APPROXIMATION ON NASH SETS WITH MONOMIAL SINGULARITIES

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Introduction and definitions

This work is devoted to the approximation of differentiable semialgebraic functions by Nash functions. This problem is well-known in case that the functions are defined on an affine Nash manifold M, and here we extend it to functions defined on Nash sets with a special kind of singularities.

Definition. Let M be an affine Nash manifold of $\mathbb{R}^n$. Let $X \subset M$ be a Nash subset and let $x \in X$. The germ X_x is a *monomial singularity* if there is a neighborhood U of x in M equipped with a Nash diffeomorphism $u : U \to \mathbb{R}^m$ with $u(x) = 0$ that maps $X \cap U$ onto a union of coordinate linear varieties. That is, there is a (finite) family Λ of subsets of indices $\lambda = \{\ell_1, \cdots, \ell_r\}$ of possibly different cardinality $r \leq m$ such that

$$X \cap U = \bigcup_{\lambda \in \Lambda} \{u_\lambda = 0\}$$

where $u = (u_1, \ldots, u_m)$ and $\{u_\lambda = 0\}$ denotes $\{u_{\ell_1} = \cdots = u_{\ell_r} = 0\}$. For simplicity we assume that there are no immersed components, that is, if $\lambda, \lambda' \in \Lambda$ are different then $\lambda \nsubseteq \lambda'$ and $\lambda' \nsubseteq \lambda$. This assures that the germs $\{u_\lambda = 0\}_x$, $\lambda \in \Lambda$, are the irreducible components of the germ X_x. We say that X has a *monomial singularity of type Λ at x*. A Nash set $X \subset M$ has *monomial singularities in M* if all germs X_x, $x \in X$, are monomial singularities.

Let us do some remarks concerning the notion of type:

- A different Nash diffeomorphism u' may provide a different type Λ' and therefore a monomial singularity has several types. Thus, two types Λ and Λ' will be called *equivalent* if the union of the coordinate linear varieties given by Λ is Nash diffeomorphic to the one given by Λ' as germs in the origin (and as we see below, it is possible to show that this equivalence is in fact global via linear isomorphisms).
- For example, if we have a union of hyperplanes $X \cap U = \{u_{\ell_1} = 0\} \cup \cdots \cup \{u_{\ell_r} = 0\}$, $1 \leq \ell_1, \ldots, \ell_r \leq m$, then it is classically called a *Nash normal crossings*. Clearly we can assume $X \cap U = \{u_1 = 0\} \cup \cdots \cup \{u_r = 0\}$ after the obvious linear change of coordinates. That is, in the context of Nash normal crossings, the number of hyperplanes determines the type up to linear isomorphism. As we will see now, for monomial singularities the characterization of the type is far more involved.

So how can we determine the equivalence class of a type?

We can do it even arithmetically. Let $\mathcal{L} = \{L_1, \ldots, L_s\}$ be a family of coordinate linear varieties of $\mathbb{R}^m$ without immersions. For each subset $I \subset \{1, \ldots, s\}$ and each $1 \leq p \leq s$ we denote $L_I = \bigcap_{j \in I} L_j$. Next, to each family of different nonempty subsets $I_1, \ldots, I_r \subset \{1, \ldots, s\}$, $r \geq 1$, we associate the number $\dim(L_{I_1} + \cdots + L_{I_r})$. The collection of all the previous dimensions will be called the *load* of $\mathcal{L}$.

Proposition. Let $\mathcal{L} = \{L_1, \ldots, L_s\}$ and $\mathcal{L}' = \{L'_1, \ldots, L'_s\}$ be coordinate linear varieties of $\mathbb{R}^m$. Then, there is a linear isomorphism f of $\mathbb{R}^m$ such that $f(L_i) = L'_i$ for all i if and only if the loads of $\mathcal{L}$ and $\mathcal{L}'$ coincide.

Properties of Nash sets with monomial singularities

From the definition it is not clear a priori if the set of points of a Nash set where the germ is a monomial singularity is a semialgebraic set. The answer is positive:

Proposition. Let $X \subset M$ be a Nash set. Fix a type Λ and put

$$T^{(\Lambda)} = \{x \in X : X \text{ has a monomial singularity of type } \Lambda \text{ at } x\}.$$

Then, the set $T^{(\Lambda)}$ is semialgebraic.

The proof of the above fact uses in a crucial way Artin's approximation theorem with bounds.

Another important fact is that following finiteness property:

Theorem. Let $X \subset M$ be a Nash set with monomial singularities. Then X can be covered with finitely many open sets U of M each one equipped with a Nash diffeomorphism $u : U \to \mathbb{R}^m$ that maps $X \cap U$ onto a union of coordinate linear varieties.

To prove the above result it is important to study first Nash functions on Nash sets with monomial singularities. Recall that if X is a Nash subset of a Nash manifold M then we say that a function $f : X \to \mathbb{R}$ is a *Nash function* if there exists an open semialgebraic neighborhood U of X and a Nash extension $F : U \to \mathbb{R}$ of f. We denote by $\mathcal{N}(X)$ the ring of Nash functions of X. We say that $f : X \to \mathbb{R}$ is c*-Nash* if its restriction to each irreducible component is a Nash function. We denote by ${}^c\mathcal{N}(X)$ the ring of ${}^c\mathcal{N}$ functions. Similarly, we define the ring $\mathcal{S}^\nu(X)$ of $\mathcal{C}^\nu$ semialgebraic functions on X and the ring ${}^c\mathcal{S}^\nu(X)$ of ${}^c\mathcal{S}^\nu$ semialgebraic functions on X for $\nu \geq 1$. By means of analytic coherence of Nash sets with monomial singularities we have the following *weak normality property* :

Theorem. If X is a Nash set with monomial singularities then $\mathcal{N}(X) = {}^c\mathcal{N}(X)$.

