APPLICATIONS OF NORMAL TRIANGULATIONS IN O-MINIMAL STRUCTURES

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Normal Triangulation Theorem. *Let $|K|$ be the realization of a simplicial complex K in R^m. Let $S_1, \dots, S_l$ be some definable subsets of $|K|$. Then there is a normal triangulation (K', ϕ) of $|K|$ compatible with $S_1, \dots, S_l$, i.e,*
(i) K' is a subdivision of K,
(ii) (K', ϕ) is a triangulation of $|K|$ compatible with each simplex $\sigma \in K$ and the subsets $S_1, \dots, S_l$, and
(iii) $\phi : |K'| \to |K|$ is definably homotopic to $id_{|K|}$.

We proved the Normal triangulation theorem to show that both the semialgebraic and the o-minimal homotopy have a strong relation between them (see [2]). Nevertheless, this theorem is a tool by itself that can be used to solve another problems

The simplicial homology functor. Let $\mathcal{R}$ be an o-minimal expansion of a real closed field. Given a closed simplicial complex K it is not difficult to define the n-homology group $H_n(|K|)$ of its realization $|K|$. However, given a continuous definable map $f : |K| \to |L|$ between the realization of two simplicial complexes K and L, it is not so easy to define the induced map $f_* : H_n(|K|) \to H_n(|L|)$ (we cannot use as in the classical case the Simplicial approximation theorem).

A. Woerheide solved this problem by using the Triangulation theorem and the Acyclic models theorem (see [5]).

Now, we can show an alternative way to solve this problem which is similar to the classical one. For, the Normal triangulation theorem fills the gap left by the lack of the Simplicial approximation theorem. Indeed, given a continuous definable map $f : |K| \to |L|$, there is a simplicial map $g : |K'| \to |L|$ which is definably homotopic to f.

The semialgebraic Hauptvermutung. *Let $|K|$ and $|L|$ be the realizations of two closed simplicial complexes K and L of a real closed field R. Let $f : |K| \to |L|$ be a semialgebraic homeomorphism in R. Then there is a simplicial isomorphism $g : |K'| \to |L'|$ such that f is semialgebraically homotopic to g, where K' and L' are some subdivisions of K and L respectively.*

M. Shiota and M. Yokoi proved the semialgebraic Hauptvermutung for the real field (see [4]). Using this result and the Tarski-Seidenberg principle, M. Coste proved a weak version for real closed fields in which no relation between the semialgebraic homeomorphism and the simplicial one is established (see [3]).

The new ingredient to prove the full statement is that, thanks to the Normal triangulation theorem, we can assume that the semialgebraic homeomorphism $f : |K| \to |L|$ is such that for every $\sigma \in K$ there is $\tau \in L$ with $f(\sigma) \subset \tau$ (see [1]).

[1] E. Baro, Normal triangulations in o-minimal structures, preprint, 2007.

[2] E. Baro and M.Otero, On o-minimal homotopy groups, preprint, 2007.

[3] M. Coste, Unicité des triangulations semi-algébraiques: validité sur un corps réel clos quelconque, et effectivité forte, C. R. Acad. Sci. Paris Sr. I Math., 312 {1991}, no. 5, 395-398.

[4] M. Shiota and M. Yokoi, Triangulations of subanalytic sets and locally subanalytic manifolds, Trans. Amer. Math. Soc., 286{1984}, no.2, 727-750.

[5] A. Woerheide, o-minimal homology, PhD Thesis, University of Illinois at Urbana-Champaign, 1996.