UNIFORM APPROXIMATION OF CONTINUOUS MAPPINGS BY SMOOTH MAPPINGS WITH NO CRITICAL POINTS ON HILBERT MANIFOLDS

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Abstract

We prove that every continuous mapping from a separable infinite-dimensional Hilbert space X into $\mathbb{R}^m$ can be uniformly approximated by C^∞-smooth mappings with no critical points. *This kind of result can be regarded as a sort of strong approximate version of the Morse-Sard theorem. Some consequences of the main theorem are as follows. Every two disjoint closed subsets of X can be separated by a one-codimensional smooth manifold that is a level set of a smooth function with no critical points. In particular, every closed set in X can be uniformly approximated by open sets whose boundaries are C^∞-smooth one-codimensional submanifolds of X. Finally, since every Hilbert manifold is diffeomorphic to an open subset of the Hilbert space, all of these results still hold if one replaces the Hilbert space X with any smooth manifold M modeled on X.*

1. Introduction and main results

A fundamental result in differential topology and analysis is the Morse-Sard theorem (see [21], [22]), which states that if $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a C^r-smooth function with $r > \max\{n-m, 0\}$ and C_f stands for the set of critical points of f (i.e., the points x at which the differential $df(x)$ is not surjective), then the set of critical values, $f(C_f)$, is of (Lebesgue) measure zero in $\mathbb{R}^m$. This result also holds true for smooth functions $f : X \longrightarrow Y$ between two smooth manifolds of dimensions n and m, respectively.

Several authors have dealt with the question of to what extent one can obtain a similar result for infinite-dimensional spaces or manifolds modeled on such spaces. Let us recall some of their results.

Smale [24] proved that if X and Y are separable connected smooth manifolds modeled on Banach spaces and $f : X \longrightarrow Y$ is a C^r Fredholm map (i.e., every differential $df(x)$ is a Fredholm operator between the corresponding tangent spaces),

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then $f(C_f)$ is meager, and, in particular, $f(C_f)$ has no interior points, provided that $r > \max\{\operatorname{index}(\mathrm{d}f(x)), 0\}$ for all $x \in X$; here $\operatorname{index}(\mathrm{d}f(x))$ stands for the index of the Fredholm operator $\mathrm{d}f(x)$, that is, the difference between the dimension of the kernel of $\mathrm{d}f(x)$ and the codimension of the image of $\mathrm{d}f(x)$, which are both finite. However, these assumptions are quite restrictive: for instance, if X is infinite-dimensional, then there is no Fredholm map $f : X \longrightarrow \mathbb{R}$. In general, the existence of a Fredholm map f from a manifold X into another manifold Y implies that Y is infinite-dimensional whenever X is.

On the other hand, one cannot dream of extending the Morse-Sard theorem to infinite dimensions without imposing strong restrictions. Indeed, as shown by Kupka's counterexample in [17], there are C^∞-smooth functions $f : X \longrightarrow \mathbb{R}$, where X is a Hilbert space, such that their sets of critical values $f(C_f)$ contain intervals and, in particular, have nonempty interiors.

More recently, S. M. Bates has carried out a deep study concerning the sharpness of the hypothesis of the Morse-Sard theorem and the geometry of the sets of critical values of smooth functions. In particular, he has shown that the above C^r-smoothness hypothesis in the statement of the Morse-Sard theorem can be weakened to $C^{r-1,1}$ (see [4]–[8]). C. G. Moreira and Bates have studied some generalizations of the Morse-Sard theorem related to Hausdorff measures and Hausdorff dimensions. They have also shown that the function f, as in Kupka's counterexample, can even be assumed to be a polymonial of degree three (see [9], [18]).

Nevertheless, for many applications of the Morse-Sard theorem, it is often enough to know that any given continuous function can be uniformly approximated by a map whose set of critical values has an empty interior. In this direction, Eells and McAlpin [16] established the following theorem: if X is a separable Hilbert space, then every continuous function from X into $\mathbb{R}$ can be uniformly approximated by a smooth function f whose set of critical values $f(C_f)$ is of measure zero. This allowed them to deduce a version of this theorem for mappings between smooth manifolds M and N modeled on X and a Banach space F, respectively, which they called an *approximate Morse-Sard theorem*: every continuous mapping from M into N can be uniformly approximated by a smooth function $f : M \longrightarrow N$ so that $f(C_f)$ has an empty interior. However, this seemingly much more general version of the result is a bit tricky: as they already observed (see [16, Rem. 3A]), when F is infinite-dimensional, the function f they obtain satisfies $C_f = M$, although $f(M)$ has an empty interior in N. Unfortunately, even though all the results of that paper seem to be true, some of the proofs are not correct.

In this paper we prove a much stronger result: if M is a C^∞-smooth manifold modeled on a separable infinite-dimensional Hilbert space X (in the sequel such a manifold is called a Hilbert manifold), then every continuous mapping from M into

$\mathbb{R}^m$ can be uniformly approximated by C^∞-smooth mappings *with no critical points*. This kind of result might be regarded as one of the strongest possible statements of approximate Morse-Sard theorems when the target space is finite-dimensional.

As a by-product, we also obtain the following: for every open set U in a separable Hilbert manifold M, there is a C^∞-smooth function f whose support is the closure of U and such that $\mathrm{d}f(x) \neq 0$ for every $x \in U$. This result could be summed up by saying that for every open subset U of M, there is a function f whose open support is U and which does not satisfy Rolle's theorem; one should compare this result with [2, Theorem 1] (see also the references therein).

Either of these results has, in turn, interesting consequences related to smooth approximation and separation of closed sets. For instance, every two disjoint closed subsets in M can be separated by a smooth one-codimensional submanifold of M which is a level set of a smooth function with no critical points. In particular, every closed subset of M can be uniformly approximated by open sets whose boundaries are smooth one-codimensional submanifolds of M.

So far, these are some *good* consequences of our main result, all of them somehow related to Morse-Sard-type theorems. But there are some *bad* consequences as well, perhaps the most noticeable one being that, since the set of smooth functions with no critical points is dense in the set of continuous functions defined on a Hilbert manifold, there are quite large sets of smooth functions for which no conceivable Morse theory could be valid.

Let us now formally state our main results. For the sake of a convenient notation in our proofs, when φ takes real values, we indistinctly use the symbols $\mathrm{d}\varphi(x) = \varphi'(x)$ to denote the derivative of φ at a point x, and we reserve $\mathrm{d}\varphi(x)$ for the derivative of a vector-valued function $\varphi : M \longrightarrow \mathbb{R}^m$ at a point $x \in M$.

