On the o-minimal LS-category

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Introduction

We introduce the o-minimal Lusternik-Schnirelmann category (in short o-minimal LS-category) of definable sets in an o-minimal expansion $\mathcal{R}$ of a real closed field R. The classical one was originally introduced to provide a lower bound on the number of critical points for any smooth function on a manifold, then it became an important subject in algebraic topology (see [5]). Recall that a definable open cover of a definable set is a finite cover of open definable subsets.

Definition. The o-minimal LS-category of a definable set X, denoted by $cat(X)^{\mathcal{R}}$, is the least integer m such that X has a definable open cover of m+1 elements with each of them definably contractible to a point in X (not necessarily definably contractible in itself).

The o-minimal LS-category has good o-minimal topological properties. For instance, if two definable sets are definable homotopy equivalent then their o-minimal LS-categories clearly coincide.

Once we have introduced the o-minimal LS-category a natural question arises: does it have any relation with the semialgebraic and the classical ones? Moreover, we would like to apply it to the study of definable groups. Is it possible?

Toolbox

Let X and Y be semialgebraic sets in R and let A and B be semialgebraic subsets of X and Y respectively. The o-minimal homotopy set $[(X,A),(Y,B)]^{\mathcal{R}}$ is the collection of definable maps $f:X\to Y$ with $f(A)\subset B$ modulo definable homotopy mapping A in B.

Fact. With the notation above, if A is relatively closed in X then,
(a) [1, Cor.3.3] the map $[(X,A),(Y,B)]^{\mathcal{R}_{sa}}\to[(X,A),(Y,B)]^{\mathcal{R}}$: $[f]\to[f]$ is a bijection,
(b) [6, Thm.III.4.1] if S a real closed field extension of R then the map $[(X,A),(Y,B)]^{\mathcal{R}_{sa}}\to[(X(S),A(S)),(Y(S),B(S))]^{\mathcal{S}_{sa}}$: $[f]\to[f(S)]$ is a bijection,
(c) [6, Thm.III.5.1] if $R=\mathbb{R}$ then the map $[(X,A),(Y,B)]^{\mathcal{R}_{sa}}\to[(X,A),(Y,B)]$: $[f]\to[f]$ is a bijection, where $[(X,A),(Y,B)]$ is the classical homotopy set.

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Main Results

We shall denote by $\mathcal{R}_{sa}$ the field structure of the real closed field R. Our aim is to relate the o-minimal LS-cateogry with the semialgebraic and the classical ones. Using the development of o-minimal homotopy in [1] (see the toolbox), we prove the following comparison theorems.

Theorem. [2, Thm.3.5] Let X be a semialgebraic subset of R^n. Then

$$cat(X)^{\mathcal{R}}=cat(X)^{\mathcal{R}_{sa}}.$$

Theorem. [2, Thm.3.6] Let X be a semialgebraic subset of R^n. Let S be a real closed field extension of R. Then

$$cat(X)^{\mathcal{R}_{sa}}=cat(X(S))^{\mathcal{S}_{sa}}.$$

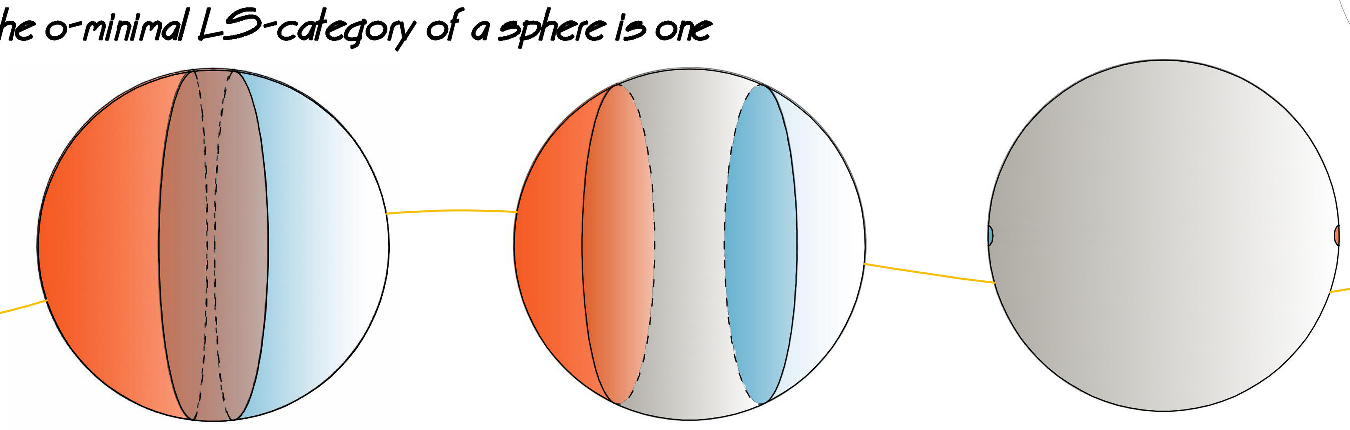
Theorem. [2, Thm.3.8] Let X be a semialgebraic subset of $\mathbb{R}^n$. Then the semialgebraic category of X equals the classical one.

