Normal triangulations in o-minimal structures

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Abstract

Let $\mathcal{R}$ be an o-minimal structure over a real closed field R. Given a simplicial complex K and some definable subsets $S_1, \dots, S_l$ of its realization $|K|$ in R we prove that there exist a subdivision K' of K and a definable triangulation $\phi' : |K'| \to |K|$ of $|K|$ partitioning $S_1, \dots, S_l$ with ϕ' definably homotopic to $id_{|K|}$. As an application of this result we obtain the semialgebraic Hauptvermutung.

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1 Introduction

Let $\mathcal{R}$ be an o-minimal structure over a real closed field R. We are specially interested in the non-Archimedean case, as for instance the field $\mathbb{R}(t)^\wedge$ of Puiseux series in t over $\mathbb{R}$. This field $\mathbb{R}(t)^\wedge$ being real closed is already an o-minimal structure. Moreover, by a recent result of L. Lipshitz and Z. Robinson in [6], $\mathbb{R}(t)^\wedge$ together with the functions $f : [-t, t]^n \to R$ induced by the formal power series in $\mathbb{R}[[\xi_1, \dots, \xi_n]]$ is also o-minimal.

In the semialgebraic or more generally in the o-minimal context, a basic tool to study definable sets is the Triangulation Theorem: given a definable set S and some definable subsets $S_1, \dots, S_l$ of S, there exist a simplicial complex K and a definable homeomorphism $\phi : |K| \to S$ partitioning $S_1, \dots, S_l$. Further study of S may lead to consider new definable subsets $S'_1, \dots, S'_{l'}$ of S. In this situation, we would like to both preserve the already obtained triangulation and partition the new sets. However, techniques as repeated barycentric subdivisions are not available in this context because of the lack of Lebesgue number. We solve this problem by proving the following.

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Theorem 1.1. *Let S be definable set and let $S_1, \dots, S_l$ be some definable subsets of S. Let (K, ϕ) be a definable triangulation of S partitioning $S_1, \dots, S_l$. Then for any $S'_1, \dots, S'_{l'}$ definable subsets of S there exist a subdivision K' of K and a definable triangulation $\phi' : |K'| \to S$ partitioning $S_1, \dots, S_l, S'_1, \dots, S'_{l'}$ such that ϕ' is definably homotopic to ϕ.*

We apply the above result (in the semialgebraic case) to prove the semialgebraic Hauptvermutung.

Theorem 1.2 (Semialgebraic Hauptvermutung). *Let R be a real closed field. Let K and L be two closed simplicial complexes in R and let $f : |K| \to |L|$ be a semialgebraic homeomorphism. Then f is semialgebraically homotopic to a simplicial isomorphism $g : |K'| \to |L'|$ between subdivisions K' and L' of K and L, respectively.*

This result is proved for the real field by M.Shiota and M.Yokoi in [7]. In [2] M. Coste proves a weaker version of the semialgebraic Hauptvermutung, but strong enough to prove the unicity and strong effectiveness of semialgebraic triangulations. Namely, he proves that under the hypotheses of Theorem 1.2 there exists a simplicial isomorphism g between two subdivisions of K and L. However no relation between f and g is established.

In [8], A. Woerheide proves the existence of an o-minimal simplicial homology theory. In the present paper we give an alternative proof to Woerheide's result by showing that the natural adaptation of the classical approach to the o-minimal context can be followed just replacing the Simplicial Approximation Theorem by a corollary (see Corollary 4.4) to Theorem 1.1 (see Remark 4.6).

In [1] we give another application of the results of this paper identifying the o-minimal and semialgebraic setting modulo definable homotopy. Namely, we prove that any two semialgebraic functions which are definably homotopic are also semialgebraically homotopic. This allows us to transfer some results of semialgebraic homotopy to the o-minimal setting.

By definable we mean definable in the o-minimal structure $\mathcal{R}$ with parameters in R. All maps and functions are assumed to be continuous. Given a definable set S and some definable subsets $S_1, \dots, S_l$ of S we say that (K, ϕ) is a triangulation in R^p of S partitioning $S_1, \dots, S_l$, and denoted by $(K, \phi) \in \Delta(S; S_1, \dots, S_l)$, if K is a simplicial complex formed by a finite number of (open) simplices in R^p and $\phi : |K| \to S$ is a definable homeomorphism, with $|K| \subset R^p$, such that each S_i is the union of the images by ϕ of some simplices of K. Recall that given a definable set S the frontier of S is the set $\partial S = \overline{S} \setminus S$.

All the results mentioned above are based on the following notion and theorem.

Definition 1.3. *Let K be a simplicial complex in R^m and $S_1, \dots, S_l$ definable subsets of $|K|$. A triangulation $(K', \phi') \in \Delta(|K|; S_1, \dots, S_l)$ is a* ***normal triangulation of the complex*** *K* ***partitioning*** *$S_1, \dots, S_l$, denoted by $(K', \phi') \in \Delta^{NT}(|K|; S_1, \dots, S_l)$, if it satisfies the following conditions:*
(i) $(K', \phi') \in \Delta(|K|; S_1, \dots, S_l, \sigma)_{\sigma \in K}$
(ii) K' is a subdivision of K, and
(iii) for every $\tau \in K'$ and $\sigma \in K$, if $\tau \subset \sigma$ then $\phi'(\tau) \subset \sigma$.

Note that the definition depends not only on $|K|$ but also on K and that the interesting case is when the subsets of $|K|$ are nonempty (otherwise (K, id) is a normal triangulation). It is easy to see that $\phi'(|L|) = |L|$ for any subcomplex L of K. Also observe that condition (iii) cannot be deduced from (i) and (ii) (we can easily find a counterexample by taking as simplicial complex the standard 2-simplex and as homeomorphism a symmetry with respect to an angle bisector).

Our aim is to prove the existence of normal triangulations.

Theorem 1.4 (**Normal Triangulation Theorem**)**.** *Let K be a simplicial complex and let $S_1, \dots, S_l$ be definable subsets of $|K|$. Then there exists a triangulation $(K', \phi') \in \Delta^{NT}(|K|; S_1, \dots, S_l)$.*

The paper is organized as follows. Section 2 contains some definitions and results from [4] that we will need in the following section. Section 3 is devoted to prove the existence of independent triangulations (see Definition 3.1 and Theorem 3.2). The existence of independent triangulations allows us to prove the Normal Triangulation Theorem in Section 4, where we also prove Theorem 1.1 and give the applications described above. The reader may skip Sections 2 and 3 at a first reading, only the statement of Theorem 3.2 is used later. We also present an Appendix with some examples which motivate the properties introduced in this paper. For basic results on o-minimality we refer to [4].

The results of this paper are part of the author's Ph.D. dissertation.