THEOREM 1.1

Let U be an open subset of a separable infinite-dimensional Hilbert space X. Then, for every continuous mapping $f : U \longrightarrow \mathbb{R}^m$ and for every continuous positive function $\varepsilon : U \longrightarrow (0, +\infty)$, there exists a C^∞-smooth mapping $\psi : U \longrightarrow \mathbb{R}^m$ such that $\|f(x) - \psi(x)\| \leq \varepsilon(x)$ and $\mathrm{d}\,\psi(x)$ is surjective for all $x \in U$ (i.e., ψ has no critical points).

We prove this result in Section 2. Let us now establish the announced consequences of Theorem 1.1.

THEOREM 1.2

Let M be a separable Hilbert manifold. Then, for every continuous mapping f : $M \longrightarrow \mathbb{R}^m$ and every continuous positive function $\varepsilon : M \longrightarrow (0, +\infty)$, there exists a

C^∞-smooth mapping $\psi : M \longrightarrow \mathbb{R}^m$ such that ψ has no critical points and $\|f(x) - \psi(x)\| \leq \varepsilon(x)$ for all $x \in M$.

Proof

One could adapt the ideas in the proof of Theorem 1.1 to extend it to the setting of Hilbert manifolds, but, for simplicity, we instead use another approach. Indeed, bearing in mind a fundamental result on Hilbert manifolds due to Eells and Elworthy [15] that every separable Hilbert manifold can be C^∞-embedded as an open subset of the Hilbert space, it is a triviality to observe that Theorem 1.1 still holds if we replace U with a separable Hilbert manifold. □

We say that an open subset U of a Hilbert manifold M is *smooth*, provided that its boundary ∂U is a smooth one-codimensional submanifold of M.

COROLLARY 1.3

Let M be a separable Hilbert manifold. Then, for every two disjoint closed subsets C_1, C_2 of M, there exists a C^∞-smooth function $\varphi : M \longrightarrow \mathbb{R}$ with no critical points such that the level set $N = \varphi^{-1}(0)$ is a one-codimensional C^∞-smooth submanifold of M which separates C_1 and C_2 in the following sense. Define $U_1 = \{x \in M : \varphi(x) < 0\}$ and $U_2 = \{x \in M : \varphi(x) > 0\}$; then U_1 and U_2 are disjoint C^∞-smooth open sets of M with common boundary $\partial U_1 = \partial U_2 = N$, so that $C_i \subset U_i$ for $i = 1, 2$.

Proof

By Urysohn's lemma, there exists a continuous function $f : M \longrightarrow [-1, 1]$ such that $C_1 \subset f^{-1}(-1)$ and $C_2 \subset f^{-1}(1)$. Taking $\varepsilon = 1/3$ and applying Theorem 1.2, we get a C^∞-smooth function $\varphi : M \longrightarrow \mathbb{R}$ which has no critical points and is such that $|f(x) - \varphi(x)| \leq 1/3$ for all $x \in M$; in particular,

$$C_1 \subseteq f^{-1}(-1) \subseteq \varphi^{-1}(-\infty, 0) := U_1$$

and

$$C_2 \subseteq f^{-1}(1) \subseteq \varphi^{-1}(0, +\infty) := U_2.$$

The open sets U_1 and U_2 are smooth because their common boundary $N = \varphi^{-1}(0)$ is a smooth one-codimensional submanifold of M (thanks to the implicit function theorem and the fact that $d\varphi(x) \neq 0$ for all x). □

A trivial consequence of this result is that every closed subset of M can be uniformly approximated by smooth open subsets of M. In fact, we have the following.

COROLLARY 1.4

Every closed subset of a separable Hilbert manifold M can be approximated by *smooth open subsets of M in the following sense: for every closed set $C \subset M$ and every open set W containing C, there is a C^∞-smooth open set U so that $C \subset U \subseteq W$.*

Finally, the following result, which also implies the above corollary, tells us that for every open set U in M there always exists a function whose open support is U and which does not satisfy Rolle's theorem.

THEOREM 1.5

For every open subset U of a Hilbert manifold M, there is a continuous function f on M whose support is the closure of U, so that f is C^∞-smooth on U and yet f has no critical point in U.

Proof

For the same reasons as in the proof of Theorem 1.2, we may assume that U is an open subset of the Hilbert space $X = \ell_2$. Let $\varepsilon : X \longrightarrow [0, +\infty)$ be the distance function to $X \setminus U$; that is, let

$$\varepsilon(x) = \operatorname{dist}(x, X \setminus U) = \inf \big\{ \|x - y\| : y \in X \setminus U \big\}.$$

The function ε is continuous on X and satisfies $\varepsilon(x) > 0$ if and only if $x \in U$. According to Theorem 1.1, and setting $f(x) = 2\varepsilon(x)$, there exists a C^∞-smooth function $\psi : U \longrightarrow \mathbb{R}$ which has no critical points on U and which ε-approximates f on U; that is, $|2\varepsilon(x) - \psi(x)| \leq \varepsilon(x)$ for all $x \in U$. This inequality implies that $\lim_{x \to z} \psi(x) = 0$ for every $z \in \partial U$. Therefore, if we set $\psi = 0$ on $X \setminus U$, the extended function $\psi : X \longrightarrow [0, +\infty)$ is continuous on the whole of X, is C^∞-smooth on U, and has no critical points on U. On the other hand, $\psi(x) \geq \varepsilon(x) > 0$ for all $x \in U$; hence the support of ψ is $\overline{U}$. □

2. Proof of the main result

The main idea behind the proof of Theorem 1.1 is as follows. First, we use a *perturbed* smooth partition of unity to approximate the given continuous mapping f. The summands of this perturbed partition of unity are functions supported on scalloped balls and carefully constructed in such a way that the critical set C_φ of the approximating sum φ is locally compact.

Then we have to eliminate all the critical points without losing much of the approximation. To this end, we compose the approximating mapping φ with a deleting diffeomorphism $h : U \longrightarrow U \setminus C_\varphi$ which extracts the critical points C_φ and is as close to the identity as we want. The existence of such a diffeomorphism is guaranteed by the following quite elaborated result of West [25]. We say that a mapping g from a subset A of M is limited by an open cover G of M if the collection $\{\{x, g(x)\} : x \in A\}$ refines G.

