SYZYGIES OF PROJECTIVE SURFACES: AN OVERVIEW

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Abstract. In this survey article we review several known results about syzygies for projective surfaces and present some new ones. We also give an outline of some of the ideas and techniques to handle these types of problems.

0. Introduction

The study of how a variety can map in projective space is central to Algebraic Geometry. An algebraic variety X is projective if it possesses an ample line bundle A, that is, a line bundle A such that $A^{\otimes n}$ is a very ample line bundle. A very ample line bundle embeds X in projective space. A projective variety has different line bundles, therefore it has many different ways to map and to embed in projective space. As a first step, given a line bundle L on X one would like to know numerical invariants of L such as the dimension of its space of global sections. One would also like to know under what conditions L is globally generated (i.e., when L induces a morphism from X to projective space) and, furthermore, under what conditions L is very ample.

Once we know that L is very ample, an interesting thing to do is to study the equations of the image of X in the projective space. One would like to find conditions on L so that the homogeneous coordinate ring R and the homogeneous ideal I of X have the "simplest" possible structure. In this sense the first property one can ask for is the normality of R.

If R is projectively normal, one can further ask for conditions under which I is as simple as possible in terms of its generators. However, to write down precise equations of a given embedding is in general very hard. A more tractable problem would be to determine the degrees of the generators for I. These questions have attracted a lot of attention in recent years, although for a smooth algebraic curve C of genus g the study of these problems can be traced back to the nineteenth century. For curves, simple conditions in terms of the degree of the line bundle L can be given so that R is normal and I is generated by equations of the smallest possible degree, which is 2. Indeed, as a corollary of the Theorem of Riemann–Roch, if the degree of L is greater than or equal to $2g+1$, then L is very ample. On the other hand, the Italian mathematician Guido Castelnuovo proved, under the same hypothesis, that the coordinate ring R of the image of C is normal and that if the degree of L is greater than or equal to $2g+2$, then I is generated by quadrics. Other important classical results in this direction are the famous theorems of Noether and Enriques–Petri (cf. [ACGH]) for a canonical curve, that is, for a curve embedded by its canonical line bundle. These results tell precisely when a canonical curve is projectively normal and its ideal generated by forms of degree two.

Castelnuovo's result was rediscovered by several people: Mumford, Mattuck, Fujita, St. Donat, among others. Recently Mark Green and Robert Lazarsfeld brought a new perspective into this problem by looking at the minimal free resolution of I, relating the study of the resolution, via Koszul cohomology, with the study of the cohomology groups of certain vector bundles (cf. [G], [L]). Besides proving several interesting theorems relating the geometry of an embedding of an algebraic curve and its syzygies, they have also posed tantalizing conjectures for curves.

Let L be a very ample line bundle on a variety X and let

$$0 \longrightarrow F_n \xrightarrow{\varphi_n} \cdots \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow R \longrightarrow 0$$

be the minimal graded free resolution of the coordinate ring R of the image of X by the embedding induced by L. The property N_p is defined as follows:

– L satisfies the property N_0 if R is normal.

– L satisfies the property N_1 if in addition I is generated by quadrics, that is, if the entries of the matrix of φ_1 have degree 2.

– L satisfies the property N_p if in addition to satisfying property N_1, the resolution is linear from the second step until the p-th step, i.e., if the matrices of $\varphi_2, \ldots, \varphi_p$ have linear entries.

Thus property N_p does not only mean that the coordinate ring R is normal, and that the homogeneous ideal I is generated in the lowest possible degree (that is, degree 2, since the variety is not contained in a hyperplane). It says in addition that the module M_1 of relations or *syzygies* among the generators of I is generated in degree 1; that the next module of syzygies (the relations among the generators of M_1) is also generated in degree 1 and so on, until arriving at the syzygy module M_{p-1}, which is also generated in degree 1.

