_{k+1}} \in \mathbb{R}$ so that, for

$$g_{k+1} := \sum_{j=0}^{N_{k+1}} \beta_{k+1,j} e_j^*$$

we have that $g_{k+1} \in S^*$ and

$$\{x \in S : \ f_{k+1}(x) > \delta_{k+1}^2\} \subset \{x \in S : \ g_{k+1}(x) > \epsilon_{k+1}\} \subset$$
$$\{x \in S : \ g_{k+1}(x) > \gamma_{k+1}\} \subset \{x \in S : \ f_{k+1}(x) > \delta_{k+1}^3\}.$$

We now set

$$U_{k+1} := \{x \in S^+ : \ g_1(x) < \epsilon_1 \, , ..., \, g_k(x) < \epsilon_k \ \text{ and } \ g_{k+1}(x) > \gamma_{k+1}\}, \tag{4.2}$$

and define

$$h_{k+1} : S^+ \longrightarrow \mathbb{R}$$
$$h_{k+1} = \theta_{k+1}(g_{k+1}) \prod_{j<k+1} (1 - \theta_j(g_j)) \, .$$

where $\theta_{k+1} : \mathbb{R} \to [0,1]$ is a C^∞ function such that

$$\theta_{k+1}(t) = 0 \quad \text{if and only if } t \leq \gamma_{k+1}$$
$$\theta_{k+1}(t) = 1 \quad \text{if and only if } t \geq \epsilon_{k+1}.$$

Clearly the interior of the support of h_{k+1} is the set U_{k+1}.
Thus a sequence $\{h_n\}_{n\in\mathbb{N}}$ of C^1 smooth functions with the above properties is well defined by induction. As in the proof of Theorem 1.6 it is not difficult to check that the family $\{U_k\}_{k\in\mathbb{N}}$ is a locally finite open covering of S^+ refining $\{P_k\}_{k\in\mathbb{N}}$. Therefore the functions

$$\psi_n := \frac{h_n}{\sum_{k=1}^\infty h_k}, \ \ n \in \mathbb{N},$$

define a C^1 partition of unity in S^+ subordinate to $\{P_k\}_{k\in\mathbb{N}}$.
We will also need the following fact.

Claim 4.1. *For every $k \in \mathbb{N}$ and every $y \in S^+$, we have*

$$h_k(y) = 0 \Longrightarrow Dh_k(y) = 0.$$

Proof. Proceed as in the proof of Claim 3.2. □

Again we have that for every $x \in S^+$ there exist a number $n = n_x$ and an open neighborhood V_x of x in S^+ such that $\psi_k(y) = 0$ for every $k > n$ and every $y \in V_x$. Let us also call $m = m_y$ the largest j such that $h_j(y) \neq 0$; this is also the largest j for which $\psi_j(y) \neq 0$. Thus, for every $y \in V_x$, we have

$$m_y = \max\{j : y \in \psi_j^{-1}((0,1])\} \leq n_x.$$

The derivatives of the functions ψ_n can be calculated as in the proof of Theorem 1.6. We have

$$D\psi_k(y)(v) = \sum_{j=1}^n \lambda_{k,j}(y) \, g_j(v) = \sum_{j=1}^{m_y} \lambda_{k,j}(y) \, g_j(v)$$

for each $v \in T_y S^+$, where the functions $\lambda_{k,j} : V_x \to \mathbb{R}$ are of class C^1.
The following lemma summarizes the properties of the partition of unity $\{\psi_n\}_{n\in\mathbb{N}}$ which will be most useful to us.

Lemma 4.2. *Given two continuous functions $f : S^+ \to F$ and $\varepsilon : S^+ \to (0, \infty)$, there exists a collection of norm-one linear functionals $\{g_k\}_{k\in\mathbb{N}} \subset Y^*$ of the form*

$$g_k = \sum_{j=0}^{N_k} \beta_{k,j} e_j^*,$$

where $N_1 \leq N_2 \leq N_3 \leq ...$, an open covering $\{P_n\}_{n\in\mathbb{N}}$ of S^+, and a C^1 partition of unity $\{\psi_n\}_{n\in\mathbb{N}}$ in S^+ such that:

(1) *$\{\psi_n\}_{n\in\mathbb{N}}$ is subordinate to $\{P_n\}_{n\in\mathbb{N}}$.*
(2) *For every $x \in S^+$ there exist a neighborhood V_x of x in S^+ and a number $n = n_x \in \mathbb{N}$ such that $\psi_m = 0$ on V_x for all $m > n$, and the derivatives of the functions $\psi_1, ..., \psi_n$ on V_x are of the form*

$$D\psi_k(y)(v) = \sum_{j=1}^{n} \lambda_{k,j}(y)\, g_j(v) = \sum_{j=1}^{m_y} \lambda_{k,j}(y)\, g_j(v),$$

for $v \in T_yS^+$, the tangent hyperplane to S^+ at $y \in S^+ \cap V_x$, where $m_y \leq n$ is the largest number such that $\psi_{m_y}(y) \neq 0$. More precisely, if L_y denotes the derivative of the function $u \mapsto \sqrt{1 - \|u\|^2}$ evaluated at the point u_y such that $y = \left(u_y, \sqrt{1 - \|u_y\|^2}\right)$, we have

$$D\psi_k(y)(v) = D\psi_k(y)(u, L_y(u)) = L_y(u)\mu_{k,0}(y) + \sum_{j=1}^{N_n} \mu_{k,j}(y)e_j^*(u) = L_y(u)\mu_{k,0}(y) + \sum_{j=1}^{N_{m_y}} \mu_{k,j}(y)e_j^*(u)$$

for every $k = 1, ..., n$, and for every $v = (u, L_y(u)) \in T_yS^+$, where the functions $\mu_{k,j} : V_x \to \mathbb{R}$ are of class C^1, $j = 1, ..., n$.
(3) *For every $n \in \mathbb{N}$ there exist a point $y_n := x_n \in P_n$ such that*

$$\frac{15}{16}\,\varepsilon(y_n) \leq \varepsilon(y) \leq \frac{17}{16}\,\varepsilon(y_n) \quad \text{for every } y \in P_n, \quad \text{and} \quad \sup_{x,y\in P_n} \|f(x) - f(y)\| \leq \frac{\varepsilon(y_n)}{16}.$$

Note also that the integer n_x can be chosen as the first such j so that $x \in \{y \in S^+ : g_j(y) > \epsilon_j\}$. Then V_x can be chosen as $\{y \in S^+ : g_j(y) > \epsilon_j\}$ and, hence, we have $R_j \subset V_x \subset P_j$ for such V_x.

Now we are ready to start the construction of our approximating function $\varphi : S^+ \to F$, which will be of the form

$$\varphi(x) = \sum_{n=1}^{\infty} \left(f(y_n) + T_n(x)\right) \psi_n(x),$$

where the y_n are the points given by condition (3) of the preceding lemma, and the operators $T_n : Y \to F$ will be defined below. We have to distinguish two cases.