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2 Preliminaries

We will make extensively use of the following notions and results from Chapter 8 of [4]. We include them here to both make this paper readable and be able to change slightly the notation. In particular note that for a triangulation (K, ϕ) of S we will use the classical notation $\phi : |K| \to S$ instead of the notation $\phi : S \to |K|$ used in [4]. Recall that given a definable set S the boundary of S is the set $bd(S) = \overline{S} \setminus int(S)$.

We recall that a *triangulated set* is the pair $(S, \phi(K))$, where S is a definable set, $(K, \phi) \in \Delta(S)$ and $\phi(K) = \{\phi(\sigma) : \sigma \in K\}$. Given a triangulated set $(S, \mathcal{P})$ and $C, D \in \mathcal{P}$ we call D a *face* of C if $D \subset cl(C)$, a *proper face* of C if $D \subset cl(C) \setminus C$ and a *vertex* of C if it has dimension 0.

A multivalued function F on a triangulated set $(S, \mathcal{P})$ is a finite collection of functions, $F = \{f_{C,i} : C \in \mathcal{P}, 1 \leq i \leq k(C)\}$, $k(C) \geq 0$, each function $f_{C,i} : C \to R$ definable and $f_{C,1} < \ldots < f_{C,k(C)}$. We set

$$F|_C = \{f_{C,i} : 1 \leq i \leq k(C)\}, \text{ for } C \in \mathcal{P},$$
$$\mathcal{P}^F = \{\Gamma(f) : f \in F\} \cup \{(f_{C,i}, f_{C,i+1}) : C \in \mathcal{P}, 1 \leq i < k(C)\},$$
$$S^F = \text{the union of the sets in } \mathcal{P}^F.$$

Such multivalued function F is called *closed* if for each pair $C, D \in \mathcal{P}$ with D a proper face of C and each $f \in F|_C$ there is $g \in F|_D$ such that $g(y) = \lim_{x \to y} f(x)$ for all $y \in D$. Note that then each $f \in F$, say $f \in F|_C$, extends continuously to a definable function $cl(f) : cl(C) \cap S \to R$ such that the restrictions of $cl(f)$ to the faces of C in $\mathcal{P}$ belong to F.

We call a multivalued function F **superfull** if it is closed, $k(C) \geq 1$ for all $C \in \mathcal{P}$, and it satisfies the following two conditions:

(A) for each pair $C, D \in \mathcal{P}$ with D a proper face of C and each $g \in F|_D$ we have $g = cl(f)|_D$ for some $f \in F|_C$, where $cl(f)$ is the continuous extension of f to $cl(C) \cap S$, and

(B) if $f_1, f_2 \in F|_C$, $f_1 \neq f_2$, then there exists at least one vertex of C where $cl(f_1)$ and $cl(f_2)$ take different values.

Note that our definition of superfull is stronger of that of *full* in [4] (where only condition (A) is required –see Definition VIII.2.5–). In general, we can convert a full multivalued function in a superfull one by taking the first barycentric subdivision. However, the properties of triangulations we want to consider are not preserved by taking barycentric subdivisions (see Example 5.2 in Appendix).

Recall that given a definable $S \subset R^m$, $(K, \phi) \in \Delta(S)$ and a definable $S' \subset S \times R$, a triangulation $(L, \psi) \in \Delta(S')$ in R^{n+1} is said to be a *lifting* of (K, ϕ) if $K = \{\pi_n(\sigma) : \sigma \in K\}$ and the diagram

$$\begin{tikzcd} {|L|} \arrow[r, "\psi"] \arrow[d, "\pi_n"'] & S' \arrow[d, "\pi_m"] \\ {|K|} \arrow[r, "\phi"'] & S \end{tikzcd}$$

commutes where π_m and π_n are the projections maps on the first m and n coordinates, respectively.

For the proof of the following technical fact see VIII.2.6 in [4]. We will show in Lemma 3.4 how to lift a triangulation via a superfull multivalued function using this fact.

Fact 2.1. *Let* $(a_0,\dots,a_n)$ *be an* n*-simplex in* R^p *and* $r_j, s_j \in R$, $r_j \leqslant s_j$, *for* $j=0,\dots,n$ *and* $r_j < s_j$ *for some* j. *Write* $b_j=(a_j,r_j)$, $c_j=(a_j,s_j)\in R^{p+1}$. *Then* $(b_0,\dots,b_j,c_j,\dots,c_n)$ *is an* $(n+1)$*-simplex in* R^{p+1} *for any* $0\leqslant j\leqslant n$ *such that* $b_j\neq c_j$. *Moreover, the collection of all* $(n+1)$*-simplices* $(b_0,\dots,b_j,c_j,\dots,c_n)$ *with* $b_j\neq c_j$ *and all their faces is a closed simplicial complex.*

3 Triangulations with the independence property

In order to prove the existence of normal triangulations, we now introduce triangulations satisfying an independence property which may be of interest by itself.

Definition 3.1. *Let* $(K,\phi)\in\Delta(S)$, *where* S *is a closed and bounded definable set in* R^m. *We say that* (K,ϕ) *is an* ***independent triangulation*** *if*
(i) for every n*-simplex* $\tau=(v_0,\dots,v_n)\in K$ *we have that* $\phi(v_0),\dots,\phi(v_n)\in R^m$ *are affinely independent, that is, they span an* n*-simplex* $\tau^\phi := (\phi(v_0),\dots,\phi(v_n))$ *in* R^m, *and*
(ii) if τ_1 *and* τ_2 *are different simplices of* K *then* τ_1^ϕ *and* τ_2^ϕ *are disjoint.*

Note that an independent triangulation induces in R^m –via the images by ϕ of the vertices of K– a copy of K. The aim of this section is to prove the following.

Theorem 3.2 (**Independent Triangulation Theorem**)**.** *Let* $S\subset R^m$ *be a closed and bounded definable set and let* $S_1,\dots,S_l$ *be definable subsets of* S. *Then there exists an independent triangulation* $(K,\phi)\in\Delta(S;S_1,\dots,S_l)$.

We will show that any closed and bounded definable set has an independent triangulation by an induction argument and following closely the scheme of the proof of the Triangulation Theorem in [4]. In the induction step we will need the existence of triangulations with the following technical property: a triangulation $(K,\phi)\in\Delta(S;S_1,\dots,S_l)$ of a closed and bounded definable set S and some definable subsets $S_1,\dots,S_l$ of S is said to be **small with respect to** $S_1,\dots,S_l$ if for every $\tau=(v_0,\dots,v_n)\in K$ with $\phi(v_0),\dots,\phi(v_n)\in\overline{S_j}$ we have that $\phi(\tau)\subset\overline{S_j}$. For examples of triangulations without these properties see Example 5.1 in Appendix.

Without mention we will use the following fact: given a definable set S and some definable subsets of S, a triangulation of S which partitions this subsets also partitions their closures and their frontiers in S.