Recall that...

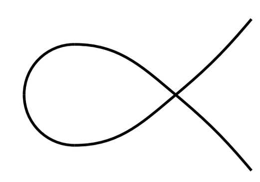
Semialgebraic = Boolean combination of sets defined by polynomial equations and inequalities.
Nash (affine) manifold = smooth submanifold of $\mathbb{R}^n$ + semialgebraic.
Nash map $f : M \to \mathbb{R}$ = smooth +semialgebraic (which implies analytic).
Nash set = zero set of a Nash map.
Smooth points of a semialgebraic set Z = points $x \in Z$ where the germ Z_x equals the germ of a Nash manifold. The set of smooth points is an open dense subset of Z. The points that are not smooth are called singular.

Approximation

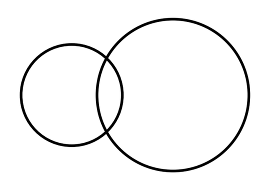
The ring $\mathcal{S}^\nu(M)$ of $\mathcal{C}^\nu$ semialgebraic functions is equipped with an $\mathcal{S}^\nu$ Whitney topology via $\mathcal{S}^\nu$ tangent fields. The fact that we have $\mathcal{S}^\nu$ bump functions as well as finite $\mathcal{S}^\nu$ partitions of unity makes a crucial difference between $\mathcal{S}^\nu$ and Nash functions and the existence of these glueing functions justify our interest in approximation. In particular, given a Nash subset X of M we can extend any $\mathcal{S}^\nu$ function on X to M and therefore $\mathcal{S}^\nu(X)$ carries the quotient topology making the restriction map $\rho : \mathcal{S}^\nu(M) \to \mathcal{S}^\nu(X)$ a quotient map. If Y is another Nash subset of $\mathbb{R}^b$ then a map $f = (f_1, \ldots, f_b) : X \to Y \subset \mathbb{R}^b$ is $\mathcal{S}^\nu$ if each component f_i is $\mathcal{S}^\nu$ and $\mathcal{S}^\nu(X, Y)$ inherits the topology from the product $\mathcal{S}^\nu(X, \mathbb{R}^b) = \mathcal{S}^\nu(X) \times \cdots \times \mathcal{S}^\nu(X)$. As we already mention if X and Y are Nash manifolds then Shiota proved in the 80's that any map in $\mathcal{S}^\nu(X, Y)$ can be also approximated by Nash maps in $\mathcal{N}(X, Y)$. Note that in this case the problem reduces to prove approximation for $\mathcal{S}^\nu(X)$ because any Nash manifold has a Nash tubular neighborhood.

> *What can we say about approximation when X and Y are Nash sets with monomial singularities?*

Since Y is not a Nash manifold and we can not use tubular neighborhoods, the situation gets more complicated. Thus we need to impose a condition in the codomain. Specifically, we say that a Nash set with monomial singularities Y is a *Nash monomial crossings* if in addition the irreducible components of Y are Nash manifolds.



Nash set with monomial singularities
but not Nash monomial crossings



Nash monomial crossings

Theorem. Let $X \subset M$ be a Nash set with monomial singularities and let $Y \subset N$ be a Nash monomial crossings. Let $m = \dim(M)$, $n = \dim(N)$ and $q = m\left(\binom{n}{[n/2]} - 1\right)$ where $[n/2]$ denotes the integer part of $n/2$. If $\nu \geq q$ then every $\mathcal{S}^\nu$ map $f : X \to Y$ that preserves irreducible components can be $\mathcal{S}^{\nu-q}$ approximated by Nash maps $g : X \to Y$.

Application: Nash manifolds with corners

We apply the approximation result above to compare $\mathcal{S}^\nu$ and Nash classifications of affine Nash manifolds with corners. This somehow complements Shiota's results on $\mathcal{C}^\nu$ classification of Nash manifolds [Sh, VI.2.2].

An *(affine) Nash manifold with corners* is a semialgebraic set $Q \subset \mathbb{R}^a$ that is a smooth submanifold with corners of (an open subset of) $\mathbb{R}^a$. In [FGR] it is proved that any Nash manifold with corners $Q \subset \mathbb{R}^a$ is a closed semialgebraic subset of a Nash manifold $M \subset \mathbb{R}^a$ of the same dimension and the Nash closure X in M of the boundary ∂Q is a Nash normal crossings. Note that we can define naturally $\mathcal{S}^\nu$ functions and maps and their topologies via the closed inclusion of Q in M; of course this does not depend on the affine Nash manifold M. A Nash manifold with corners Q has *divisorial corners* if it is contained in a Nash manifold M as before such that the Nash closure X of ∂Q in M is a normal crossing divisor (i.e. the irreducible components are Nash manifolds). As one can expect this is not always the case and a careful study can be found in [FGR, 1.12]. For example, the intersection of the two circles above is a Nash manifold with divisorial corners. However, the oval described by the cubic curve on the left is a Nash manifold with corners but without divisorial corners.

Theorem. Let Q_1 and Q_2 be two m-dimensional affine Nash manifolds with divisorial corners. If Q_1 and Q_2 are $\mathcal{S}^\nu$ diffeomorphic for some $\nu > m^2$ then they are Nash diffeomorphic.

References

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