We illustrate these notions by some examples. The simplest variety to study is $\mathbf{P}^1$. The complete linear series of the line bundle $\mathcal{O}_{\mathbf{P}^1}(n)$ embedds $\mathbf{P}^1$ as a rational normal curve of degree n. There is not much to say for the case $\mathcal{O}_{\mathbf{P}^1}(1)$: it satisfies property N_0. The line bundle $\mathcal{O}_{\mathbf{P}^1}(2)$ embedds $\mathbf{P}^1$ in $\mathbf{P}^2$ as a smooth conic. Its coordinate ring is normal and its ideal is generated by a form of degree 2, hence according to the above definition $\mathcal{O}_{\mathbf{P}^1}(2)$ satisfies property N_1. If the conic is for example $x^2+y^2+z^2=0$, we will write the resolution of the cooriditate ring R as:

$$0 \longrightarrow S(-2) \xrightarrow{x^2+y^2+z^2} S \longrightarrow R \longrightarrow 0\ .$$

By $S(-2)$ we mean the graded module S shifted by -2, that is $S(-2)$ is generated in degree 2. We also observe that the map between $S(-2)$ and S is homogeneous of degree 0. We will describe the maps between the free modules F_i and F_{i-1} with a matrix where the rows are the images of the generators of F_i, so that in the matrix we can read off the minimal generators of the $(i-1)$-th syzygy module of R.

The next case we look at is $\mathcal{O}_{\mathbf{P}^1}(3)$. The line bundle $\mathcal{O}_{\mathbf{P}^1}(3)$ embedds $\mathbf{P}^1$ as a rational normal curve of degree 3 in $\mathbf{P}^3$. A rational normal curve of degree 3 is cut out by 3 quadrics. Take for example $I=(y^2-xz, yz-xt, z^2-yt)$, the ideal of the twisted cubic. This is the resolution of $R=S/I$:

$$0 \longrightarrow S^2(-3) \xrightarrow{\begin{pmatrix} t & -z & y \\ z & -y & x \end{pmatrix}} S^3(-2) \xrightarrow{\begin{pmatrix} y^2-xz \\ yz-xt \\ z^2-yt \end{pmatrix}} S \longrightarrow R \longrightarrow 0.$$

Therefore the ideal I is generated by quadratic equations and the resolution is linear in the next step. Hence $\mathcal{O}_{\mathbf{P}^1}(3)$ satisfies property N_2. In fact the resolution does not have any more stages. The same happens for a rational normal curve of degree n: its ideal is generated by quadrics and the resolution is linear from that point to the end. Thus $\mathcal{O}_{\mathbf{P}^1}(n)$ satisfies property N_{n-1} (we might even say it satisfies N_∞, since there is no nonlinear step in the resolution, the resolution ending after the $(n-1)$-th step).

To see a situation in which a line bundle satisfies N_p but fails to satisfy N_{p+1} we should look at higher genus curves. Let C be an elliptic curve. On an elliptic curve a degree 1 line bundle is effective but has a base-point, and a line bundle of degree 2 induces a double cover of $\mathbf{P}^1$, hence it is not very ample nor it satisfies property N_0. If the degree of L is greater than or equal to 3, L induces an embedding. If L has degree 3, L embedds C in $\mathbf{P}^2$ as a smooth cubic curve C'. Then its homogeneous coordinate ring is normal, for it is the ring of a smooth hypersurface, but the ideal of C' is generated by a cubic form. Therefore L satisfies N_0 but not N_1.

If L has degree 4, L embedds C as an elliptic normal quartic in $\mathbf{P}^3$, the complete intersection of two quadrics F, G. The resolution of its coordinate ring R is then the Koszul complex:

$$0 \longrightarrow S(-4) \xrightarrow{(G \quad -F)} S^2(-2) \xrightarrow{\begin{pmatrix} F \\ G \end{pmatrix}} S \longrightarrow R \longrightarrow 0.$$

Thus L satisfies N_1 but not N_2 because the entries $G, -F$ of the matrix in the second step are not linear entries.

If L has degree 5, the resolution of the ring R of the image of C by the embedding induced by L is this (we only write the Betti numbers):

$$0 \longrightarrow S(-5) \longrightarrow S^5(-3) \longrightarrow S^5(-2) \longrightarrow S \longrightarrow R \longrightarrow 0.$$

Since the maps in the resolution are homegeneous of degree 0 we can read off what maps are linear, from the twists of each free module in the resolution. Then if L has degree 5, L satisfies N_2 but not N_3. This pattern continues as the degree of L increases and in fact a degree $p+3$ line bundle L on an elliptic curve satisfies property N_p but not property N_{p+1}.