Remark 3.3. Let S be a closed and bounded definable set and let $S_1,\dots,S_l$ be definable subsets of S. Let $(K,\phi)\in\Delta(S;S_1,\dots,S_l)$ and let $(\widetilde{K},\widetilde{\phi})\in\Delta(S;\phi(\sigma))_{\sigma\in K}$ small w.r.t. $\partial\phi(\sigma)$, $\sigma\in K$. Then,
(a) if $\tau\in\widetilde{K}$ and $\sigma\in K$ are such that $\widetilde{\phi}(\tau)\subset\phi(\sigma)$ then there exists a

vertex $v \in Vert(\tau)$ with $\widetilde{\phi}(v) \in \phi(\sigma)$ (in particular if σ is not a vertex then $\widetilde{\phi}(\tau) \not\subseteq \phi(\sigma)$), and
(b) $(\widetilde{K}, \widetilde{\phi}) \in \Delta(S; S_1, \dots, S_l)$ and is small w.r.t. $S_1, \dots, S_l$.

Proof. (a) Let $\tau \in \widetilde{K}$ and $\sigma \in K$. Suppose that $\widetilde{\phi}(\tau) \subset \phi(\sigma)$. Exclude the trivial case of τ being a vertex and assume for all $v \in Vert(\tau)$ we have that $\widetilde{\phi}(v) \notin \phi(\sigma)$, i.e., $\widetilde{\phi}(v) \in \partial\phi(\sigma)$. By smallness $\widetilde{\phi}(\tau) \subset \overline{\partial\phi(\sigma)} = \partial\phi(\sigma)$, a contradiction.
(b) It is clear that $(\widetilde{K}, \widetilde{\phi}) \in \Delta(S; S_1, \dots, S_l)$. To show smallness w.r.t. $S_1, \dots, S_l$, let $\tau = (v_0, \dots, v_n) \in \widetilde{K}$ and $j \in \{1, \dots, l\}$ be such that $\widetilde{\phi}(v_i) \in \overline{S_j}$, for all $i = 0, \dots, n$. Let $\sigma \in K$ be such that $\widetilde{\phi}(\tau) \subset \phi(\sigma)$. By (a) there exists a vertex $v_{i_0} \in Vert(\tau)$ with $\widetilde{\phi}(v_{i_0}) \in \phi(\sigma)$. Hence, since (K, ϕ) also partitions $\overline{S_j}$, $\phi(\sigma) \subset \overline{S_j}$. Therefore $\widetilde{\phi}(\tau) \subset \overline{S_j}$.

□

The following lemma will be useful in the induction step. Its proof is an adaptation –taking care of independence– of that of Lemma VIII.2.8 in [4].

Lemma 3.4. *Let $A \subset R^{m+1}$ be a closed and bounded definable set and let $(K, \phi) \in \Delta(A)$ be an independent triangulation in R^p. Let F be a superfull multivalued function on $(A, \phi(K))$. Then (K, ϕ) can be lifted to an independent triangulation $(L, \psi) \in \Delta(A^F; D)_{D \in \phi(K)^F}$ in R^{p+1}.*

Proof. The construction of (L, ψ) is that of Lemma VIII.2.8 in [4]. Unfortunately we will need to introduce the notation to check that (L, ψ) is independent. We construct L and ψ above each $C \in \phi(K)$. Let $C \in \phi(K)$ and let $a_0, \dots, a_n$ be the vertices of $\phi^{-1}(C)$ listed in some prefixed order on $Vert(K)$. Let $f < g$ be two successive members of $F|_C$. Put $r_j = cl(f)(\phi(a_j))$, $s_j = cl(g)(\phi(a_j))$, $b_j = (a_j, r_j)$, $c_j = (a_j, s_j)$. Let $L(f, g)$ be the complex in R^{p+1} constructed in Fact 2.1 (since F satisfies condition (B) of superfullness we do not need to take a barycentric subdivision as in [4]). Define the homeomorphism $\psi_{f,g}^{-1} : [cl(f), cl(g)] \to |L(f, g)|$ by $\psi_{f,g}^{-1}(x, tcl(f)(x) + (1-t)cl(g)(x)) = t\Phi_b(x) + (1-t)\Phi_c(x), 0 \le t \le 1$, where $\Phi_b(x)$ and $\Phi_c(x)$ are the points of $(b_0, \dots, b_n)$ and $(c_0, \dots, c_n)$ with the same affine coordinates with respect to $b_0, \dots, b_n$ and $c_0, \dots, c_n$ as $\phi^{-1}(x)$ has with respect to $a_0, \dots, a_n$. We also define for each $f \in F|_C$ the simplicial complex $L(f)$ in R^{p+1} as the n-simplex $(b_0, \dots, b_n)$ with $b_j = (a_j, cl(f)(\phi(a_j)))$, and all its faces. Then $\psi_f^{-1} : \Gamma(cl(f)) \to |L(f)|$ is by definition the homeomorphism given by $\psi_f^{-1}(x, cl(f)(x)) = \Phi_b(x)$, where $\Phi_b(x)$ is defined as before. Finally we consider the simplicial complex L which is the union of all simplicial complexes $L(f, g)$ and $L(f)$ and the definable homeomorphism $\psi : |L| \to A^F$ such that $\psi|_{L(f,g)} = \psi_{f,g}$ and $\psi|_{L(f)} = \psi_f$.

Let us show that the triangulation (L, ψ) is independent. Let $C \in \phi(K)$ and two successive functions $f, g \in F|_C$. Consider the triangulation $(L(f, g), \psi_{f,g})$ constructed at the beginning of the proof (and with the same notation). Given $v \in Vert(L(f,g))$ then either $v = b_j$ or $v = c_j$ for some j. Hence either $\psi_{f,g}(v) = \psi_{f,g}(b_j) = (\phi(a_j), r_j)$ or $\psi_{f,g}(v) = \psi_{f,g}(c_j) = (\phi(a_j), s_j)$. Denote $(\phi(a_j), r_j)$ by $\widetilde{b}_j$ and $(\phi(a_j), s_j)$ by $\widetilde{c}_j$. Observe that by independence of (K, ϕ) we have that $\phi(a_0), \ldots, \phi(a_n)$ are affinely independent and therefore, by Fact 2.1, $(\widetilde{b}_0, \ldots, \widetilde{b}_j, \widetilde{c}_j, \ldots, \widetilde{c}_n)$ are also affinely independent for any j such that $c_j \neq b_j$. Moreover, also by Fact 2.1, the $(n+1)$-simplices $(\widetilde{b}_0, \ldots, \widetilde{b}_j, \widetilde{c}_j, \ldots, \widetilde{c}_n)$ with $\widetilde{b}_j \neq \widetilde{c}_j$ and all their faces is a closed simplicial complex which we will denote by $L(f,g)^{\psi_{f,g}}$. In a similar way, we construct a closed simplicial complex $L(f)^{\psi_f}$ for each $f \in F|_C$, $C \in \phi(K)$. Since (K, ϕ) is independent, a routine argument shows that the collection L^{ψ} of all simplices of all closed simplicial complexes $L(f,g)^{\psi_{f,g}}$ and $L(f)^{\psi_f}$ is a closed simplicial complex. To finish the proof it is enough to observe, using the notation of Definition 3.1, that σ^{ψ} is a simplex of L^{ψ} for every $\sigma \in L$. $\Box$

From the proof of Lemma 3.4 we get the following.