Case 1: Assume that F is infinite-dimensional.
In order to define the operators T_n, we work with the infinite subset $\mathbb{P}$ of $\mathbb{N}$ given by assumption (3) of Theorem 1.7, and we take a countable pairwise disjoint family of infinite subsets of $\mathbb{P}$ which *goes to infinity.* More precisely, we write

$$\bigcup_{n=1}^{\infty} I_n \subseteq \mathbb{P},$$

in such a way that:

(1) $I_n := \{n_i : i \in \mathbb{N}\}$ is infinite for each $n \in \mathbb{N}$;
(2) $I_n \cap I_m = \emptyset$ for all $n \neq m$; and
(3) $\{1, \ldots, N_n\} \cap I_n = \emptyset$ for all $n \in \mathbb{N}$.

Here $\{N_n\}_{n\in\mathbb{N}}$ is the non-decreasing sequence of positive integers that appears in the construction of the functionals g_n of Lemma 4.2.
Now, by using assumption (3) of the statement, we can find, for each number $n \in \mathbb{N}$, a linear continuous surjection $S_n : E \to F$ of the form

$$S_n = A_n \circ P_n,$$

where A_n is a bounded linear operator from $\overline{\text{span}}\{e_{n_k} : k \in \mathbb{N}\} = \overline{\text{span}}\{e_m : m \in I_n\}$ onto F, and $P_n : E \to \overline{\text{span}}\{e_{n_k} : k \in \mathbb{N}\}$ is the natural projection associated to the unconditional basis $\{e_j\}_{j\in\mathbb{N}}$.
Now we finally define $T_n : Y \to F$ by

$$T_n(u,t) = \frac{\varepsilon(y_n)}{4\|S_n\|}\, S_n(u),$$

and $\varphi : S^+ \to F$ by

$$\varphi(x) = \sum_{n=1}^{\infty} \left(f(y_n) + T_n(x)\right) \psi_n(x).$$

It is clear that φ is well defined and of class C^1.

Claim 4.3. *We have that $\|\varphi(x) - f(x)\| \le \varepsilon(x)$ for every $x \in S^+$.*

Proof. This is shown exactly as in Claim 3.5. □

Let us now calculate the derivative of our function φ. For every $v = (u, L_y(u)) \in T_y S^+$ we have that

$$T_n(v) = T_n(u, L_y(u)) = \frac{\varepsilon(y_n)}{4\|S_n\|}\, S_n(u), \tag{4.3}$$

and we have that $D(T_{n|_{S^+}})(y)$ is the restriction of $DT_n(y) = T_n$ to $T_y S^+$, that is to say, if $v = (u, L_y(u)) \in T_y S^+$ then

$$DT_n(y)(u, L_y(u)) = \frac{\varepsilon(y_n)}{4\|S_n\|}\, S_n(u). \tag{4.4}$$

We can now compute the derivative of φ on S^+. Recall that, by condition (2) of Lemma 4.2, for every $x \in S^+$ there is a neighborhood V_x of x in S^+ and a number $n = n_x \in \mathbb{N}$ such that

$$\varphi(y) = \sum_{j=1}^{n} \left(f(y_j) + T_j(y)\right) \psi_j(y)$$

for every $y \in V_x$. Fix $y \in V_x$ and recall that for $m = m_y$ (the largest number j for which $\psi_j(y) \neq 0$), we have

$$\varphi(y) = \sum_{j=1}^{m} \left(f(y_j) + T_j(y)\right) \psi_j(y).$$

By using (4.4) and the expression for $D\psi_j(y)$ given in Lemma 4.2, we see that

$$D\varphi(y)(u, L_y(u)) = \left(\sum_{j=1}^{n} \psi_j(y)\, \frac{\varepsilon(y_j)}{4\|S_j\|}\, S_j(u)\right) + L_y(u)\alpha_{N_n,0}(y) + \sum_{j=1}^{N_n} \alpha_{N_n,j}(y) e_j^*(u) =$$

$$= \left(\sum_{j=1}^{m} \psi_j(y)\, \frac{\varepsilon(y_j)}{4\|S_j\|}\, S_j(u)\right) + L_y(u)\alpha_{N_m,0}(y) + \sum_{j=1}^{N_m} \alpha_{N_m,j}(y) e_j^*(u) \tag{4.5}$$

for every $(u, L_y(u)) \in T_y S^+$, $y \in V_x$, where the functions $\alpha_{N_n,j} : V_x \subset S^+ \to F$ are of class C^1 (because we have $\alpha_{N_n,j}(y) = \sum_{i=1}^n (f(y_i) + T_i(y))\, \mu_{i,j}(y)$ and $\alpha_{N_n,0}(y) = \sum_{i=1}^n (f(y_i) + T_i(y))\, \mu_{i,0}(y)$, where $\mu_{i,j}$ are as in Lemma 4.2).

Let us now prove that the critical set of φ is relatively small.

Lemma 4.4. *The set $C_\varphi := \{x \in S^+ : D\varphi(x)$ is not surjective$\}$ is of the form*

$$C_\varphi = \{\left(w, \sqrt{1 - \|w\|^2}\right) : w \in A\},$$

where $A \subset E$ is a relatively closed subset of the open unit ball of E that is locally contained in a complemented subspace of infinite codimension in E.

Proof. Observe that if $n = n_x = 1$ then $\varphi(y) = f(y_1) + T_1(y)$ for every $y \in V_x$, and because T_1 is surjective φ does not have any critical point in V_x. Now let us assume that $n = n_x \geq 2$. Let $y = (w, \sqrt{1 - \|w\|^2})$ be point of V_x. Let $m = m_y$ be the largest number such that $\psi_m(y) \neq 0$. Recall that $m \leq n$. We only need to show that if

$$w \notin \overline{\operatorname{span}}\, \{e_j : j \in \mathbb{P} \text{ or } j = 1, \ldots, N_n\}$$

then for every $v \in F$ there exists $u \in E$ such that

$$D\varphi(w, \sqrt{1 - \|w\|^2})(u, L_y(u)) = v,$$

since this will mean that the set

$$A := \{w \in E : (w, \sqrt{1 - \|w\|^2}) \in C_\varphi\}$$

will be locally contained in subspaces of the form $\overline{\operatorname{span}}\, \{e_i : i \in \mathbb{P} \text{ or } i = 1, \ldots, N_n\}$, which are complemented, and of infinite codimension, in E.
We will need to use the following.

Fact 4.5. *For every $w = \sum_{j=1}^\infty w_j e_j \in E \setminus \{0\}$ and every $j_0 \in \mathbb{N}$ we have that*

$$w_{j_0} \neq 0 \implies \langle J(w), e_{j_0} \rangle \neq 0,$$

where $J(w)$ denotes $D\|\cdot\|(w)$, and $\langle J(w), u\rangle := J(w)(u)$.