THEOREM 2.1 (West [25, Th. 1])
Let C be a closed, locally compact subset of a Hilbert manifold M, let U be an open subset of M with $C \subset U$, and let G be an open cover of M. Then there is a C^∞-diffeomorphism h of M onto $M \setminus C$ which is the identity outside U and is limited by G.

In this way we obtain a smooth mapping ψ which has no critical points and which happens to approximate the function φ (which in turn approximates the original f) because the perturbation brought on φ by the composition with h is not very important. (Recall that h is arbitrarily closed to the identity.)

The following proposition shows the existence of a function φ with the above properties. Recall that C_φ stands for the set of critical points of φ.

PROPOSITION 2.2
Let U be an open subset of the separable Hilbert space X. Let $f : U \longrightarrow \mathbb{R}^m$ be a continuous mapping, and let $\varepsilon : U \longrightarrow (0, \infty)$ be a continuous positive function. Then there exists a C^∞-smooth mapping $\varphi : U \longrightarrow \mathbb{R}^m$ such that
(a) *C_φ is locally compact and closed (relatively to U),*
(b) *$\|\varphi(x) - f(x)\| \leq \varepsilon(x)/2$ for all $x \in U$.*

Assume for a while that Proposition 2.2 is already established, and let us see how we can deduce Theorem 1.1.

Proof of Theorem 1.1
For the given continuous mappings f and ε, take a mapping φ with the properties of Proposition 2.2. Since φ and ε are continuous, for every $z \in U$ there exists $\delta_z > 0$ such that if $x, y \in B(z, \delta_z)$, then $\|\varphi(y) - \varphi(x)\| \leq \varepsilon(z)/4 \leq \varepsilon(x)/2$.

Let $G = \{B(x, \delta_x) : x \in U\}$, let $M = U$, and for the critical set $C = C_\varphi$, use Theorem 2.1 to find a C^∞-diffeomorphism $h : U \longrightarrow U \setminus C$ such that h is limited by G. Define $\psi = \varphi \circ h$.

Since h is limited by G, we have that, for any given $x \in U$, there exists $z \in U$ such that $x, h(x) \in B(z, \delta_z)$, and therefore $\|\varphi(h(x)) - \varphi(x)\| \leq \varepsilon(z)/4$; that is, we have

$$\|\psi(x) - \varphi(x)\| \leq \frac{\varepsilon(z)}{4} \leq \frac{\varepsilon(x)}{2}.$$

Hence, by combining this inequality with Proposition 2.2(b), we obtain

$$\|\psi(x) - f(x)\| \leq \varepsilon(x) \tag{1}$$

for all $x \in U$.

Finally, it is clear that ψ does not have any critical points. Indeed, since $h(x) \notin C = C_\varphi$, we have that the linear map $d\varphi(h(x))$ is surjective, and $dh(x)$ is a linear isomorphism, so

$$d\,\psi(x) = d\,\varphi\big(h(x)\big) \circ d\,h(x) \tag{2}$$

is a linear surjection from X onto $\mathbb{R}^m$, for every $x \in U$. □

Remark 2.3
In the case when $f : U \longrightarrow \mathbb{R}$, we do not need to use the full power of West's result (whose proof is quite involved and extremely technical), and we can obtain a more self-contained proof that we briefly outline next.

When the function f takes values in the real line, one can show that C_φ is contained in a countable union of small compact sets K_n which are separated from one another by pairwise disjoint small open sets U_n; that is, $C_\varphi \subseteq \bigcup_{n=1}^\infty K_n \subset \bigcup_{n=1}^\infty U_n$ with $U_n \cap U_m = \emptyset$ whenever $n \neq m$, and the oscillation of φ on each U_n is very small. The proof of this stronger statement of Proposition 2.2 is similar to the one we give below (Case I), but it requires taking two extra precautions to define the K_n's and U_n's. First, one has to slightly perturb the radii r_n in such a way that, for any finite selection of centers y_n, the spheres that are the boundaries of the balls $B(y_n, r_n)$ have empty intersection with the affine subspace spanned by those centers, and no center belongs to any other sphere. (Of course, these are generic properties of radii and centers.) And second, one has to use Remark 2.7 in order to inductively select the λ_n's in the proof of Proposition 2.2 in a way that allows one to define the K_n's and U_n's. (Making a picture with only three balls helps one to guess how these sets can be defined by induction together with the λ_n's.)

Thanks to this more accurate alternative statement of Proposition 2.2, we can replace West's theorem with a much more elementary result that tells us that for every compact subset K and every open subset U of X with $K \subset U$, there exists a C^∞-diffeomorphism $h : X \longrightarrow X \setminus K$ such that h restricts to the identity outside U. (This fact is well known and was probably first proved by Moulis [20]. We gave another proof in an earlier version of the present paper which has recently been extended to the class of Banach spaces having Schauder bases and C^p-smooth bump functions; see [3].)

In this case, to eliminate the critical points of the approximating function φ, we may compose φ with a sequence of deleting diffeomorphisms $h_n : X \longrightarrow X \setminus K_n$ which extract each of the compact sets of critical points K_n and restrict to the identity outside each of the open sets U_n. The infinite composition of deleting diffeomorphisms with our function, $\psi = \varphi \circ \bigcirc_{n=1}^\infty h_n$, is locally finite in the sense that only a finite number (in fact, at most one) of the diffeomorphisms are acting on some neighborhood of each point, while all the rest restrict to the identity on that neighborhood.

As in the proof above, it follows that ψ has no critical points (the same argument works locally), and it still approximates f. (Recall that each h_n restricts to the identity outside the set U_n, on which φ has a very small oscillation, and that the U_n are pairwise disjoint.)

Proof of Proposition 2.2
We assume that $U = X$ since the proof is completely analogous in the case of a general open set. One has only to take some (easy but rather rambling) technical precautions in order to make sure that the different balls considered in the argument are in U.

In order to avoid bearing an unnecessary burden of notation, we make the proof of this proposition for the case of a constant $\varepsilon > 0$. Later on, we briefly explain what additional technical precautions must be taken in order to deduce the general form of this result (see Rem. 2.9).

Let $B(x, r)$ and $\overline{B}(x, r)$ stand for the open ball and closed ball, respectively, of center x and radius r, with respect to the usual Hilbertian norm $\|\cdot\|$ of X.