Corollary 3.5. *There is a lifting (L, ψ) as in the conclusion of Lemma 3.4 satisfying the following. For any $\tau \in L$ with $\psi(\tau) \subset D$ for some $D \in \phi(K)^F$, $C = \pi(D) \in \phi(K)$, and $\phi^{-1}(C) = (a_0, \ldots, a_n)$, we have*
(i) $\pi(\psi(\tau)) = C$,
(ii) if $D = (f,g)_C$ for some successive $f, g \in F|_C$, $f < g$, then τ is either an $(n+1)$-simplex $(b_0, \ldots, b_j, c_j, \ldots, c_n)$, $b_j \neq c_j$, or is an n-simplex $(b_0, \ldots, b_{j-1}, c_j, \ldots, c_n)$, $b_j \neq c_j$, where $b_i = (a_i, cl(f)(\phi(a_i)))$ and $c_i = (a_j, cl(g)(\phi(a_i)))$, $i = 0, \ldots, n$, and
(iii) if $D = \Gamma(f)$ for some $f \in F|_C$, then τ is an n-simplex $(b_0, \ldots, b_n)$, where $b_i = (a_i, cl(f)(\phi(a_i)))$, $i = 0, \ldots, n$. Moreover, $\psi(\tau) = \Gamma(f) = D$.

We will also use next lemma in the induction step. We show how smallness allow us to modify a multivalued function satisfying (A) of superfullness to achieve a superfull one.

Lemma 3.6. *Let A be a closed and bounded definable set, let $(K, \phi) \in \Delta(A)$ and F be a closed multivalued function on the triangulated set $(A, \phi(K))$ satisfying condition (A) of superfull. Let $(K_0, \phi_0) \in \Delta(A; \phi(\sigma))_{\sigma \in K}$ small w.r.t. $\partial\phi(\sigma)$, $\sigma \in K$. Then, the multivalued function F_0 on $(A, \phi_0(K_0))$, obtained by the restrictions of the functions in F to the sets of $\phi_0(K_0)$, is superfull.*

Proof. Clearly the multivalued function F_0 is closed and satisfies (A) of superfullness. Let us check that F_0 satisfies also (B) of superfullness. Let $C = \phi_0(\tau)$, where $\tau \in K_0$, and let $f_1, f_2 \in F_0|_C$ be two different functions. By construction, there exist $\sigma \in K$ and two different $\widetilde{f}_1, \widetilde{f}_2 \in F|_{\phi(\sigma)}$ such that $C \subset \phi(\sigma)$ and $\widetilde{f}_i|_C = f_i$, $i = 1, 2$. By Remark 3.3(a), there exists a vertex $\phi_0(v)$ of C such that $\phi_0(v) \in \phi(\sigma)$. Then $cl(f_i)(\phi_0(v)) = \widetilde{f}_i(\phi_0(v))$, $i = 1, 2$, so $cl(f_1)$ and $cl(f_2)$ take different values on $\phi_0(v)$. $\Box$

We are now ready to prove the main result of this section.

Proof of Theorem 3.2. In fact, we will prove by induction on m a stronger result: Given a closed and bounded definable set $S \subset R^m$ and some definable subsets $S_1, \ldots, S_l$ of S, there exists a triangulation $(K, \phi) \in \Delta(S; S_1, \ldots, S_l)$ which is both independent and small w.r.t. $S_1, \ldots, S_l$.

The case $m = 0$ is trivial. Suppose that the theorem holds for a certain m and let us prove it for $m + 1$. Consider the definable sets $T = bd(S) \cup bd(S_1) \cup \cdots \cup bd(S_l)$ (which has dimension less than $m + 1$) and $A = \pi(T)$, where π denotes the projection on the first m coordinates. By the proof of the Triangulation Theorem in [4] we can assume that:
(a) for every $a \in A$ the fiber $T_a = \{x \in R : (a, x) \in T\}$ is finite, and
(b) there exist a triangulation $(K, \phi) \in \Delta(A; \pi(S_1), \ldots, \pi(S_l))$ and a closed multivalued function F on the triangulated set $(A, \phi(K))$ satisfying condition (A) of superfullness and such that S, S_i, $\overline{S_i}$ and $\overline{S}$ are finite unions of sets in $\phi(K)^F$.

Moreover, by our induction hypothesis we can assume in (b) that (K, ϕ) is independent. Note that (a) is obtained in [4] by applying a certain linear automorphism which preserves independence and smallness. In our case – aiming for independence– we now first modify the multivalued function (see M1 below) to obtain a superfull one and hence being able to apply our lifting Lemma 3.4. Unfortunately, to be able to execute M1 we make essential use of the smallness property of the induction hypothesis. A trivial second modification (see M2 below) will allows us to prove smallness (see Example 5.3 in the Appendix for a justification of this modification).
M1. By induction hypothesis there exists an independent triangulation $(K_1, \phi_1) \in \Delta(A; \phi(\sigma))_{\sigma \in K}$ small w.r.t. $\partial\phi(\sigma)$, $\sigma \in K$. Let F_1 be the multivalued function on $(A, \phi_1(K_1))$ obtained by the restrictions of the functions in F to the sets of $\phi_1(K_1)$. By Lemma 3.6 the multivalued function F_1 is superfull.
M2. Given $C \in \phi_1(K_1)$ and two successive functions $f, g \in F_1|_C$ with $f < g$, we consider the function $\frac{f+g}{2}$ on C. Let F_2 be the multivalued function obtained by adding to F_1 the new functions $\frac{f+g}{2}$ for each pair of successive functions $f, g \in F_1|_C$, $C \in \phi_1(K_1)$, $f < g$. The new multivalued function F_2 on $\phi_1(K_1)$ is also superfull.

By Lemma 3.4, we can lift (K_1, ϕ_1) to an independent triangulation $(L_0, \psi_0) \in \Delta(A^{F_2}; D)_{D \in \phi_1(K_1)^{F_2}}$. Finally, let $L = \{\sigma \in L_0 : \psi_0(\sigma) \subset S\}$. Clearly $(L, \psi) \in \Delta(S; S_1, \ldots, S_l)$ is independent, where $\psi = \psi_0|_{|L|}$.