Proof. If $w_{j_0} \neq 0$ then, by assumption (2) of the statement of Theorem 1.7, we have that

$$\|\sum_{j=1,\, j \neq j_0}^\infty w_j e_j\| \leq \|\sum_{j=1}^\infty w_j e_j\|.$$

This means that the convex function $\theta : \mathbb{R} \to \mathbb{R}$ defined by

$$\theta(t) = \|w + t e_{j_0}\|$$

has a minimum at $t = -w_{j_0}$. On the other hand, if we had $\langle J(w), e_{j_0}\rangle = 0$, then the same function θ would have another minimum at the point $t = 0$. But since $\|\cdot\|$ is strictly convex the function θ can only attain its minimum at a unique point. Therefore we must have $\langle J(w), e_{j_0}\rangle \neq 0$. □

So let us pick a point $w \in E \setminus \overline{\operatorname{span}}\, \{e_j : j \in \mathbb{P} \text{ or } j = 1, \ldots, N_n\}$ and a vector $v \in F$, and let us construct a vector $u \in E$ such that $D\varphi(w, \sqrt{1 - \|w\|^2})(u, L_y(u)) = v$, where $y = (w, \sqrt{1 - \|w\|^2})$. By assumption, there exists $j_0 \in \mathbb{N}$, $j_0 \in \mathbb{N} \setminus \mathbb{P}$, such that $j_0 > N_n \geq N_m$ and $w_{j_0} \neq 0$. According to the fact just shown, we have $\langle J(w), e_{j_0}\rangle \neq 0$. Now, since S_m is surjective and $\psi_m(y) \neq 0$, we may find a sequence $(u_{m_i})_{i \in \mathbb{N}}$ (indexed by the subsequence $(m_i)_{i \in \mathbb{N}}$ defined by I_m) such that

$$\psi_m(y) \frac{\varepsilon(y_m)}{4\|S_m\|}\, S_m \left(\sum_{i=1}^\infty u_{m_i} e_{m_i} \right) = v.$$

Note that $j_0 \notin I_m = \{m_i : i \in \mathbb{N}\}$, because $j_0 \in \mathbb{N} \setminus \mathbb{P}$ and $I_m \subset \mathbb{P}$. Then we can set

$$u_{j_0} := -\frac{\langle J(w), \sum_{i=1}^{\infty} u_{m_i} e_{m_i} \rangle}{\langle J(w), e_{j_0} \rangle},$$

so that we have

$$\langle J(w),\, u_{j_0} e_{j_0} + \sum_{i=1}^{\infty} u_{m_i} e_{m_i} \rangle = 0,$$

which bearing in mind that

$$L_y = -\frac{\|w\|}{\sqrt{1-\|w\|^2}} \langle J(w),\, \cdot \rangle$$

also implies that

$$L_y \left(u_{j_0} e_{j_0} + \sum_{i=1}^{\infty} u_{m_i} e_{m_i} \right) = 0.$$

So if we set $u_j = 0$ for all $j \notin I_m \cup \{j_0\}$ and we define

$$u := \sum_{j=1}^{\infty} u_j e_j$$

then we have that

$$\psi_m(y) \frac{\varepsilon(y_m)}{4\|S_m\|}\, S_m\,(u) = v, \;\; L_y(u) = 0, \;\; \sum_{j=1}^{N_m} \alpha_{N_m,j}(y) e_j^*(u) = 0, \quad \text{and also } S_j(u) = 0 \text{ for } j < m,$$

because $j_0 > N_n \geq N_m$, $I_m \cap \{1, 2, ..., N_m\} = \emptyset$, and the sets I_j are pairwise disjoint. In view of (4.5) these equalities imply that $D\varphi(y)(u) = v$. □

Now, according to Lemma 4.4 and Theorem 1.4, we can extract the set C_φ, since it is C^1 diffeomorphic (via the projection of the graph S^+ of the function $w \mapsto \sqrt{1-\|w\|^2}$ onto the open unit ball of E) to a subset which can be extracted. Therefore we can finish the proof of Theorem 1.7 exactly as we did with Theorem 1.6.

Case 2: Assume that $F = \mathbb{R}^m$. The proof is almost identical, but with the following important difference: now the set $\mathbb{P}$ is by definition the set of *even* positive integers, and the sets I_n are *finite* subsets of $\mathbb{P}$ such that:

(1) $\sharp I_n = m$ for each $n \in \mathbb{N}$;
(2) $I_n \cap I_j = \emptyset$ for all $n \neq j$; and
(3) $\{1, \ldots, N_n\} \cap I_n = \emptyset$ for all $n \in \mathbb{N}$.

Here $\{N_n\}_{n\in\mathbb{N}}$ is the non-decreasing sequence of positive integers that appears in the construction of the functionals g_n of Lemma 4.2.
Of course in this case we can always find linear surjections $A_n : \text{span}\{e_i \,:\, i \in I_n\} \to \mathbb{R}^m$. □

5. Technical versions of Theorems 1.6 and 1.7, examples, and remarks

In this section we will give some examples, make some remarks and establish more technical variants of our results which follow by the same method of proof. We will also prove Proposition 1.8.

The proof of Theorem 1.6 can be easily adjusted to obtain more general results with more complicated statements. Namely, the following two results are true.

Theorem 5.1. *Let E and F be Banach spaces. Assume that:*

(1) *E is infinite-dimensional, with a separable dual E^*.*
(2) *There exist three sequences $\{E_{n,1}\}_{n\geq 1}$, $\{E_{n,2}\}_{n\geq 1}$, $\{E_{n,3}\}_{n\geq 2}$ of subspaces of E such that*

$$E = E_{1,1} \oplus E_{1,2}, \quad E_{1,2} = (E_{2,1} \oplus E_{2,2}) \oplus ... \oplus (E_{n,1} \oplus E_{n,2} \oplus E_{n,3}), \quad E_{n,3} = E_{n+1,1} \oplus E_{n+1,2} \oplus E_{n+1,3},$$

with either $E_{n,3}$ infinite-dimensional and reflexive and $\dim E_{n,2} \geq 1$, or else $E_{n,2}$ infinite-dimensional and reflexive for all $n \geq 2$. Suppose also that there exists a bounded linear operator from $E_{n,1}$ onto F for every $n \in \mathbb{N}$.

Then, for every continuous mapping $f : E \to F$ and every continuous function $\varepsilon : E \to (0, \infty)$ there exists a C^1 mapping $g : E \to F$ such that $\|f(x) - g(x)\| \leq \varepsilon(x)$ and $Dg(x) : E \to F$ is a surjective linear operator for every $x \in E$.

Observe that if $E_{n,3}$ is infinite-dimensional and reflexive, the spaces $E_{n,2}$ can be taken to be of dimension 1 for every $n \in \mathbb{N}$. The proof is almost the same as that of Theorem 1.6. Here, up to finite-dimensional perturbations of the subspaces $E_{k,j}$, we can arrange that $E_{1,2} \subset Ker g_1^1$ and that $E_{n,1} \oplus E_{n,3} \subseteq \bigcap_{j=1}^n Ker g_j^1$, and we may set $A_x = \{y \in S^+ : E_{n,3} \subset Ker L_y\}$.