Case I. We first consider the case of a real-valued function $f : U \longrightarrow \mathbb{R}$. Fix $\varepsilon > 0$. By continuity, for every $x \in X$ there exists $\delta_x > 0$ such that $|f(y) - f(x)| \leq \varepsilon/200$ whenever $y \in B(x, 2\delta_x)$. Since $X = \bigcup_{x \in X} B(x, \delta_x/2)$ is separable, there exists a countable subcovering, $X = \bigcup_{n=1}^{\infty} B(x_n, r_n/2)$, where $r_n = \delta_{x_n}$ for some sequence of centers (x_n). By induction (and using the fact that every finite-dimensional subspace of X has an empty interior in X), we can choose a sequence of *linearly independent* vectors (y_n) with $y_n \in B(x_n, r_n/2)$, so that

$$X = \bigcup_{n=1}^{\infty} B(y_n, r_n). \tag{3}$$

Moreover, we have

$$|f(y) - f(y_n)| \leq \frac{\varepsilon}{100} \quad \text{whenever } \|y - y_n\| \leq r_n. \tag{4}$$

Now, we define the scalloped balls B_n which are the basis for our perturbed partition of unity: set $B_1 = B(y_1, r_1)$, and for $n \geq 2$, define

$$B_n = B(y_n, r_n) \setminus \Big(\bigcup_{j=1}^{n-1} \overline{B}(y_j, \lambda_n r_j) \Big),$$

where $1/2 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \lambda_{n+1} < \cdots < 1$ with $\lim_{n\to\infty} \lambda_n = 1$.

Taking into account the fact that $\lim_{n\to\infty} \lambda_n = 1$, it is easily checked that the B_n form a locally finite open covering of X, with the nice property that $|f(y) - f(y_n)| \leq \varepsilon/100$ whenever $y \in B_n$.

Next, pick a C^∞-smooth function $g_1 : \mathbb{R} \longrightarrow [0, 1]$ such that
(i) $g_1(t) = 1$ for $t \le 0$,
(ii) $g_1(t) = 0$ for $t \ge {r_1}^2$,
(iii) $g_1'(t) < 0$ if $0 < t < {r_1}^2$;
and then define $\varphi_1 : X \longrightarrow \mathbb{R}$ by $\varphi_1(x) = g_1(\|x - y_1\|^2)$ for all $x \in X$. Note that φ_1 is a C^∞-smooth function whose open support is B_1, and $B_1 \cap C_{\varphi_1} = \{y_1\}$; that is, y_1 is the only critical point of φ_1 which lies inside B_1.

Now, for $n \ge 2$, pick C^∞-smooth functions $\theta_{(n,j)} : \mathbb{R} \longrightarrow [0, 1]$, $j = 1, \dots, n$, with the following properties. For $j = 1, \dots, n-1$, $\theta_{(n,j)}$ satisfies
(i) $\theta_{(n,j)}(t) = 0$ for $t \le (\lambda_n r_j)^2$,
(ii) $\theta_{(n,j)}(t) = 1$ for $t \ge {r_j}^2$,
(iii) $\theta'_{(n,j)}(t) > 0$ if $(\lambda_n r_j)^2 < t < {r_j}^2$;
while for $j = n$, the function $\theta_{(n,n)}$ is such that
(i) $\theta_{(n,n)}(t) = 1$ for $t \le 0$,
(ii) $\theta_{(n,n)}(t) = 0$ for $t \ge {r_n}^2$,
(iii) $\theta'_{(n,n)}(t) < 0$ if $0 < t < {r_n}^2$.
Then define the function $g_n : \mathbb{R}^n \longrightarrow [0, 1]$ as

$$g_n(t_1, \dots, t_n) = \prod_{i=1}^{n} \theta_{(n,i)}(t_i)$$

for all $t = (t_1, \dots, t_n) \in \mathbb{R}^n$. This function is clearly C^∞-smooth on $\mathbb{R}^n$ and satisfies the following properties:
(i) $g_n(t_1, \dots, t_n) > 0$ if and only if $t_j > (\lambda_n r_j)^2$ for all $j = 1, \dots, n-1$, and $t_n < {r_n}^2$; and g_n vanishes elsewhere;
(ii) $g_n(t_1, \dots, t_n) = \theta_{(n,n)}(t_n)$ whenever $t_j \ge {r_j}^2$ for all $j = 1, \dots, n-1$;
(iii) $\nabla g_n(t_1, \dots, t_n) \ne 0$, provided $(\lambda_n r_j)^2 < t_j$ for all $j = 1, \dots, n-1$, and $0 < t_n < {r_n}^2$.
Moreover, under the same conditions as in (iii) just above, we have

$$\frac{\partial g_n}{\partial t_n}(t_1, \dots, t_n) = \frac{\partial \theta_{(n,n)}}{\partial t_n}(t_n) \prod_{i=1}^{n-1} \theta_{(n,i)}(t_i) < 0 \tag{5}$$

since no function in this product vanishes on the specified set; while for $j < n$, according to the corresponding properties of the functions $\theta_{(n,j)}$, we have

$$\frac{\partial g_n}{\partial t_j}(t_1, \dots, t_n) = \frac{\partial \theta_{(n,j)}}{\partial t_j}(t_j) \prod_{i=1, i\ne j}^{n} \theta_{(n,i)}(t_i) > 0. \tag{6}$$

If we are not in the conditions of (iii), then the corresponding inequalities do still hold but are not strict.

Let us now define $\varphi_n : X \longrightarrow [0, 1]$ by

$$\varphi_n(x) = g_n\big(\|x - y_1\|^2, \ldots, \|x - y_n\|^2\big).$$

It is clear that φ_n is a C^∞-smooth function whose open support is precisely the scalloped ball B_n.

As above, let us denote by C_{φ_n} the critical set of φ_n; that is, let $C_{\varphi_n} = \{x \in X : \varphi_n'(x) = 0\}$. Since our norm $\|\cdot\|$ is Hilbertian, we have that if $x \in C_{\varphi_n} \cap B_n$, then x belongs to the affine span of $y_1, \ldots, y_n$. Indeed, if for $x \in B_n$,

$$\varphi_n'(x) = \sum_{j=1}^{n} \frac{\partial g_n}{\partial t_j}\big(\|x - y_1\|^2, \ldots, \|x - y_n\|^2\big)\, 2(x - y_j) = 0, \tag{7}$$

then, since the y_j's are independent and the coefficients of the $x - y_j$'s are not all zero by (5), it follows that x is in the affine span of $y_1, \ldots, y_n$. Here, as is usual, we identify the Hilbert space X with its dual X^*, and we make use of the fact that the derivative of the function $x \mapsto \|x\|^2$ is the mapping $x \mapsto 2x$.