It remains to prove that (L, ψ) is small w.r.t. $S_1, \ldots, S_l$. Let $\tau = (v_0, \ldots, v_n) \in L$ and $i \in \{1, \ldots, l\}$ be such that $\psi(v_r) \in \overline{S_i}$ for all $r = 0, \ldots, n$. We show that $\psi(\tau) \subset \overline{S_i}$. Let $D \in \phi_1(K_1)^{F_2}$ be such that $\psi(\tau) \subset D$ and denote by $C = \pi(D) \in \phi_1(K_1)$. Note that by Corollary 3.5(i), $\pi(\psi(\tau)) = \pi(D) = C \in \phi_1(K_1)$. We first consider the case that D is the graph of a function of F_2. By Corollary 3.5(iii), we have that $D = \psi(\tau)$.

Claim 1. *For any $C \in \phi_1(K_1)$, $f \in F_2|_C$ and $i \in \{1,\dots,l\}$ if $(w, cl(f)(w)) \in \overline{S_i}$ for every vertex w of C then $\Gamma(f) \subset \overline{S_i}$.*

Then $D \subset \overline{S_i}$ modulo Claim 1. Now consider the case that $D = (f,g)_C$ for two successive functions $f, g \in F_2|_D$, $f < g$, and suppose that $D \not\subseteq \overline{S_i}$. Then $D \subset \overline{S_i}^c$. By definition of F_2 we can assume that there exist two successive functions $f_1, g_1 \in F_1|_C$, $f_1 < g_1$, such that $f = f_1$ and $g = \frac{f_1|_C + g_1|_C}{2}$. Therefore $D \subset \widetilde{D} := (f_1, g_1)_C$. Since $D \subset \overline{S_i}^c$ then $\widetilde{D} \subset \overline{S_i}^c$. At this point we claim that

Claim 2. *The set*
$\widetilde{D}_{cil} = \{(x,y) : x \in \overline{C} \setminus C, cl(f_1)(x) \neq cl(g_1)(x), cl(f_1)(x) < y < cl(g_1)(x)\}$
is contained in $\overline{S_i}^c$.

Assume also that we have proved Claim 2. Then by Corollary 3.5(ii), and following its notation, we have two cases: either τ is an n-simplex $(b_0, \dots, b_{j-1}, c_j, \dots, c_n)$ with $b_j \neq c_j$ and $\phi_1^{-1}(C) = (a_0, \dots, a_n)$, or τ is an n-simplex $(b_0, \dots, b_j, c_j, \dots, c_{n-1})$ with $b_j \neq c_j$ and $\phi_1^{-1}(C) = (a_0, \dots, a_{n-1})$. In both cases, since $b_j \neq c_j$, we have that $cl(f)(\phi_1(a_j)) \neq cl(g)(\phi_1(a_j))$ and therefore $cl(f_1)(\phi_1(a_j)) \neq cl(g_1)(\phi_1(a_j))$. Since

$$cl(g)(\phi_1(a_j)) = \frac{cl(f_1)|_C + cl(g_1)|_C}{2}(\phi_1(a_j)),$$

we deduce that $cl(f_1)(\phi_1(a_j)) < cl(g)(\phi_1(a_j)) < cl(g_1)(\phi_1(a_j))$ and hence $\psi(c_j) = (\phi_1(a_j), cl(g)(\phi_1(a_j))) \in \widetilde{D}_{cil}$. By Claim 2, $\psi(c_j) \notin \overline{S_i}$, a contradiction. We conclude that $\psi(\tau) \subset D \subset \overline{S_i}$ as required.

It remains to prove the two claims.

Proof of Claim 1. Let $f \in F_2|_C$, $C \in \phi_1(K_1)$ and $i \in \{1, \dots, l\}$ be such that $(w, cl(f)(w)) \in \overline{S_i}$ for every vertex w of C. Suppose first that $f \in F_1|_C$ and $\Gamma(f) \not\subseteq \overline{S_i}$. Since F_1 is the restrictions of the functions of F to the sets of $\phi_1(K_1)$, there exists $\widetilde{f} \in F|_{\widetilde{C}}$, $\widetilde{C} \in \phi(K)$, $C \subset \widetilde{C}$, such that $\widetilde{f}|_C = f$. Since $\Gamma(f) \not\subseteq \overline{S_i}$, we have that $\Gamma(\widetilde{f}) \not\subseteq \overline{S_i}$. Therefore $\Gamma(\widetilde{f}) \subset \overline{S_i}^c$. Since (K_1, ϕ_1) is small w.r.t. $\partial\phi(\sigma)$, $\sigma \in K$, by Remark 3.3(a) there exists one vertex w_0 of C such that $w_0 \in \widetilde{C}$. Therefore $(w_0, cl(f)(w_0)) = (w_0, \widetilde{f}(w_0)) \in \Gamma(\widetilde{f})$ does not lie in $\overline{S_i}$, a contradiction. Suppose now that $f = \frac{f_0+g_0}{2}$, for two successive functions $f_0, g_0 \in F_1|_C$, $f_0 < g_0$. Then $\Gamma(f) \subset (f_0, g_0)_C$. Since F_1 is the restrictions of the functions of F to the sets of $\phi_1(K_1)$, there exist $\widetilde{f}_0, \widetilde{g}_0 \in F|_{\widetilde{C}}$, $\widetilde{C} \in \phi(K)$, $C \subset \widetilde{C}$, such that $\widetilde{f}_0|_C = f_0$ and $\widetilde{g}_0|_C = g_0$. Suppose $\Gamma(f) \not\subseteq \overline{S_i}$. Then $(f_0, g_0)_C \not\subseteq \overline{S_i}$ and therefore $(\widetilde{f}_0, \widetilde{g}_0)_{\widetilde{C}} \not\subseteq \overline{S_i}$. Hence $(\widetilde{f}_0, \widetilde{g}_0)_{\widetilde{C}} \subset \overline{S_i}^c$. By Remark 3.3(a), there exists one vertex w_0 of C such that $w_0 \in \widetilde{C}$. Therefore $(w_0, cl(f)(w_0)) = (w_0, \frac{cl(f_0)+cl(g_0)}{2}(w_0)) = (w_0, \frac{\widetilde{f}_0+\widetilde{g}_0}{2}(w_0)) \in (\widetilde{f}_0, \widetilde{g}_0)_{\widetilde{C}}$ does not lie in $\overline{S_i}$, a contradiction.

Proof of Claim 2. Suppose there is $(x_0, y_0) \in \widetilde{D}_{cil}$ such that $(x_0, y_0) \in \overline{S_i}$.