We can summarize the comparison theorems above in the following corollary.

Corollary. Let X be a semialgebraic subset of R^n defined without parameters. Then $cat(X)^{\mathcal{R}}=cat(X(\mathbb{R}))$, where $cat(X(\mathbb{R}))$ denotes the classical LS-category of $(X(\mathbb{R}))$.

An example:

the o-minimal LS-category of a sphere is one



We can apply the comparison theorems to transfer classical results concerning the LS-category to the o-minimal setting.

Proposition. [2, Cor.3.10] Let X be a definably connected definable set. Then $cat(X)^{\mathcal{R}}\leq dim(X)$.

Proposition. [2, Cor.3.11] Let X be a definable set and let $n\geq 1$ such that $\pi_r(X)^{\mathcal{R}}=0$ for all $r=0,\dots,n-1$. Then $cat(X)^{\mathcal{R}}\leq dim(X)/n$.

Proposition. [2, Cor.3.12] Let X be a definable set. Let $cuplength_{\mathbb{Q}}(X)^{\mathcal{R}}$ be the least integer k such that all (k+1)-fold cup products vanish in the reduced cohomology $\tilde{H}^*(X;\mathbb{Q})^{\mathcal{R}}$. Then $cat(X)^{\mathcal{R}}\geq cuplength_{\mathbb{Q}}(X)^{\mathcal{R}}$.

Motivation

Recall that given a definably compact d-dimensional definable group G, the work of several authors (e.g. A. Berarducci, E. Hrushovski, Y. Peterzil, A. Pillay, M. Otero and others) in the positively resolution of Pillay's conjecture has shown that there exist a smallest type-definable subgroup G^{00} of G with bounded index such that $\mathbb{L}(G):=G/G^{00}$ with the logic topology is a compact d-dimensional Lie group (e.g., see [8]).

The main motivation to study the o-minimal LS-category is to establish a topological analogy between a definably compact definably connected group G and the connected compact Lie group $\mathbb{L}(G)$ associated to it. In this direction it has been already proved that their homotopy groups are isomorphic (see [3]). Our aim is to prove that $cat(G)^{\mathcal{R}}$ and $cat(\mathbb{L}(G))$ are equal. To do this we establish the following stronger result.

Theorem. [2, Thm.1.2] Let G be a definably connected definably compact definable group whose underlying set is a semialgebraic set defined without parameters. Then $G(\mathbb{R})$ is homotopy equivalent to $\mathbb{L}(G)$.

Note that by the triangulation theorem, we can always assume that the underlying set of a definable group is a semialgebraic set defined without parameters. There are two key results to prove this theorem. The first one is a recent result by E. Hrushovski, Y. Peterzil and A. Pillay (see [7]) which establishes that the commutator subgroup G' of G is semisimple and definable. Moreover, G is the almost direct product of $Z(G)^0$ and G'. The second one is a classical result by A. Borel concerning Lie groups which establishes that $\mathbb{L}(G)$ is homeomorphic to $Z(\mathbb{L}(G))^0\times\mathbb{L}(G)'$ (see [4]).

As a consequence of the theorem above we get the following.

Corollary. [2, Cor.1.3] Let G and H be definably connected definably compact definable groups. Then G and H are definable homotopy equivalent if and only if $\mathbb{L}(G)$ and $\mathbb{L}(H)$ are homotopy equivalent.

Corollary. [2, Cor.4.10] Let G be a definably connected definably compact definable group. Then $cat(G)^{\mathcal{R}}=cat(\mathbb{L}(G))$.