Theorem 5.2. *Let E, X, and F be Banach spaces. Assume either that E is infinite-dimensional, separable, and reflexive, and F is finite-dimensional, or that:*
(1) *E is infinite-dimensional, with a separable dual E^*.*
(2) *There exists a decomposition of E,*

$$E = G \oplus E_1 \oplus X,$$

such that G is infinite-dimensional and reflexive, and E_1 is isomorphic to E.
(3) *There exists a bounded linear operator from $G \oplus X$ onto F (equivalently, F is a quotient of $G \oplus X$).*

Then, for every continuous mapping $f : E \to F$ and every continuous function $\varepsilon : E \to (0, \infty)$ there exists a C^1 mapping $g : E \to F$ such that $\|f(x) - g(x)\| \leq \varepsilon(x)$ and $Dg(x) : E \to F$ is a surjective linear operator for every $x \in E$.

If, additionally, X is isomorphic to E, then F can be taken as a quotient of E.

Observe that each of these two results imply Theorem 1.6, with Theorem 5.1 being the most general one.
For instance, Theorem 5.2 can be applied to the James space J and to its dual J^*. Indeed, both spaces have separable dual. It is known that J has many reflexive infinite-dimensional complemented subspaces G [22]. Since J is prime [21], for each such G, we can write $J = G \oplus J$ (for instance we have $J = l_2 \oplus J$). Now, recalling the fact that J has a separable dual, apply Theorem 5.2 to $E = E_1 = J$ and $X = \{0\}$ to see that every continuous function $f : J = G \oplus J \to F$, where F is a quotient of G, can be uniformly approximated by C^1 smooth mappings without critical points. Similar arguments work for the dual of the James space J^*.

It also follows that the conclusion of Theorem 1.6 is true for *composite* spaces of the form $c_0 \oplus \ell_p$ or $c_0 \oplus L^p$, $1 < p < \infty$, with k being the order of smoothness of ℓ_p or L^p. More generally, if E is any finite direct sum of the classical Banach spaces c_0, ℓ_p or L^p, $1 < p < \infty$, and there is a bounded linear operator from E onto F then the conclusion of Theorem 1.2 is true with k being the minimum of the orders of smoothness of the spaces appearing in this decomposition of E.

Remark 5.3. Notice that that in the case that E is a separable Hilbert space, we have that the function $w \mapsto \|w\|^2$ is of class C^∞, hence all the mappings appearing in the proof of Theorem 1.6 are of class C^∞, and we directly obtain an approximating function g of class C^∞ with no critical points.

In fact, in the Hilbertian case we do not need to use a partition of unity in the upper sphere S^+. We can directly construct a partition of unity $\{\psi_n\}_{n\in\mathbb{N}}$ in E subordinated to an open covering by open balls with linearly independent centers $\{y_j\}$, as in [6]. Then, choosing an orthonormal basis $\{e_j\}$ for which $span\{y_1, \ldots, y_n\} = span\{e_1, \ldots, e_n\}$ for every $n \in \mathbb{N}$, we define operators $T_n : E \to F$ as in the proof of Theorem 1.7, where $\mathbb{P}$ can be any infinite subset of $\mathbb{N}$ such that $\mathbb{N} \setminus \mathbb{P}$ is also infinite. Then one can easily check that the function

$$\varphi(y) = \sum_{n=1}^{\infty} \left(f(y_n) + T_n(y - y_n)\right) \psi_n(y)$$

approximates f and the set C_φ of its critical points is locally contained in a subspace of infinite codimension in E, specifically in subspaces of the form $\overline{span}\{e_j : j \in \mathbb{P} \text{ or } j = 1, \ldots, n\}$. Then one can extract C_φ by means of a C^∞ diffeomorphism $h : E \to E \setminus C_\varphi$ which is sufficiently close to the identity, and conclude that the function $g := \varphi \circ h$ approximates f and has no critical points.

The same proof as that of Theorem 1.7, with some adjustments, allows us to obtain a more general (and also more technical) result as follows.

Theorem 5.4. *Let E be an infinite-dimensional Banach space, and F be a Banach space such that:*

(1) *E has an equivalent norm $\|\cdot\|$ which is C^1 and locally uniformly convex.*
(2) *E has a (normalized) Schauder basis $\{e_n\}_{n\in\mathbb{N}}$ which is shrinking.*
(3) *There exists an infinite subset $\mathbb{I}$ of $\mathbb{N}$ such that the subspace $\overline{span}\{e_j : j \in \mathbb{N}\setminus\mathbb{I}\}$ is complemented in E, and for every $x = \sum_{j=}^{\infty} x_j e_j$ and every $j_0 \in \mathbb{I}$ we have that*

$$\left\| \sum_{j\in\mathbb{N},\, j\neq j_0} x_j e_j \right\| \leq \left\| \sum_{j\in\mathbb{N}} x_j e_j \right\|.$$

(4) *In the case that F is infinite-dimensional, there exists an infinite subset $\mathbb{P}$ of $\mathbb{N}$ such that $\mathbb{I} \setminus \mathbb{P}$ is infinite and for every infinite subset J of $\mathbb{P}$ the subspace $E' = \overline{span}\{e_j : j \in J \cup (\mathbb{N} \setminus \mathbb{I})\}$ is complemented in E, and there exists a linear bounded operator from E' onto F.*

Then, for every continuous mapping $f : E \to F$ and for every continuous function $\varepsilon : E \to (0, \infty)$ there exists a C^1 mapping $g : E \to F$ such that $\|f(x) - g(x)\| \leq \varepsilon(x)$ and $Dg(x) : E \to F$ is a surjective linear operator for every $x \in E$.

Proof. The only important difference with the proof of Theorem 1.7 is that now we have to use an analogue of Fact 4.5 which is true if we just pick $j_0 \in \mathbb{I}$. Therefore, if we take $w = \sum_{j=1}^{\infty} w_j e_j \notin \overline{span}\{e_j : j \in \mathbb{P} \cup (\mathbb{N} \setminus \mathbb{I}) \text{ or } j = 1, \ldots, N_n\}$, since $\mathbb{I} \setminus \mathbb{P}$ is infinite, there will exist $j_0 \in \mathbb{I}$, $j_0 > N_n$ such that $w_{j_0} \neq 0$ and thus $\langle J(w), e_{j_0}\rangle \neq 0$. The operators S_n have supports in complemented subspaces of the form $\overline{span}\{e_j : j \in I_n \cup (\mathbb{N} \setminus \mathbb{I})\}$, where the sets $I_n \subset \mathbb{P}$ are defined as in the proof of Theorem 1.7. □

Remark 5.5. It is clear that the spaces ℓ_p and L^p, $1 < p < \infty$ satisfy the assumptions of Theorem 1.6. It may not be so obvious why the space c_0 satisfy the assumptions of Theorem 1.7; let us clarify this point. If we repeat the proof of [26, Theorem V.1.5] in the particular case that $\Gamma = \mathbb{N}$, since all the operations that are made in this proof are coordinate-wise monotone, we see that the C^1 and LUR renorming $\|\cdot\|$ that we obtain for c_0 has the property that

$$\left\| \sum_{j\in\mathbb{N},\, j\neq j_0} x_j e_j \right\| \leq \left\| \sum_{j\in\mathbb{N}} x_j e_j \right\|$$

for every $j_0 \in \mathbb{N}$ and every $x = (x_1, x_2, x_3, ...) \in c_0$, where $\{e_n\}$ is the canonical basis of c_0. This shows that this norm $\|\cdot\|$ satisfies assumptions (1) and (2) of Theorem 1.7. On the other hand, for every infinite subset J of $\mathbb{N}$ we have that $\overline{\text{span}}\{e_j : j \in J\}$ is isomorphic to c_0, so it is clear that assumption (3) is satisfied as well, provided that there exists a continuous linear operator from c_0 onto F. Therefore $E = c_0$ satisfies the conclusion of Theorem 1.2.