Since $\phi_1(K_1)^{F_2}$ partitions $\overline{S_i}$, for all $y \in (cl(f_1)(x_0), cl(g_1)(x_0))$ we have that $(x_0, y) \in \overline{S_i}$. Moreover, we will show that $(x_0, y) \in bd(S_i)$ for every $y \in (cl(f_1)(x_0), cl(g_1)(x_0))$. Fix $y \in (cl(f_1)(x_0), cl(g_1)(x_0))$. Since $x_0 \in \overline{C} \setminus C$, by the Curve Selection Lemma, VI.1.5 in [4], there exists a definable curve $\gamma(t)$, $t \in (0,1)$, such that $\lim_{t\to 1} \gamma(t) = x_0$ and $\gamma(t) \in C$, for all $t \in (0,1)$. Consider the curve

$$\gamma_y(t) = (\gamma(t), f_1(\gamma(t)) + (g_1(\gamma(t)) - f_1(\gamma(t)))\left(\frac{y - f_1(x_0)}{g_1(x_0) - f_1(x_0)}\right)), t \in (0,1).$$

Note that $\gamma_y(t) \in \widetilde{D}$ for all $t \in (0,1)$ and $\lim_{t\to 1} \gamma_y(t) = (x_0, y)$. Therefore $(x_0, y) \in \overline{\widetilde{D}} \subset \overline{\overline{S_i}^c} \subset \overline{int(S_i)^c} = int(S_i)^c$. We conclude that $(x_0, y) \in bd(S_i)$. We have shown that $(x_0, y) \in bd(S_i)$ for all $y \in (cl(f_1)(x_0), cl(g_1)(x_0))$, which is a contradiction because the fiber of T in x_0 is finite (see (a) at the beginning of the proof). $\square$

We have the following corollary to the proof of Theorem 3.2.

Corollary 3.7. *Let $S \subset R^m$ be a closed and bounded definable set and let $S_1, \dots, S_l$ be definable subsets of S. Then there exists a triangulation $(K, \phi) \in \Delta(S; S_1, \dots, S_l)$ which is both independent and small with respect to $S_1, \dots, S_l$.*

4 Normal triangulations

We begin this section with a key lemma for the proof of the Normal Triangulation Theorem. It can be easily proved making use of o-minimal homology. However, we only use the Open Mapping Theorem which has been proved (independently of o-minimal homology) by J. Johns in [5].

Lemma 4.1. *Let $\sigma \subset R^n$ be an n-simplex and let $f : \overline{\sigma} \to \overline{\sigma}$ be an injective definable map. If $f(\partial\sigma) \subset \partial\sigma$ then $f(\sigma) = \sigma$ (and hence $f(\partial\sigma) = \partial\sigma$).*

Proof. Since $\overline{\sigma}$ is closed and bounded, $f(\overline{\sigma})$ is closed. By the Open Mapping Theorem, $f(\sigma)$ is open in R^n. This implies that $\sigma \cap f(\sigma)$, which is closed in σ since $\sigma \cap f(\sigma) = \sigma \cap f(\overline{\sigma})$, is also open in σ. On the other hand σ is definably connected and, since f is injective, $dim(f(\sigma)) = n$, so $f(\sigma)$ cannot be included in $\partial\sigma$ which has smaller dimension. Hence $\sigma \cap f(\sigma) = \sigma$. Finally, since $f(\sigma) \subset int(f(\overline{\sigma})) \subset \sigma$, we conclude that $f(\sigma) = \sigma$. $\square$

Remark 4.2. It is enough to prove the Normal Triangulation Theorem for closed simplicial complexes. For let K be a simplicial complex and $S_1, \dots, S_l$ some definable subsets of $|K|$. Let $(K_0, \phi_0) \in \Delta^{NT}(|\overline{K}|; S_1, \dots, S_l)$, where $\overline{K}$ denotes the closed simplicial complex which is the collection of faces of simplices of K. Then $(K', \phi') \in \Delta^{NT}(|K|; S_1, \dots, S_l)$, where $K' = \{\sigma \in K_0 : \sigma \subset |K|\}$ and $\phi' = \phi_0|_{|K'|}$.

Proof of Theorem 1.4. By Remark 4.2 we can assume that K is closed. We first apply Theorem 3.2 to get an independent triangulation $(K_0, \phi_0) \in \Delta(|K|; S_1, \dots, S_l, \sigma)_{\sigma \in K}$. Now consider the collection of simplices $K' = \{\tau^{\phi_0} : \tau \in K_0\}$ (with the notation of Definition 3.1). By definition of independent triangulation, K' is a closed simplicial complex. Moreover, the map between the set of vertices $g_{vert} : Vert(K_0) \to Vert(K') : v \mapsto \phi_0(v)$ induce a simplicial isomorphism $g : |K_0| \to |K'|$. Note that $g(\tau) = \tau^{\phi_0}$ for $\tau \in K_0$. We also observe that given $\tau \in K_0$ if $\phi_0(\tau) \subset \sigma \in K$ then the images by ϕ_0 of the vertices of τ lie in $\overline{\sigma}$ and therefore $g(\tau) = \tau^{\phi_0} \subset \overline{\sigma}$. Hence, given $\sigma \in K$ take $\tau_1, \dots, \tau_m \in K_0$ be such that $\sigma = \phi_0(\tau_1)\dot{\cup} \cdots \dot{\cup}\phi_0(\tau_m)$ and then we get $g(\tau_1)\dot{\cup} \cdots \dot{\cup} g(\tau_m) \subset \overline{\sigma}$.

Claim. $g(\tau_1)\dot{\cup} \cdots \dot{\cup} g(\tau_m) = \sigma$.

Once we have proved the Claim, we can assure that (a) K' is a subdivision of K, and (b) for every $\tau \in K_0$ and every $\sigma \in K$ we have that $\phi_0(\tau) \subset \sigma$ if and only if $g(\tau) \subset \sigma$. Now we consider the following map

$$\phi' := \phi_0 \circ g^{-1} : |K'| \to |K|.$$

Finally we are ready to show that $(K', \phi') \in \Delta^{NT}(|K|; S_1, \dots, S_l)$ as required. We have to check the three conditions of Definition 1.3. The fact that g is a simplicial isomorphism and that the triangulation $(K_0, \phi_0) \in \Delta(S_1, \dots, S_l, \sigma)_{\sigma \in K}$ give us (i); by (a) above we get (ii), and to check (iii), given a simplex $\tau^{\phi_0} \in K'$ such that $\tau^{\phi_0} \subset \sigma \in K$ we have that $\phi'(\tau^{\phi_0}) = \phi_0 \circ g^{-1}(\tau^{\phi_0}) = \phi_0(\tau) \subset \sigma$ by (b). It remains to prove the Claim.
Proof of the Claim. By induction on the dimension n of the simplex $\sigma \in K$, the case $n = 0$ being trivial. Let $\sigma \in K$ be an $(n+1)$-simplex. Let $\tau_1, \dots, \tau_m \in K_0$ be such that $\sigma = \phi_0(\tau_1)\dot{\cup} \cdots \dot{\cup}\phi_0(\tau_m)$. We may assume $\sigma \subset R^{n+1}$. Now consider the injective definable map $(g \circ \phi_0^{-1})|_{\overline{\sigma}} : \overline{\sigma} \to \overline{\sigma}$. Applying the induction hypothesis to each simplex in $\partial\sigma$ we get $(g \circ \phi_0^{-1})(\partial\sigma) = \partial\sigma$. Therefore by Lemma 4.1 we have that $(g \circ \phi_0^{-1})(\sigma) = \sigma$, i.e., $\sigma = g(\tau_1)\dot{\cup} \cdots \dot{\cup} g(\tau_m)$. □

In the rest of this section we will give applications of the Normal Triangulation Theorem.