We go back now to the study in more generality of the property N_p for curves. Generalizing the result of Castelnuovo, Green proved that if L is a line bundle of degree greater than or equal to $2g+p+1$ on a smooth curve C of genus g, then L satisfies the property N_p. Note that the previous examples, rational and elliptic curves, are particular cases of this.

The theorems of Castelnuovo, Green and the theorems of Noether, Enriques, Petri and the famous Green's conjecture for canonical curves unveil a deep relation between the topological and analytical invariants (genus, degree of the line bundle, Clifford index) and the extrinsic algebraic and geometric properties of a curve. Green's conjecture, a central open question in the theory of curves, tell us that the Clifford index of a curve – a purely geometric quantity – can be read off from the structure of the free resolution of the canonical ring.

Having reviewed some of the classical and modern results known for curves, a natural thing to do is to look for higher dimensional analogues of these results. The corollary of Riemann–Roch regarding very ampleness and the theorem of Castelnuovo and Green can be restated as follows: Let A be an ample line bundle on C, let K_C be the canonical bundle of C and let $L = K_C \otimes A^{\otimes n}$. If $n \geq 3$, then L is very ample. If $n \geq p+3$, then L satisfies property N_p.

The above reformulation suggests a direction to generalize the results of Castelnuovo, Green and others, to higher dimensions. For higher dimensional varieties Fujita and Mukai have made two conjectures which have attracted wide attention during the last few years. Let X be a variety of dimension d and let L be a line bundle of the form $K_X \otimes A^{\otimes n}$, where K_X is the canonical bundle of X and A is an ample line bundle on X. Then:

Conjecture (Fujita). *If $n \geq d+1$, L is globally generated (i.e., L induces a morphism to projective space); and if $n \geq d+2$, L is very ample.*

For an algebraic surface, we have the following:

Conjecture (Mukai). *If $n \geq p+4$, L satisfies property N_p.*

Fujita's conjecture holds in dimension 1 (as already said, it is an easy consequence of Riemann–Roch) and in dimension 2 (Reider, 1988, [R]). The part referring to global generation was proved in dimension 3 (Ein and Lazarsfeld, 1993, [EL2]; see also [F] and [Ka] for a finer version of the conjecture) and in dimension 4 (Kawamata, 1997, [Ka]) and is open in dimension greater than 4. Concerning very ampleness, Fujita's conjecture is open in dimension greater than or equal to 3.

Much less is known regarding Mukai's conjecture. It is true in dimension 1 (Green's theorem), but it is not known in general for higher dimension. Not even the easiest case is settled in general: dimension 2 and $p = 0$. For surfaces and higher dimensional varieties some partial results were known: a result of Butler (cf. [B]) for ruled varieties; Mukai's conjecture for elliptic ruled surfaces for $p = 0$, which was proved by Y. Homma (in fact she completely characterized those line bundles on an elliptic ruled surface satisfying property N_0); a result of Kempf for Abelian varieties (implying in particular Mukai's conjecture for $p = 0, 1$ for Abelian surfaces; cf. [Ke]). Moreover, if one imposes the extra assumption of very ampleness on A, Ein and Lazarsfeld proved a beautiful general result (cf. [EL1]), that we state here for the case of surfaces different from $\mathbf{P}^2$:

Theorem 1.1.1. *Let X be a projective variety and let A be a very ample line bundle such that $(X, A) \neq (\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(1))$. Then $L = K_X \otimes A^{\otimes p+n}$ satisfies N_p.*

We remark that the knowledge of the N_p properties of the Veronese embedding of $\mathbf{P}^2$ is complete (see [OP]). The problem considered in 1.1.1 becomes considerably hard, even for algebraic surfaces, if we relax the condition of very ampleness on A to base-point-freeness. We have proved results along these lines for algebraic surfaces (see [GP4]).