Similarly, it can be shown that $x \in C_{\varphi_1 + \cdots + \varphi_m} \cap (B_1 \cup \cdots \cup B_m)$ implies that x belongs to the affine span of $y_1, \ldots, y_m$. Indeed, write $(\varphi_1 + \cdots + \varphi_n)'(x)$ as in (7), and choose j such that $x \in B_j$ and $x \notin B_k$ for $j < k \le m$. Then the coefficient of $x - y_j$ in the sum is nonzero by (5).

In order that our approximating function have a small critical set, we cannot use the standard approximation provided by the partition of unity associated with the functions $(\varphi_j)_{i \in \mathbb{N}}$, namely,

$$x \mapsto \frac{\sum_{n=1}^{\infty} \alpha_n \varphi_n(x)}{\sum_{n=1}^{\infty} \varphi_n(x)},$$

where $\alpha_n = f(y_n)$. Indeed, such a function would have a huge set of critical points since it would be constant (equal to α_n) on a lot of large places (at least on each B_n minus the union of the rest of the B_j). Instead, we modify this standard approximation by letting the α_n be functions (and not mere numbers) of very small oscillation and with only one critical point (namely, y_n). So, for every $n \in \mathbb{N}$, let us pick a C^∞-smooth real function $a_n : [0, +\infty) \longrightarrow \mathbb{R}$ with the following properties:

(i) $a_n(0) = f(y_n)$,

(ii) $a_n'(t) < 0$ whenever $t > 0$,

(iii) $|a_n(t) - a_n(0)| \le \varepsilon/100$ for all $t \ge 0$;

and define $\alpha_n : X \longrightarrow \mathbb{R}$ by $\alpha_n(x) = a_n(\|x - y_n\|^2)$ for every $x \in X$. It is clear that α_n is a C^∞-smooth function on X whose only critical point is y_n, and

$$|\alpha_n(x) - f(y_n)| \le \frac{\varepsilon}{100} \quad \text{for all } x \in X.$$

Now, we can define our approximating function $\varphi : X \longrightarrow \mathbb{R}$ by

$$\varphi(x) = \frac{\sum_{n=1}^{\infty} \alpha_n(x)\varphi_n(x)}{\sum_{n=1}^{\infty} \varphi_n(x)}$$

for every $x \in X$. Since the sums are locally finite, it is clear that φ is a well-defined C^{∞}-smooth function.

FACT 2.4
The function φ approximates f nicely. Namely, we have
(i) $|\varphi(x) - f(x)| \leq \varepsilon/50$ *for all $x \in X$, and*
(ii) $|\varphi(y) - f(x)| \leq \varepsilon/25$ *for all $x, y \in B(y_n, r_n)$ and each $n \in \mathbb{N}$.*

Proof
For every n, we have $|\alpha_n(x) - f(y_n)| \leq \varepsilon/100$ for all $x \in X$. On the other hand, by (4), we know that $|f(x) - f(y_n)| \leq \varepsilon/100$ whenever $x \in B(y_n, r_n)$. Then, by the triangle inequality, it follows that

$$|\alpha_n(x) - f(x)| \leq \frac{\varepsilon}{50} \tag{8}$$

whenever $x \in B(y_n, r_n)$. Since $\varphi_m(y) = 0$ when $y \notin B(y_m, r_m)$, from (8) we get

$$|\varphi(x) - f(x)| = \left| \frac{\sum_{m=1}^{\infty} (\alpha_m(x) - f(x))\varphi_m(x)}{\sum_{m=1}^{\infty} \varphi_m(x)} \right| \leq \frac{\sum_{m=1}^{\infty} (\varepsilon/50)\varphi_m(x)}{\sum_{m=1}^{\infty} \varphi_m(x)} = \frac{\varepsilon}{50}$$

for all $x \in X$, which shows (i).

The proof of (ii) is trivial: $|\varphi(y) - f(x)| \leq |\varphi(y) - f(y)| + |f(y) - f(y_n)| + |f(y_n) - f(x)|$. The first term is smaller than $\varepsilon/50$ by (i), and each of the other terms is smaller than $\varepsilon/100$ by (4). □

Now, let us have a look at the derivative of φ. To this end, let us introduce the auxiliary functions f_n defined by

$$f_n(x) = \frac{\sum_{k=1}^{n} \alpha_k(x)\varphi_k(x)}{\sum_{k=1}^{n} \varphi_k(x)} \quad \text{for all } x \in \bigcup_{i=1}^{n} B_i.$$

Since the B_j's form a locally finite cover, it follows that for each x there are a neighborhood V_x and an $n = n_x$ such that $V_x \subseteq B_n$ and $V_x \cap B_j = \emptyset$ for $j > n$. In particular, $\varphi = f_n = f_m$ for $m \geq n$ on V_x.

Then the expression for the derivative of φ is given by

$$\varphi'(x) = \frac{\sum_{j=1}^{n} [\alpha_j'(x)\varphi_j(x) + \alpha_j(x)\varphi_j'(x)] \sum_{i=1}^{n} \varphi_i(x) - \sum_{j=1}^{n} \varphi_j'(x) \sum_{i=1}^{n} \alpha_i(x)\varphi_i(x)}{\left(\sum_{j=1}^{n} \varphi_j(x)\right)^2}.$$

Therefore, for x and $n = n_x$ as above, we have $\varphi'(x) = 0$ if and only if

$$\sum_{j=1}^{n}\sum_{i=1}^{n}\varphi_i(x)\big[\alpha_j'(x)\varphi_j(x) + (\alpha_j(x) - \alpha_i(x))\varphi_j'(x)\big] = 0. \tag{9}$$

Notation 2.5
In the sequel, $\mathscr{A}[z_1, \dots, z_k]$ stands for the affine subspace spanned by a finite sequence of points $z_1, \dots, z_k \in X$.

FACT 2.6
Let x and $n = n_x$ be as above. Then there are numbers $\beta_j(x)$, not all of them zero, such that

$$\varphi'(x) = \sum_{k=1}^{n}\beta_k(x)(x - y_k).$$

In particular, if $x \in C_\varphi$, then $x \in \mathscr{A}_n := \mathscr{A}[y_1, \dots, y_n]$.