Proof of Theorem 1.1. By the Normal Triangulation Theorem there exists $(K', \psi) \in \Delta^{NT}(|K|; \phi^{-1}(S'_1), \dots, \phi^{-1}(S'_{l'}))$. Consider the following definable map $H : |K'| \times I \to |K'| : (x, s) \mapsto (1-s)x + s\psi(x)$. The map H is well-defined because, by normality of (K', ψ), we have that $\psi(x) \in \sigma$ for each $x \in \sigma \in K$. Also observe that H is clearly continuous. Therefore H is a definable homotopy between ψ and $id_{|K|}$. Finally, define $\phi' = \phi \circ \psi$. Clearly $(K', \phi') \in \Delta(S; S_1, \dots, S_l, S'_1, \dots, S'_{l'})$ and ϕ' is definably homotopic to ϕ. □

Corollary 4.3. *Let K be a simplicial complex and $S_1, \dots, S_l$ definable subsets of $|K|$. Then there exists $(K', \phi') \in \Delta(|K|; S_1, \dots, S_l, \sigma)_{\sigma \in K}$ such that K' is a subdivision of K and ϕ' is definably homotopic to $id_{|K|}$.*

Proof. By applying Theorem 1.1 to $|K|$, $(K, id_{|K|})$, $S_1, \dots, S_l$ and the simplices of K. $\square$

Recall that given a definable function $f : |K| \to |L|$ between the realizations of two simplicial complexes K and L we say that f is *compatible* if for every $\sigma \in K$ there is $\tau \in L$ such that $f(\sigma) \subset \tau$.

Corollary 4.4. *Let K and L be two simplicial complexes and let $f : |K| \to |L|$ be a definable map. Then there exist a subdivision K' of K and a definable homeomorphism $\phi : |K'| \to |K|$ such that both ϕ and $f \circ \phi$ are compatible.*

Proof. It is enough to observe that by the Normal Triangulation Theorem there exists $(K', \phi) \in \Delta^{NT}(|K|, f^{-1}(\tau))_{\tau \in L}$. $\square$

Theorem 4.5. *Let R be a real closed field. Let $X \subset R^n$ and $Y \subset R^m$ be two semialgebraic sets defined without parameters. Then any semialgebraic map (homeomorphism) $f : X \to Y$ is semialgebraically homotopic to a semialgebraic map (resp. homeomorphism) $g : X \to Y$ defined without parameters.*

Proof. We denote by $\overline{\mathbb{Q}}$ the real algebraic numbers. We can assume that X and Y are the realization of two simplicial complexes K and L respectively, whose vertices lie in $\overline{\mathbb{Q}}$. Taking a subdivision, by Corollary 4.3 we can assume that for each $\sigma \in K$ there exists $\tau_\sigma \in L$ such that $f(\sigma) \subset \tau_\sigma$. We denote f by f_c to stress the fact that c is a tuple of parameters in R such that f is defined over. Consider the first order formula $\psi(y)$ defined without parameters which says that f_y is a map (resp. homeomorphism) between $|K|$ and $|L|$ and for each $\sigma \in K$, $f_y(\sigma) \subset \tau_\sigma$. By completeness of the theory of real closed fields, since R satisfies $\exists y \psi(y)$, then $\overline{\mathbb{Q}}$ satisfies $\exists y \psi(y)$. Therefore there exists a tuple of parameters a in $\overline{\mathbb{Q}}$ such that $f_a : |K| \to |L|$ is a semialgebraic map (resp. homeomorphism) and for each $\sigma \in K$, $f_a(\sigma) \subset \tau_\sigma$. Denote f_a by g. Finally the map $H : |K| \times I \to |L| : (x, t) \to (1-t) f_c(x) + t g(x)$, is well-defined because for each $\sigma \in K$ we have that both $g(\sigma)$ and $f_c(\sigma) \subset$ are contained in τ_σ. Hence H is a semialgebraic homotopy between f and g. $\square$

Proof of Theorem 1.2. We can assume that the vertices of K and L lie in $\overline{\mathbb{Q}}$ (the real algebraic numbers). By Theorem 4.5 we can also assume that f is defined without parameters. We now show that, since f is defined without parameters, the theorem follows from the real Semialgebraic Hauptvermutung. Indeed, by Theorem 4.1 in [7] there exists a simplicial isomorphism

$\widetilde{g} : |K'|(\mathbb{R}) \to |L'|(\mathbb{R})$ between the realizations in $\mathbb{R}$ of two subdivision K' and L' of K and L respectively, such that $\widetilde{g}$ is semialgebraically homotopic to the map $f^{\mathbb{R}} : |K|(\mathbb{R}) \to |L|(\mathbb{R})$ induced by f. Note that we can express with a first order sentence ψ defined without parameters the existence of some parameters and some points in K and L which are the vertices of two simplicially isomorphic subdivisions of K and L and that the simplicial isomorphism is semialgebraically homotopic to f. By completeness of the theory of real closed fields, since $\mathbb{R}$ satisfies ψ, then $\overline{\mathbb{Q}}$ also satisfies ψ. □

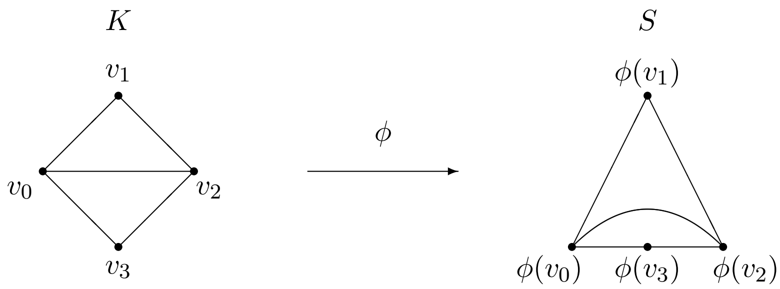
We finish this section with another application of the Normal Triangulation Theorem.