The latter fact can be generalized to Banach spaces with a shrinking basis which contain copies of c_0.

Theorem 5.6. *Let E be a Banach space that contains the space c_0 and admits a shrinking Schauder basis. Let F be a quotient of E.*
Then, for every continuous mapping $f : E \to F$ and for every continuous function $\varepsilon : E \to (0, \infty)$ there exists a C^1 mapping $g : E \to F$ such that $\|f(x) - g(x)\| \leq \varepsilon(x)$ and $Dg(x) : E \to F$ is a surjective linear operator for every $x \in E$.

Proof. We will show that E satisfies the assumptions of Theorem 5.4.
First, by Sobczyk's Theorem [64], c_0 is complemented in E, that is, E is isomorphic to $G \oplus c_0$, for a certain Banach space G. Since c_0 is isomorphic to $c_0 \oplus c_0$, G may be taken as E. So, we can and will assume that

$$E = c_0 \oplus E.$$

Let $\{e_j\}_{j \in N}$ be the canonical Schauder basis in c_0. Equip c_0 with the C^1 and LUR norm $||\cdot||$ which was described in Remark 5.5. That is, for every $j_0 \in \mathbb{N}$, we have

$$|| \sum_{j=1, j\neq j_0}^{\infty} \alpha_j e_j || \leq || \sum_{j=1}^{\infty} \alpha_j e_j ||,$$

for every $x = \sum_{j=1}^{\infty} \alpha_j e_j \in c_0$.
Similarly, let $\{d_n\}_{n \in \mathbb{N}}$ be a shrinking Schauder basis in E. Equip E with a C^1 and LUR norm $|\cdot|$. Define a new norm in $c_0 \oplus E$, by letting

$$|||x + y||| = \sqrt{||x||^2 + |y|^2},$$

for every $x + y \in c_0 \oplus E$. This norm is C^1 and LUR as well. Define $\{f_k\}_{k \in \mathbb{N}} \subset c_0 \oplus E$, where $f_{2j-1} = e_j + 0$ and $f_{2n} = 0 + d_n$ for every $j, n \in \mathbb{N}$. It is also easy to check that $\{f_k\}_{k \in \mathbb{N}}$ is a shrinking Schauder basis for $c_0 \oplus E$.
For $x + y \in c_0 \oplus E$, let $x = \sum_{j=1}^{\infty} \alpha_j e_j \in c_0$ and $y = \sum_{n=1}^{\infty} \beta_n d_n \in E$ be their basis expansions. Then, writing $z_{2j-1} = \alpha_j$ and $z_{2n} = \beta_n$, we obtain the expansion of $z = x + y = \sum_{k=1}^{\infty} z_k f_k = \sum_{j=1}^{\infty} z_{2j-1} e_j + \sum_{n=1}^{\infty} z_{2n} d_n \in c_0 \oplus E$. For every $j_0 \in \mathbb{N}$, we have

$$\begin{aligned} ||| \sum_{k=1, k\neq 2j_0}^{\infty} z_k f_k ||| = ||| \sum_{j=1, j\neq j_0}^{\infty} z_{2j-1} e_j + \sum_{n=1}^{\infty} z_{2n} d_n ||| \leq ||| \sum_{j=1, j\neq j_0}^{\infty} \alpha_j e_j + \sum_{n=1}^{\infty} \beta_n d_n ||| \leq \\ ||| \left(\sum_{j=1, j\neq j_0}^{\infty} \alpha_j e_j \right) + y ||| = \left(|| \sum_{j=1, j\neq j_0}^{\infty} \alpha_j e_j ||^2 + |y|^2 \right)^{\frac{1}{2}} \leq \sqrt{||x||^2 + |y|^2} = \\ |||x + y||| = ||| \sum_{k=1}^{\infty} z_k f_k |||, \end{aligned}$$

where in the second line we have used the fact that $|| \sum_{j=1, j\neq j_0}^{\infty} \alpha_j e_j ||^2 \leq || \sum_{j=1}^{\infty} \alpha_j e_j ||^2 = ||x||^2$. Now, we are in a position to apply Theorem 5.4. Namely, let $\mathbb{I} = \{2n : n \in \mathbb{N}\}$ and $\mathbb{P} = \{4n : n \in \mathbb{N}\}$. $\square$

Corollary 5.7. *Let $C(K)$ be the Banach space of continuous functions, where K is a metrizable countable compactum and F be a quotient of $C(K)$.*
Then, for every continuous mapping $f : C(K) \to F$ and for every continuous function $\varepsilon : C(K) \to (0, \infty)$ there exists a C^∞ mapping $g : E \to F$ such that $\|f(x) - g(x)\| \leq \varepsilon(x)$ and $Dg(x) : E \to F$ is a surjective linear operator for every $x \in E$.

Proof. By an application of [43, Theorem 1.4], which states that a Banach space has a shrinking basis provided its dual has a Schauder basis, we obtain that $C(K)$ has a shrinking basis (because $C(K)^* = l_1$). Moreover, using the fact that c_0 is a subspace of $C(K)$, we infer that $C(K)$ is isomorphic to $c_0 \oplus G$ for some Banach space G, which yields (as in the above proof) that $C(K)$ is isomorphic to $c_0 \oplus C(K)$. Hence, by Theorem 5.6, the C^1 version of our assertion holds. The C^∞ version requires the fact that $C(K)$ has an equivalent C^∞ norm, which is due to Haydon [42], and Proposition 1.8. □

For more information about the spaces $C(K)$ we refer the reader to [60]. The space $C(K)$ is an example of *isometric* predual of ℓ_1 (meaning a Banach space E with an equivalent norm $\|\cdot\|$ such that the dual $(E^*, \|\cdot\|^*)$ is isometric to ℓ_1). The class of isomorphic predual spaces for ℓ_1 is larger that the class of isometric predual spaces (the space constructed by Bourgain and Delbaen [17] is such an example), which in turn is smaller than the class of $C(K)$ spaces for metrizable countable compactum K, see [13].

Remark 5.8. Since every isometric predual space E of ℓ_1 contains c_0 (see for instance [70, Corollary 1]) and admits an equivalent real-analytic norm [25, Corollary 3.3], the above corollary is valid for E. Even more, the corollary is valid for any infinite-dimensional separable Banach space E which has a shrinking basis and which admits an equivalent polyhedral norm (equivalently, with a countable James boundary). This follows from the facts that, being polyhedral, E must contain c_0, and that a space with a countable James boundary admits an equivalent real-analytic norm (see [25] or [38, Chapter 5, section 6] for reference).

As we noted in the introduction our main results imply that continuous functions between many Banach spaces can be arbitrarily well approximated by smooth open mappings.