Proof
As above, in all the subsequent calculations, we identify the Hilbert space X with its dual X^*, and we identify the derivative of $\|\cdot\|^2$ with the mapping $x \mapsto 2x$. To save notation, let us simply write

$$\frac{\partial g_n}{\partial t_j}\big(\|x - y_1\|^2, \dots, \|x - y_n\|^2\big) = \mu_{(n,j)}$$

and

$$a_j'\big(\|x - y_j\|^2\big) = \eta_j.$$

Notice that, according to (5) and (6), $\mu_{(n,j)} \geq 0$ for $j = 1, \dots, n-1$, while $\mu_{(n,n)} \leq 0$; and $\mu_{(n,n)} \neq 0$, provided $x \in B_n$ and $x \neq y_n$. On the other hand, it is clear that $\eta_j < 0$ for all j unless $x = y_j$ (in which case, $\eta_j = 0$).

Assuming $x \in C_{f_n} \cap B_n$, and taking into account the expression for $\varphi_j'(x)$ from (7) and the fact that $\alpha_j'(x) = 2\eta_j(x - y_j)$, we can write condition (9) in the form

$$2\sum_{j=1}^{n}\sum_{i=1}^{n}\varphi_i(x)\Big[\eta_j\varphi_j(x)\,(x - y_j) + \big(\alpha_j(x) - \alpha_i(x)\big)\sum_{\ell=1}^{j}\mu_{(j,\ell)}\,(x - y_\ell)\Big] = 0,$$

which in turn is equivalent (taking the common factors of each $(x - y_j)$ together) to the following one:

$$\sum_{j=1}^{n}\beta_j(x - y_j) = 0,$$

where

$$\beta_j := \Big[\eta_j\varphi_j(x)\sum_{i=1}^{n}\varphi_i(x) + \sum_{k=j}^{n}\Big(\sum_{i=1}^{n}\big(\alpha_k(x)-\alpha_i(x)\big)\varphi_i(x)\Big)\mu_{(k,j)}\Big]. \tag{10}$$

We next show that at least one of the β_j's is strictly negative and hence nonzero. We can obviously assume that x is not any of the points $y_1, \dots, y_n$ (which are already in $\mathscr{A}_n$). In this case, we have $\mu_{(n,n)} < 0$ and $\eta_j < 0$ for all $j = 1, \dots, n$. For simplicity, we make the argument only in the case of $n = 3$; giving a proof in a more general case would be little instructive and tedious to read.

Let us first assume that $\varphi_j(x) \neq 0$ for $j = 1, 2, 3$. We begin by looking at

$$\beta_3 := \eta_3\varphi_3(x)\sum_{i=1}^{3}\varphi_i(x) + \sum_{i=1}^{3}\big(\alpha_3(x)-\alpha_i(x)\big)\varphi_i(x)\mu_{(3,3)}.$$

If $\sum_{i=1}^{3}(\alpha_3(x)-\alpha_i(x))\varphi_i(x) \geq 0$, we are done since in this case we easily see that $\beta_3 < 0$. (Remember that $\mu_{(3,3)} \leq 0$, $\eta_3 < 0$, and $\varphi_3(x) > 0$.) Otherwise, we have

$$\sum_{i=1}^{3}\big(\alpha_3(x)-\alpha_i(x)\big)\varphi_i(x) < 0,$$

and then we look at the term

$$\beta_2 := \eta_2\varphi_2(x)\sum_{i=1}^{3}\varphi_i(x) + \sum_{k=2}^{3}\Big(\sum_{i=1}^{3}\big(\alpha_k(x)-\alpha_i(x)\big)\varphi_i(x)\Big)\mu_{(k,2)}.$$

Now, since $\mu_{(3,2)} \geq 0$, we have $\sum_{i=1}^{3}(\alpha_3(x)-\alpha_i(x))\varphi_i(x)\mu_{(3,2)} \leq 0$. On the other hand, $\eta_2\varphi_2(x)\sum_{i=1}^{3}\varphi_i(x) < 0$, so that if $\sum_{i=1}^{3}(\alpha_2(x)-\alpha_i(x))\varphi_i(x)$ happens to be nonnegative, then we also have $\sum_{i=1}^{3}(\alpha_2(x)-\alpha_i(x))\varphi_i(x)\mu_{(2,2)} \leq 0$. Then we are done since β_2, being a sum of negative terms (one of them strictly negative), must be negative as well. Otherwise,

$$\sum_{i=1}^{3}\big(\alpha_2(x)-\alpha_i(x)\big)\varphi_i(x)$$

is negative, and then we finally pass to the term

$$\beta_1 := \eta_1\varphi_1(x)\sum_{i=1}^{3}\varphi_i(x) + \sum_{k=1}^{3}\Big(\sum_{i=1}^{3}\big(\alpha_k(x)-\alpha_i(x)\big)\varphi_i(x)\Big)\mu_{(k,1)}.$$

Here, by the assumptions we have made so far and taking into account the signs of $\mu_{(k,j)}$ and η_j, we see that $\sum_{i=1}^{3}(\alpha_k(x)-\alpha_i(x))\varphi_i(x)\mu_{(k,1)} \leq 0$ for $k = 2, 3$. Having arrived at this point, it is sure that $\sum_{i=1}^{3}(\alpha_1(x)-\alpha_i(x))\varphi_i(x)$ must be nonnegative $\big($otherwise, the numbers $\sum_{i=1}^{3}(\alpha_k(x)-\alpha_i(x))\varphi_i(x)$ should be strictly negative for all $k=1,2,3$, which is impossible if one takes $\alpha_k(x)$ to be the maximum of the $\alpha_i(x)\big)$, and now we can deduce as before that $\beta_1<0$.

Finally, let us consider the case when some of the $\varphi_i(x)$ vanish for $i=1,2$. (Remember that $\varphi_3(x)\neq 0$ since $x\in B_3$, the open support of φ_3.) From the definitions of $\mu_{(k,j)}$, g_n, and φ_n, it is clear that $\mu_{(k,j)}=0$ for $j>k$, and bearing this fact in mind, we can simplify equality (10) to a great extent by dropping all the terms that now vanish.

If $\varphi_1(x)=\varphi_2(x)=0$, then (10) reads

$$\varphi_3(x)^2\eta_3\,(x-y_3)=0,$$

which cannot happen since we assumed $x\neq y_j$. (This means that the only critical point that f_n can have in $B_3\setminus(B_1\cup B_2)$ is y_3.)