Remark 4.6. In both the semialgebraic and o-minimal setting it is possible to develop a homology theory over real closed fields as it was proved by M. Knebusch–H. Delfs and A. Woerheide respectively (see e.g. [3] and [8]). When adapting the classical development to the o-minimal setting, the lack of the Simplicial Approximation Theorem lead us to the problem of verifying the excision axiom in the singular case and to construct a well-defined functor in the simplicial one. M. Knebusch and H. Delfs avoid this problem developing their semialgebraic homology theory via cohomology of sheaves. In his PhD dissertation A. Woerheide returns to the classical line solving the problem of constructing a well-defined o-minimal simplicial homology functor applying the Triangulation Theorem and the Acyclic Models Theorem. Then he uses this o-minimal simplicial homology theory to verify the excision axiom of the o-minimal singular homology. On the other hand, our Normal Triangulation Theorem fills the gap left by the lack of the Simplicial Approximation Theorem, which allows us to follow the classical proof. Therefore it gives an alternative proof of the existence of a functor for o-minimal simplicial homology. To see this consider a definable map $f : |K| \to |L|$ between the realizations of two closed simplicial complexes K and L. To define the group homomorphism $f_* : H_*(K) \to H_*(L)$ we proceed as follows. Let K' and ϕ be respectively the subdivision of K and the definable homeomorphism given by Corollary 4.4. Since $f \circ \phi$ and ϕ are both compatible, then both satisfy the star condition. Hence they both have a simplicial approximation and we can define $(f \circ \phi)_* : H_*(K') \to H_*(L)$ and $\phi_* : H_*(K') \to H_*(K)$. We define the homomorphism $f_* : H_*(K) \to H_*(L)$ by $f_* = (f \circ \phi)_* \circ \phi_*^{-1}$. It remains to prove that ϕ_* is indeed an isomorphism, but this is a classical result since, by Corollary 4.4, $\phi(\sigma') \subset \sigma$ for each pair $\sigma' \in K'$ and $\sigma \in K$ with $\sigma' \subset \sigma$, and therefore any simplicial approximation to ϕ is a simplicial approximation to $id : |K'| \to |K|$. Using the classical techniques it is easy to check that with these induced homomorphisms we have a well-defined functor for o-minimal simplicial homology. A similar approach allow us to define the o-minimal simplicial functor in the relative case and, adapting the classical techniques, it is easy to verify the o-minimal Eilenberg-Steenrod homology axioms.

5 Appendix

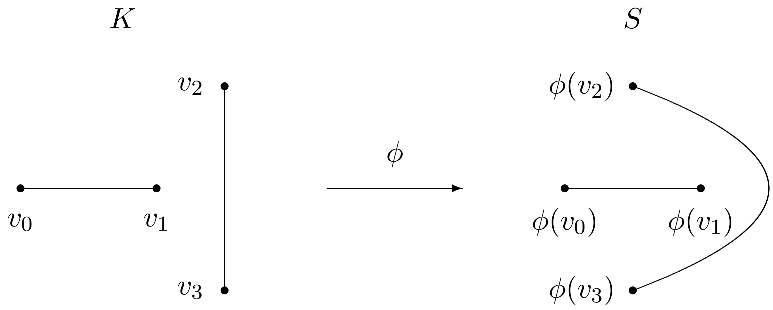
All the following examples are in dimension 2.

Example 5.1. 1) Example of a triangulation (K, ϕ) of a closed and bounded definable set S which is not independent because (i) fails.

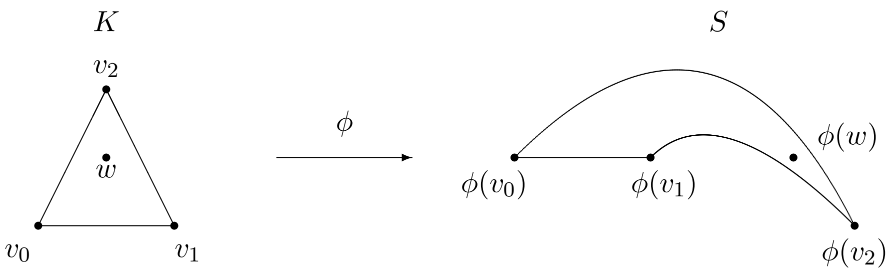


Observe that if we denote by $S_1 = \phi((v_0, v_2))$ then (K, ϕ) is small w.r.t. S_1.

2) Example of a triangulation (K, ϕ) of a closed and bounded definable set S which is not independent because it satisfies (i) but it does not satisfies (ii).



Example 5.2. This example shows that the independence property of triangulations is not preserved by taking barycentric subdivisions.



Example 5.3. Let S be the closed and bounded 2-dimensional definable set of Figure 1, where the union of the curves in its interior is the subset S_1. If we follow the proof of the Independence Triangulation Theorem and we make the modification M1 but we do not make the modification M2 (see Figure 2) then we will obtain a triangulation which is not small w.r.t. S_1.

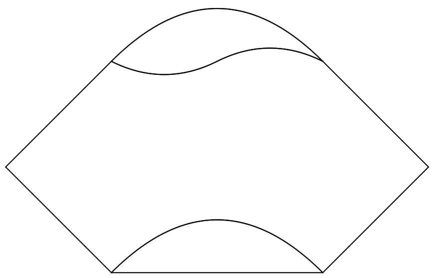


Figure 1

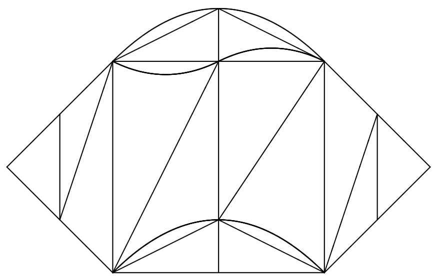


Figure 2

References

[1] E. Baro and M.Otero, On o-minimal homotopy groups, preprint, 19pp., 2007, `www.uam.es/elias.baro/articulos/ominhomotopy.pdf`.

[2] M. Coste, Unicité des triangulations semi-algébriques: validité sur un corps réel clos quelconque, et effectivité forte, C. R. Acad. Sci. Paris Sr. I Math., 312 (1991), no. 5, 395–398.

[3] H. Delfs and M. Knebusch, *Locally semialgebraic spaces*, Lecture Notes in Mathematics, 1173, Springer-Verlag, Berlin, 1985.

[4] L. van den Dries, *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series, 248, Cambridge University Press, 1998.

[5] J. Johns, An open mapping theorem for o-minimal structures, J. Symbolic Logic, 66 (2001), 1817-1820.

[6] L. Lipshitz and Z. Robinson, Overconvergent real closed quantifier elimination, Bull. London Math. Soc., 38 (2006), no.6, 897-906.

[7] M. Shiota and M. Yokoi, Triangulations of subanalytic sets and locally subanalytic manifolds, Trans. Amer. Math. Soc., 286(1984), no.2, 727-750.

[8] A. Woerheide, O-minimal homology, PhD Thesis, University of Illinois at Urbana-Champaign, 1996.

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