Remark 5.9. *Let (E, F) be a pair of Banach spaces with the property that for every continuous mapping $f : E \to F$ and for every continuous function $\varepsilon : E \to (0, \infty)$ there exists a C^k mapping $g : E \to F$ with no critical points such that $\|f(x) - g(x)\| \leq \varepsilon(x)$, $x \in E$. Then the pair (E, F) also has the following property: for every continuous mapping $f : E \to F$ and for every continuous function $\varepsilon : E \to (0, \infty)$ there exists an open mapping $g : E \to F$ of class C^k such that $\|f(x) - g(x)\| \leq \varepsilon(x)$, $x \in E$.*

This follows trivially from [49, Theorem XV.3.5]. Recall that $g : E \to F$ is said to be open if for every open subset U of E we have that $g(U)$ is open in F. Notice that the approximation of arbitrary continuous maps by smooth (or even merely continuous) open maps is impossible for $E = \mathbb{R}^n$: for instance, if $E = \mathbb{R}^n$, $F = \mathbb{R}$, $f(x) = e^{-\|x\|^2}$, $\varepsilon(x) = 1/3$, every continuous function g which ε-approximates f must attain a global maximum in $\mathbb{R}^n$, hence $g(\mathbb{R}^n)$ is not open in $\mathbb{R}$.

Example 5.10. In view of Theorem 1.1 it is perhaps natural to ask whether in the case $E = F$ one can get C^∞ approximations $g : E \to E$ such that $Dg(x) : E \to E$ is a linear isomorphism for every $x \in E$. This is not possible, as the following example shows.
Let $f : \ell_2 \to \ell_2$ be defined by

$$f\left(\sum_{n=1}^{\infty} x_n e_n\right) = \left(\sum_{n=1}^{\infty} |x_n| e_n\right),$$

where $\{e_n\}_{n\in\mathbb{N}}$ denotes the usual basis of ℓ_2 (that is, $e_1 = (1, 0, 0, ...)$, $e_2 = (0, 1, 0, ...)$, etc). Assume that there exists $g \in C^\infty(E, E)$ such that $Dg(x) : \ell_2 \to \ell_2$ is an isomorphism and $\|f(x) - g(x)\| \leq 1/3$ for every $x \in \ell_2$. Consider the projection $P_1 : \ell_2 \to \mathbb{R}$ given by $P_1(x) = x_1$, and the function $g_1 = P_1 \circ g$. Since $Dg(x)$ is an isomorphism for every x, we must have $Dg_1(x) = P_1 \circ Dg(x) = Dg(x)(e_1) \neq 0$ for every $x \in E$, and in particular, considering the curve $\gamma_1(t) = te_1$, $t \in \mathbb{R}$, and the function

$$\theta(t) := g_1(\gamma_1(t)), \ \ t \in \mathbb{R},$$

we must have

$$\theta'(t) = Dg_1(\gamma_1(t))(e_1) \neq 0 \tag{5.1}$$

for all $t \in \mathbb{R}$. However,

$$|P_1(g(\gamma_1(t))) - P_1(f(\gamma_1(t)))| \leq \|g(\gamma_1(t)) - f(\gamma_1(t))\| \leq 1/3,$$

hence

$$\theta(1) = P_1(g(\gamma_1(1))) \geq 1 - 1/3 = 2/3,$$

and similarly

$$\theta(-1) \geq 2/3 > 1/3 \geq \theta(0).$$

Thus θ must attain a minimum at some point t_0 of the interval $(-1, 1)$, which implies that $\theta'(t_0) = 0$ and contradicts (5.1).

Proof of Theorem 1.5. As said in the introduction, by the results of [30, 46], it is sufficient to show Theorem 1.5 for functions $f : U \to V$, where $U \subset E$ and $V \subset F$ are open subsets of two separable Hilbert spaces E, F, respectively. Observe that we can assume $V = F$. Indeed, if $f : U \to V \subset F$, $\varepsilon : U \to (0, \infty)$ are continuous functions then, by taking $\widetilde{\varepsilon}(x) = \frac{1}{2}\min\{\varepsilon(x), \mathrm{dist}(f(x), F \setminus V)\}$, if we are able to $\widetilde{\varepsilon}$-approximate $f : U \to F$ by a smooth function $g : U \to F$ with no critical points, then we also have that $\|g(x) - f(x)\| < \mathrm{dist}(f(x), F \setminus V)$, which implies that $g(x) \in V$ for every $x \in U$; that is, we really have $g : U \to V$. On the other hand, showing the result for $f : U \to F$ is not more difficult than proving it in the case $U = E$ (though it does encumber the notation). For example, it requires a version of the extractibility fact (a counterpart of Theorem 1.3) where the whole space E, its closed subset X, and an open cover $\mathcal{G}$ of E must be replaced with an open subset U (of E), a closed subset of U, and an open cover of U, respectively. Such a fact can be proved by mimicking the technique of the proof of Theorem 1.3); one just has to make some easy adjustments in the appropriate places. We leave the details to the interested reader.

Throughout the paper the "limiting" function $\varepsilon(x)$ is assumed to be positive. The following remark explains what can be said if we merely require that $\varepsilon(x) \geq 0$.

Remark 5.11. Let H be a separable, infinite-dimensional Hilbert space and $f : H \to H$ be a continuous mapping. Then, for every continuous function $\varepsilon : H \to [0, \infty)$, there exists a continuous mapping $g : H \to H$ such that the restriction $g_{|_{H\setminus\varepsilon^{-1}(0)}}$ is C^∞ smooth and has no critical points, and $\|f(x) - g(x)\| \leq \varepsilon(x)$ for every $x \in H$ (hence, $f(x) = g(x)$ provided $\varepsilon(x) = 0$). This a consequence of Theorem 1.5 applied to $U = H \setminus \varepsilon^{-1}(0)$ and $\varepsilon_{|_U}$.

Let us conclude this paper with the proof of Proposition 1.8.

Proof of Proposition 1.8. Let $f : E \to F$ and $\varepsilon : E \to (0, \infty)$ be continuous. By assumption (1) there exists a C^1 function $\varphi : E \to F$ without critical points so that

$$\|f(x) - \varphi(x)\| \leq \varepsilon(x)/2.$$

It is well known that the set of continuous linear surjections from a Banach space E onto a Banach space F is open; see [49, Theorem XV.3.4] for instance. Therefore, for each $x \in E$ there exists $r_x > 0$ such that if $S : E \to F$ is a bounded linear operator then

$$\|S - D\varphi(x)\| < 2r_x \implies S \text{ is surjective.} \tag{5.2}$$

By continuity of $D\varphi$, for every x we may find a number $s_x \in (0, r_x)$ such that if $y \in B(x, s_x)$ then

$$\|D\varphi(y) - D\varphi(x)\| < r_x.$$

Since E is separable, we can extract a countable subcovering

$$E = \bigcup_{n=1}^{\infty} B(x_n, s_n),$$

where $s_n := s_{x_n}$. Let us also denote $r_n := r_{x_n}$, and define $\eta : E \to (0, 1)$ by

$$\eta(y) = \min\left\{\frac{\varepsilon(y)}{2}, \sum_{n=1}^{\infty} \frac{s_n}{2}\psi_n(y)\right\},$$

