EXPLICIT DESCRIPTIONS OF CONNECTED COMPONENTS OF THE MODULI SPACE OF SURFACES WITH $p_g = 0$: BURNIAT 'S AND KEUM-NAIE SURFACES

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Abstract. This is a joint work with Ingrid Bauer. The classification and the moduli space of surfaces with with $p_g = 0$ were on the one side motivated in the 70's by the problem of pluricanonical maps of surfaces of general type, which was finally solved by Bombieri, with final touches by Miyaoka, Benveniste and the present author.

The classical Godeaux and Campedelli surfaces, and the more recent Burniat surfaces were then rediscovered. The work by Miyaoka and Reid motivated several interesting and important researches in the field. The discovery of new surfaces with $p_g = 0$ went on in the 80's, motivated on the one side by the Bloch conjecture, and on the other side by differential topology: for instance a big achievement was the result that the Barlow surface is homeomorphic but not diffeomorphic to the rational surface X obtained by blowing up the plane in 8 points. This in turn had an important byproduct concerning the disproof of the Besse conjecture: together with Claude Le Brun, we showed that the self product of X with itself admits Kaehler Einstein metrics.

In the last five years there has been an explosion of new results, constructions and partial classification of surfaces with $p_g = 0$. Due to the celebrated Bogomolov–Miyaoka–Yau inequality, the minimal models of these surfaces have K^2 at most 9. Due to Yau's uniformization theorem, the case where $K^2 = 9$ corresponds to ball quotients and, using a result of Klingler stating the arithmeticity of the fundamental groups, a complete classification seems to have been found by Prasad and Yeung, with the help of Steger, Cartwright and a lot of computer calculations.

While Park and Lee have been working on the construction of surfaces with $p_g = 0$ which are simply connected, in the last three or four years I have been working with Ingrid Bauer and Fritz Grunewald (later also with Roberto Pignatelli) in order to construct new such surfaces, classifying all such surfaces which can be constructed as the quotient of a product of curves by the action of a finite group. This project also needed computer assistance.

We succeed in describing the fundamental groups of such surfaces, and in this way we could construct more than 40 new connected components of the moduli space of surfaces with $p_g = 0$, distinguished by the fundamental group. Looking at the case where the fundamental groups were not new has motivated an investigation of the moduli spaces of certain surfaces with $p_g = 0$. This is joint work with Ingrid Bauer, and I am going to report on this.

We solve the moduli problem for the primary Keum-Naie surfaces and the primary Burniat surfaces, showing that we get in both cases an irreducible connected component of the moduli space to which any surface with the same homotopy type belongs. In the case of primary Burniat surfaces the Inoue description plays a crucial role. More delicate and interesting is the case of secondary Burniat surfaces, having $K^2 = 5, 4$. We show that they form three connected components of the moduli space.

In the case of $K^2 = 4$, there are two components, the nodal ones yield a Gieseker moduli space (i.e., moduli space of the canonical models) which is everywhere non reduced (a fortiori, the order of nilpotency for the moduli space of canonical models is even higher).

While the Burniat surface with $K^2 = 2$ was already recognized as being a very special Campedelli surface, we find that tertiary Burniat surfaces, i.e. those with $K^2 = 3$, do not form an irreducible component of the moduli space. We are able to describe explicitly the other surfaces which appear in this component. This work is still in progress and we have not yet decided whether this is also a connected component of the moduli space.