If $\varphi_1(x)=0$ and $\varphi_2(x)\neq 0$, then the term β_1 accompanying $(x-y_1)$ in (10) vanishes, and hence (10) is reduced to

$$\sum_{j=2}^{3}\Big[\eta_j\varphi_j(x)\sum_{i=2}^{3}\varphi_i(x)+\sum_{k=j}^{3}\Big(\sum_{i=2}^{3}\big(\alpha_k(x)-\alpha_i(x)\big)\varphi_i(x)\Big)\mu_{(k,j)}\Big](x-y_j)=0.$$

Since at least one of the numbers $\sum_{i=2}^{3}(\alpha_k(x)-\alpha_i(x))\varphi_i(x)$, $k=2,3$, is nonnegative, the same reasoning as in the first case allows us to conclude that either β_3 or β_2 is strictly negative. Finally, in the case of $\varphi_1(x)\neq 0$ and $\varphi_2(x)=0$, it is β_2 that vanishes, and (10) reads $\beta_1\,(x-y_1)+\beta_3\,(x-y_3)=0$, where

$$\beta_3=\eta_3\varphi_3(x)\sum_{i=1,i\neq 2}^{3}\varphi_i(x)+\sum_{i=1,i\neq 2}^{3}\big(\alpha_3(x)-\alpha_i(x)\big)\varphi_i(x)\mu_{(3,3)}$$

and

$$\beta_1=\eta_1\varphi_1(x)\sum_{i=1,i\neq 2}^{3}\varphi_i(x)+\sum_{k=1,i\neq 2}^{3}\sum_{i=1,i\neq 2}^{3}\big(\alpha_k(x)-\alpha_i(x)\big)\varphi_i(x)\mu_{(k,1)}.$$

Again, at least one of the numbers $\sum_{i=1,i\neq 2}^{3}(\alpha_k(x)-\alpha_i(x))\varphi_i(x)$, $k=1,3$, is nonnegative, and the same argument as above applies. □

Remark 2.7

The above discussion actually shows the following inclusions:

$$C_{f_3}\cap B_3\subseteq\mathscr{A}[y_1,y_2,y_3],$$
$$C_{f_3}\cap(B_3\setminus B_1)\subseteq\mathscr{A}[y_2,y_3]\quad\text{and}\quad C_{f_3}\cap(B_3\setminus B_2)\subseteq\mathscr{A}[y_1,y_3],$$
$$C_{f_3}\cap(B_3\setminus(B_1\cup B_2))\subseteq\mathscr{A}[y_3].$$

More generally, for each $n \in \mathbb{N}$ and for every finite sequence of positive integers $k_1 < k_2 < \cdots < k_m < n$, we have $C_{f_n} \cap \left(B_n \setminus \bigcup_{j=1}^m B_{k_j}\right) \subseteq \mathscr{A}[\{y_1, \dots, y_n\} \setminus \{y_{k_1}, \dots, y_{k_m}\}]$. This is a crucial fact when one wants to give a self-contained proof of Theorem 1.1 (in the case of $m = 1$); see Remark 2.3.

Since φ has a continuous derivative, it is obvious that its critical set C_φ is closed in U. From Fact 2.6 it follows that C_φ is locally contained in finite-dimensional subspaces; that is, for each $x \in C_\varphi$, there is an open bounded neighborhood V_x of x such that $C_\varphi \cap \overline{V_x}$ is contained in a finite-dimensional subspace F_x of X and hence is compact (since it is closed and bounded as well). This means that C_φ is locally compact, and this concludes the proof of Proposition 2.2 in the case when $m = 1$.

Case II. Let us now deal with the case when $f : X \longrightarrow \mathbb{R}^m$ with $m \geq 2$. We denote $f = (f^1, \dots, f^m)$, where $f^1, \dots, f^m$ are the coordinate functions of f. In this case, we have to construct C^∞-smooth functions $\varphi^1, \dots, \varphi^m$ such that each φ^j uniformly approximates f^j and the set of points $x \in X$ at which the derivatives $d\varphi^1(x), \dots, d\varphi^m(x)$ are linearly dependent is locally compact. If we succeed in doing so, then it is clear that the function $\varphi = (\varphi^1, \dots, \varphi^m) : X \longrightarrow \mathbb{R}^m$ will approximate f and its set C_φ of critical points will be closed and locally compact.

Let us define $\varepsilon_j = \varepsilon/\sqrt{4m}$, $j = 1, \dots, m$. As each of the functions f^j, with $j = 1, \dots, m$, is continuous, for every $x \in X$ there exists $\delta_x^j > 0$ such that

$$|f^j(y) - f^j(x)| \leq \frac{\varepsilon_j}{200} \quad \text{for all } \ y \in B(x, 2\delta_x).$$

Since $X = \bigcup_{x \in X} B(x, \delta_x^j/2)$ is separable, we may take a countable subcovering

$$X = \bigcup_{n=1}^{\infty} B\left(x_n^j, \frac{r_n^j}{2}\right),$$

where $r_n^j = \delta_{x_n}^j$ for each $j = 1, \dots, m$.

Now, we can slightly perturb the centers x_n^j of the balls so that the union of all the m sequences of centers forms a set of linearly independent vectors. Indeed, bearing in mind that the complement of every finite-dimensional subspace of X is dense in the infinite-dimensional space X, we may inductively choose (taking m points $y_k^1, \dots, y_k^m$ at each kth step of the induction process) sequences of points $(y_n^j)_{n=1}^\infty$, $j = 1, \dots, m$, with $y_n^j \in B(x_n^j, r_n^j/2)$, such that

(i) $\{y_n^j : n \in \mathbb{N},\ j = 1, \dots, m\}$ is a set of linearly independent vectors,

(ii) $X = \bigcup_{n=1}^\infty B(y_n^j, r_n^j)$ for every $j = 1, \dots, m$, and

(iii) $|f^j(y) - f^j(y_n^j)| \leq \varepsilon_j/100$ whenever $\|y - y_n^j\| \leq r_n^j$.

Next, for each collection of balls $\{B(y_n^j, r_n^j)\}_{n\in\mathbb{N}}$ and each function f^j, define scalloped balls B_n^j, and construct a function φ^j exactly as in Case I, so that

$$|\varphi^j(y) - f^j(y)| \leq \varepsilon_j \quad \text{for all } \ y \in B(y_n^j, r_n^j).$$

This function φ^j is of the form

$$\varphi^j(x) = \frac{\sum_{n=1}^{\infty} \alpha_n^j(x)\varphi_n^j(x)}{\sum_{n=1}^{\infty} \varphi_n^j(x)} = \lim_{n\to\infty} f_n^j(x),$$

where

$$f_n^j(x) = \frac{\sum_{k=1}^{n} \alpha_k^j(x)\varphi_k^j(x)}{\sum_{k=1}^{n} \varphi_k^j(x)} \quad \text{for all } \ x \in \bigcup_{i=1}^{n} B_i^j,$$

the domains of the f_n^j form increasing towers of open sets whose union is X, and for each $x \in X$, there is some open neighborhood V_x^j of x and some $n_x^j \in \mathbb{N}$ such that $\varphi^j(y) = f_n^j(y)$ for all $y \in V_x^j$ and all $n \geq n_x^j$.