where $\{\psi_n\}$ is a partition of unity such that the open support of ψ_n is contained in $B(x_n, s_n)$. Now we may apply assumption (2) to find a C^k function $g : E \to F$ such that

$$\|\varphi(y) - g(y)\| \leq \eta(y), \text{ and } \|D\varphi(y) - Dg(y)\| \leq \eta(y)$$

for all $y \in E$. Then for every $y \in E$ there exists $n = n_y \in \mathbb{N}$ such that $y \in B(x_n, s_n)$ and $\eta(y) \leq s_n/2$. It follows that $\|Dg(y) - D\varphi(y)\| \leq s_n/2 < r_n$ and $\|D\varphi(y) - D\varphi(x_n)\| < r_n$, hence $\|Dg(y) - D\varphi(x_n)\| < 2r_n$, and according to (5.2) this implies that $Dg(y)$ is surjective. This shows that g has no critical points. On the other hand, since $\eta \leq \varepsilon/2$, it is clear that

$$\|f(y) - g(y)\| \leq \|f(y) - \varphi(y)\| + \|\varphi(y) - g(y)\| \leq \varepsilon(y)/2 + \varepsilon(y)/2 = \varepsilon(y),$$

so g also approximates f as required. □

References

[1] F. Albiac and J. L. Ansorena, *Characterization of* 1-*almost greedy bases*, Rev. Mat. Complut. 30 (2017) no. 1, 13-24.

[2] F. Albiac and N. Kalton, *Topics in Banach space theory*, Graduate Texts in Mathematics, 233, Springer, 2016.

[3] D.E. Alspach and Y. Benyamini, *Primariness of spaces of continuous functions on ordinals*, Israel J. Math. 27 (1977), 64-92.

[4] D. Azagra, *Diffeomorphisms between spheres and hyperplanes in infinite-dimensional Banach spaces*, Studia Math. 125 (1997) no. 2, 179–186.

[5] D. Azagra and T. Dobrowolski, *Smooth negligibility of compact sets in infinite-dimensional Banach spaces, with applications*, Math. Annalen 312 (1998), no. 3, 445–463.

[6] D. Azagra and M. Cepedello, *Uniform approximation of continuous mappings by smooth mappings with no critical points on Hilbert manifolds*, Duke Math. J. 124 (2004) no. 1, 47–66.

[7] D. Azagra and M. Jiménez-Sevilla, *Approximation by smooth functions with no critical points on separable infinite-dimensional Banach spaces*, J. Funct. Anal. 242 (2007), 1–36.

[8] D. Azagra and A. Montesinos, *Starlike bodies and deleting diffeomorphisms in Banach spaces*, Extracta Math. 19 (2004), no. 2, 171–213.

[9] D. Azagra, J. Ferrera, J. Gómez-Gil, *The Morse-Sard theorem revisited*, Quarterly J. Math., doi 10.1093/qmath/hay004

[10] D. Azagra and M. García-Bravo, *Some remarks about the Morse-Sard theorem and approximate differentiability*, preprint, 2017. arXiv:1705.05624

[11] S. M. Bates, *Toward a precise smoothness hypothesis in Sard's theorem*, Proc. Amer. Math. Soc. 117 (1993), no. 1, 279-283.

[12] S. M. Bates and C. G. Moreira, *De nouvelles perspectives sur le théorème de Morse-Sard*, C.R. Acad. Sci. Paris, t. 332, Série I (2001), 13-17.

[13] Y. Benyamini and J. Lindenstrauss, *A predual of l_1 which is not isomorphic to a $C(K)$ space*, Israel J. Math. 13 (1972), 246-254.

[14] C. Bessaga, *Every infinite-dimensional Hilbert space is diffeomorphic with its unit sphere*, Bull. Acad. Polon. Sci., Sér. Sci. Math. Astr. et Phys.14 (1966), 27-31.

[15] R. Bonic and J. Frampton, *Smooth functions on Banach manifolds*, J. Math. Mech. 15 (1966), 877-898.

[16] J.M. Borwein and J.D. Vannderwerff, *Convex Functions. Constructions, Characterizations and Counterexamples.* Encyclopedia of Mathematics and Its Applications 109. Cambridge University Press, 2010.

[17] J. Bourgain and F. Delbaen, *A class of special $\mathcal{L}_\infty$ spaces*, Acta Math. 145 (1980) 155-176.

[18] J. Bourgain, M. V. Korobkov and J. Kristensen, *On the Morse-Sard property and level sets of Sobolev and BV functions*, Rev. Mat. Iberoam. 29 (2013), no. 1, 1-23.

[19] J. Bourgain, M. V. Korobkov and J. Kristensen, *On the Morse-Sard property and level sets of $W^{n,1}$ Sobolev functions on $\mathbb{R}^n$*, J. Reine Angew. Math. 700 (2015), 93-112.

[20] D. Burghelea and N. H. Kuiper, *Hilbert manifolds*, Annals of Math. 90 (1969), 379-417.

[21] P.G. Cassaza, *James quasi-reflexive space is primary*, Israel J. Math. 26 (1977), 294âĂŞ305.

[22] P.G. Cassaza, B.L. Lin and R.H. Lohman, *On James' quasi-reflexive Banach space*, Proc. Amer. Math. Soc. 67 (1977), 265-271.

[23] S. D'Alessandro and P. Hájek, *Polynomial algebras and smooth functions in Banach spaces*, J. Funct. Anal. 266 (2014), 1627–1646.

[24] L. De Pascale, *The Morse-Sard theorem in Sobolev sapces*, Indiana Univ. Math. J. 50 (2001), 1371-1386.

[25] R. Deville, V. Fonf and P. Hájek, *Analytic and polyhedral approximation of convex bodies in separable polyhedral Banach spaces*, Israel J. Math. 105 (1998), 139-154.

[26] R. Deville, G. Godefroy, and V. Zizler, *Smoothness and renormings in Banach spaces*, vol. 64, Pitman Monographs and Surveys in Pure and Applied Mathematics, Longman Scientific & Technical, Harlow, 1993.

[27] T. Dobrowolski, *Smooth and R-analytic negligibility of subsets and extension of homeomorphism in Banach spaces*, Studia Math. 65 (1979), 115-139.

[28] T. Dobrowolski, *Every Infinite-Dimensional Hilbert Space is Real-Analytically Isomorphic with Its Unit Sphere*, Journal of Functional Analysis, 134 (1995), 350-362.

[29] T. Dobrowolski, *Relative Classification of Smooth Convex Bodies*, Bull. Acad. Polon. Sci., Sér. Sci. Math. 25 (1977), 309-312.

[30] J. Eells and K. D. Elworthy, *Open embeddings of certain Banach manifolds*, Ann. of Math. 91 (1970), 465-485.

[31] K. D. Elworthy, *Embeddings, isotopy and stability of Banach manifolds*, Compositio Math. 24 (1972), 175-226.

[32] J. Eells and J. McAlpin, *An approximate Morse-Sard theorem*, J. Math. Mech. 17 (1967/1968), 1055-1064.

[33] M. Fabian, P. Habala, P, Hájek, V. Montesinos, and V. Zizler, *Banach space theory. The basis for linear and nonlinear analysis.* CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, New York, 2011.

[34] T. Fiegel, *An example of infinite dimensional reflexive space non-isomorphic to its cartesian square*, MÃl'moires de la S. M. F., 31-32 (1972), 165-167.

[35] A. Figalli, *A simple proof of the Morse-Sard theorem in Sobolev spaces*, Proc.Amer. Math. Soc. 136 (2008), no. 10, 3675-3681.

[36] J. Frontisi, *Smooth partitions of unity in Banach spaces*, Rocky Mountain J. Math. 25 (1995) 1295–1304.

[37] P. Hajek and M. Johanis, *Smooth approximations without critical points*, Cent. Eur. J. Math. **1** (2003), no. 3, 284–291.

[38] P. Hajek and M. Johanis, *Smooth analysis in Banach spaces*, De Gruyter Series in Nonlinear Analysis and Applications, Berlin/Boston 2014.

[39] P. Hajek, V. Montesinos, J. Vanderwerff and V. Zizler, *Biorthogonal systems in Banach spaces*, CMS Books in Mathematics, Canadian Mathematical Society, Springer, 2007.

[40] P. Hajlasz, M.V. Korobkov and J. Kristensen, *A bridge between Dubovitskiĭ-Federer theorems and the coarea formula*, Journal of Functional Analysis, 2017, vol 272, no 3, p. 1265-1295.

[41] P. Hajlasz and S. Zimmerman, *Dubovitskiĭ-Sard theorem for Sobolev mappings*, preprint, arXiv:1506.00025.

[42] R. Haydon, *Normes indéfiniment différentiables sur certains espaces de Banach*, Note aux C. R. Acad. Sci. Paris 315 (1992), 1175-1178.

[43] W.B. Johnson, H.P. Rosenthal and M. Zippin, *On bases, finite-dimensional decompositions and weaker structures in Banach spaces*, Israel J. Math. 9 (1971), 488-506.

[44] V. L. Klee, *Convex bodies and periodic homeomorphisms in Hilbert space*, Trans. Amer. Math. Soc. 74 (1953), 10-43.

[45] M.V. Korobkov and J. Kristensen, *On the Morse-Sard theorem for the sharp case of Sobolev mappigs*, Indiana Univ. Math. J. 63 (2014), no. 6, 1703-1724.

[46] N. Kuiper, *The homotopy type of the unitary group of Hilbert spaces*, Topology 3 (1965) 19-30

[47] I. Kupka, *Counterexample to the Morse-Sard theorem in the case of infinite-dimensional manifolds*, Proc. Amer. Math. Soc. 16 (1965), 954-957.

[48] S. Lang, *Introduction to differentiable manifolds.* Interscience, New York, 1962.

[49] S. Lang, *Real and Functional Analysis*, Third Edition. Graduate Texts in Mathematics 142. Springer, New York, 1993.

[50] J. Lindenstrauss, and L. Tzafriri, *Classical Banach spaces. I. Sequence spaces.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 92. Springer-Verlag, Berlin-New York, 1977.

[51] N. Moulis, *Sur les variétés hibertiennes et les fonctions non-dégénérées*, Indagationes Math. 30 (1968), 497–511.

[52] N. Moulis, *Approximation de fonctions différentiables sur certains espaces de Banach*, Ann. Inst. Fourier, Grenoble 21 (1971), 293–345.

[53] C. G. Moreira, *Hausdorff measures and the Morse-Sard theorem*, Publ. Mat. 45 (2001), 149-162.

[54] A. Morse, *The behavior of a function on its critical set*, Annals of Math. 40 (1939), 62-70.

[55] A. Norton, *A critical set with nonnull image has large Hausdorff dimension*, Trans. Amer. Math. Soc. 296 (1986), 367-276.

[56] A. Pelczyński and M. Wojciechowski, *Sobolev spaces*, Handbook of the Geometry of Banach spaces Vol. 2, W.B. Johnson and J. Lindenstrauss editors, Elsevier Amsterdam (2003), 1361-1423.

[57] R.R. Phelps, *Uniqueness of Hahn-Banach Extensions and Unique Best Approximation*, Transactions of the Amer. Math. Soc. 95 (1960) no. 2, 238–255.

[58] P.L. Renz, *Smooth extensions and extractions in infinite dimensional Banach spaces.* Thesis (Ph.D.)–University of Washington, 1969.

[59] P. Renz, *Smooth extensions in infinite dimensional Banach spaces*, Trans. Amer. Math. Soc. 168 (1972), 121–132.

[60] H.P. Rosenthal, *The Banach spaces C(K)*, Handbook of the Geometry of Banach spaces Vol. 2, W.B. Johnson and J. Lindenstrauss editors, Elsevier Amsterdam (2003), 1547-1602.

[61] M.E. Rudin, *A new proof that metric spaces are paracompact*, Proc. Amer. Math. Soc. 20, no. 2 (1969), 603.

[62] A. Sard, *The measure of the critical values of differentiable maps*, Bull. Amer. Math. Soc. 48 (1942), 883-890.

[63] S. Smale, *An infinite dimensional version of Sard's theorem*, Am. J. Math. 87 (1965), 861-866.

[64] A. Sobczyk, *Projection of the space* (m) *on its subspace* c_0, Bull. Amer. Math. Soc. 47 (1941), 938–947.

[65] J. Vanderwerff, *Smooth approximations in Banach spaces*, Proc. Amer. Math. Soc. 115 (1992) 113–120

[66] J.C. Wells, *Smooth Banach spaces and approximations.* Thesis (Ph.D.)–California Institute of Technology, 1969.

[67] J.E. West, *The diffeomorphic excision of closed local compacta from infinite-dimensional Hilbert manifolds*, Compositio Math. 21 1969 271–291.

[68] H. Whitney, *A function not constant on a connected set of critical points*, Duke Math. J. 1 (1935), 514-517.

[69] Y. Yomdin, *The geometry of critical and near-critical values of differentiable mappings*, Math. Ann. 264 (1983), 495-515.

[70] M. Zippin, *On some subspaces of Banach spaces whose duals are* L_1 *spaces*, Proc. Amer. Math. Soc. 23 (1969), 378-385.

ICMAT (CSIC-UAM-UC3-UCM), Departamento de Análisis Matemático y Matemática Aplicada, Facultad Ciencias Matemáticas, Universidad Complutense, 28040, Madrid, Spain. **DISCLAIMER:** The first-named author is affiliated with Universidad Complutense de Madrid, but this does not mean this institution has offered him all the support he expected. On the contrary, the Biblioteca Complutense has hampered his research by restricting his access to books.
E-mail address: azagra@mat.ucm.es

Department of Mathematics, Pittsburgh State Uuniversity, 1701 South Broadway Street, Pittsburg, KS 66762, United States of America
E-mail address: tdobrowolski@pittstate.edu

ICMAT (CSIC-UAM-UC3-UCM), Calle Nicolás Cabrera 13-15. 28049 Madrid, Spain
E-mail address: miguel.garcia@icmat.es