Now, define the mappings $\varphi : X \longrightarrow \mathbb{R}^m$ and $f_n : \bigcap_{j=1}^m \bigcup_{i=1}^n B_i^j \longrightarrow \mathbb{R}^m$ by

$$\varphi(x) = \left(\varphi^1(x), \ldots, \varphi^m(x)\right) \qquad \text{and} \qquad f_n(x) = \left(f_n^1(x), \ldots, f_n^m(x)\right).$$

By the choice of the ε_j and the construction of the functions φ^j, it is clear that

$$\|\varphi(x) - f(x)\| \leq \frac{\varepsilon}{2} \quad \text{for all } \ x \in X;$$

that is, φ approximates f, as is required.

FACT 2.8

If $x \in C_{f_n} \cap \left[\bigcap_{j=1}^m \bigcup_{i=1}^n B_i^j\right]$, *then* $x \in \mathscr{A}[y_i^j : 1 \leq i \leq n, 1 \leq j \leq m]$.

Proof

According to Fact 2.6, for each j and each $x \in \bigcup_{i=1}^n B_i^j$ we can assign numbers $\beta_1^j(x), \ldots, \beta_n^j(x)$ such that at least one of them does not vanish, and

$$\mathrm{d}\, f_n^j(x) = \sum_{k=1}^{n} \beta_k^j(x)(x - y_k^j). \tag{11}$$

Suppose now that $x \in \bigcap_{j=1}^m \bigcup_{i=1}^n B_i^j$, and suppose that the linear map $\mathrm{d} f_n(x) : X \longrightarrow \mathbb{R}^m$ is not surjective (i.e., suppose that x is a critical point of f_n); this means that there are numbers $\gamma_1(x), \ldots, \gamma_m(x)$, not all of them zero, such that

$$\sum_{j=1}^{m} \gamma_j(x)\, \mathrm{d}\, f_n^j(x) = 0. \tag{12}$$

Then, by combining (11) and (12), we get

$$\sum_{j=1}^{m}\sum_{k=1}^{n}\gamma_j(x)\beta_k^j(x)(x-y_k^j)=0, \tag{13}$$

where not all of the numbers $\gamma_j(x)\beta_k^j(x)$ vanish. Since the vectors y_k^j are all linearly independent, it follows from (13) that x is in the affine span of the vectors y_k^j with $j=1,\dots,m$ and $k=1,\dots,n$. □

As $\mathrm{d}\varphi$ is continuous, the set of critical points C_φ is closed in U. Now, we can easily show that C_φ is locally compact as well. Indeed, take $x \in X$. For every $j=1,\dots,m$, we know that there exist a neighborhood V_x^j of x and some $n_x^j \in \mathbb{N}$ such that $\varphi^j(y) = f_n^j(y)$ for all $y \in V_x^j$ and every $n \geq n_x^j$. Fix $n = n_x := \max\{n_x^1,\dots,n_x^m\}$, and take W_x an open bounded neighborhood of x such that $\overline{W_x} \subset V_x := \bigcap_{j=1}^m V_x^j$. Then we have

$$\varphi(y) = \big(\varphi^1(y),\dots,\varphi^m(y)\big) = \big(f_n^1(y),\dots,f_n^m(y)\big) = f_n(y)$$

for all $y \in V_x$, and, in particular, $V_x \subset \bigcap_{j=1}^m \bigcup_{i=1}^n B_i^j$. Now, according to Fact 2.8, it follows that $C_\varphi \cap V_x = C_{f_n} \cap V_x$ is contained in an affine subspace of dimension nm. In particular, $C_\varphi \cap \overline{W_x}$ is compact because it is closed, bounded, and contained in a finite-dimensional subspace. □

Remark 2.9

Let us say a few words as to the way one has to modify the above proof in order to establish Proposition 2.2 when ε is a continuous positive function. At the beginning of the proof of Case I of Proposition 2.2, before choosing the δ_x's, we have to take some number $\alpha_x > 0$ such that $|\varepsilon(y)-\varepsilon(x)| \leq \varepsilon(x)/4$ whenever $\|y-x\| \leq 2\alpha_x$, and then we can find some $\delta_x \leq \alpha_x$ such that $|f(y)-f(x)| \leq \varepsilon(x)/200$ whenever $y \in B(x,2\delta_x)$. In particular, after choosing the $r_n = \delta_{x_n}$ as in the proof of Case I, we have

$$|f(y)-f(y_n)| \leq \frac{\varepsilon(y_n)}{200} \qquad \text{and} \qquad \varepsilon(y_n) \leq \frac{4}{3}\varepsilon(y) \tag{14}$$

for all $y \in B(y_n,r_n)$. Then we can go on with the proof, with appropriate modifications, to construct the functions φ and f_n. Some obvious changes must be made in the definition of the functions a_n and α_n. Fact 2.4 now tells us that

$$|\varphi(y)-f(y_n)| \leq \frac{\varepsilon(y_n)}{4} \tag{15}$$

for all $y \in B(y_n,r_n)$. Then, by combining (14) and (15), we get

$$|\varphi(y)-f(y)| \leq |\varphi(y)-f(y_n)|+|f(y_n)-f(y)| \leq \frac{\varepsilon(y_n)}{4}+\frac{\varepsilon(y_n)}{8} = \frac{3}{8}\varepsilon(y_n) \leq \frac{\varepsilon(y)}{2}$$

for all $y \in B(y_n, r_n)$ and, since these balls cover X, this proves that $|\varphi(y) - f(y)| \leq \varepsilon(y)/2$ for all $y \in X$.

In Case II, it is enough to define the functions $\varepsilon_j(x) = \varepsilon(x)/\sqrt{4m}$ for $j = 1, \ldots, m$. The rest of the proof applies just by replacing ε_j and ε with $\varepsilon_j(x)$ and $\varepsilon(x)$, and by making some obvious minor modifications as in Case I.

Open Problem 2.10
The above proof works only in Hilbert space (because Fact 2.6 fails whenever the norm is not Hilbertian). We do not know if Theorem 1.1 is true for more general Banach spaces X.

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