The questions we have posed are interesting in themselves and not just for being generalizations of the results already known for curves. For a fixed projective variety X and an ample line bundle A on X, $K_X \otimes A^{\otimes n}$ is very ample for large n, by the very definition of very ampleness. Thus the problem of finding explicit bounds for n arises naturally. Of course, one would like to find bounds as sharp as possible, to reach eventually an optimal bound. Furthermore, if one considers all ample line bundles on a given variety X, or even, if one considers varieties belonging to a given class, it is natural to look for explicit uniform bounds for n. In dimension 1 the uniform and optimal bound is given by the previously mentioned corollary of Riemann–Roch. In dimension 2 the uniform and optimal bound is given by Reider's Theorem and, in general, the optimal bound when one considers any variety of a given dimension should be the one conjectured by Fujita.

It is natural and interesting to ask higher syzygy analogues of the above questions. Indeed, if A is ample, $K_X \otimes A^{\otimes n}$ satisfies property N_p when n is large enough (cf. [EL1], [I] and [N]). Thus one can ask for explicit and uniform bounds on n for the adjunction bundle $K_X \otimes A^{\otimes n}$ to satisfy property N_p. The uniform and optimal result for any variety of dimension 1 is Green's theorem. One expects the uniform and optimal bound for surfaces to be the conjectured bound of Mukai. Since this is not yet known, a less ambitious but still worthy task is to look for uniform bounds for interesting classes of surfaces.

Conventions and notation. Although some of the result we will discuss hold in greater generality for the purpose of this exposition we will work over an algebraically closed field $\mathbf{k}$ of characteristic 0 and we will restrict ourselves to smooth surfaces. Hence, when we write surface we will mean smooth surface.

1. An overview of the results on syzygies of surfaces

In this section we review some of the known results regarding the syzygies of algebraic surfaces. First we present the knowledge about Mukai's conjecture on surfaces. After that, we introduce a new philosophy concerning the kind of results one should expect to hold for surfaces, regarding the syzygies of embeddings induced by line bundles which are not necessarily adjunction bundles.

1.1. Results regarding Mukai's conjecture.

As mentioned in the introduction, Mukai's conjecture for surfaces is not known in general. There are however several classes of surfaces for which the conjecture is known to hold. For the remaining classes of surfaces explicit bounds are known on n so that the line bundle $K_X \otimes A^{\otimes n}$ satisfies property N_p, for any ample line bundle A on X. We summarize here these results.

We start with **surfaces with negative Kodaira dimension**, i.e., those surfaces which are birationally equivalent to geometrically ruled surfaces. For geometrically ruled surfaces, the following bound is known:

Theorem 1.1.2 (Butler, cf. [B]). *Let X be a geometrically ruled surface, let A be an ample line bundle on X and let $L = K_X \otimes A^{\otimes n}$. If $n \geq 5$, then L satisfies property N_0. If $n \geq 4p + 4$, $p \geq 1$, then L satisfies property N_p.*

This bound has been sharpened for elliptic ruled surfaces:

Theorem 1.1.3 (Homma, cf. [Ho1] and [Ho2]). *Let X be a surface, geometrically ruled over a smooth elliptic curve. Let A be an ample line bundle on X and let $L = K_X \otimes A^{\otimes n}$. If $n \geq 4$, then L satisfies property N_0.*

Theorem 1.1.4 (cf. [GP1], Corollary 4.6 and [GP2], Corollary 6.2). *Let X be a surface, geometrically ruled over a smooth elliptic curve. Let A be an ample line bundle on X and let $L = K_X \otimes A^{\otimes n}$. If $n \geq 2p + 3$ and $p \geq 1$, then L satisfies property N_p.*

These theorems show in particular that Mukai's conjecture for projective normality and normal presentation holds for elliptic ruled surfaces.

The previous results are for geometrically ruled surfaces. We now pay attention to anticanonical rational surfaces, which are birationally ruled, but not necessarily minimal. For these surfaces, we prove a higher syzygy analogue of Reider's theorem, which yields as an easy corollary an affirmative answer to Mukai's conjecture:

Theorem 1.1.5 (cf. [GP6]). *Let X be an anticanonical surface (i.e., a surface supporting an effective anticanonical divisor). Let A be an ample bundle and let $L = K_X \otimes A^{\otimes n}$. If $n \geq p + 4$, then L satisfies property N_p.*

We now focus on **surfaces with Kodaira dimension** 0. We have the following bound:

Theorem 1.1.6 (cf. [GP4], Theorem 0.3; [GP5]). *Let X be a minimal surface with Kodaira dimension* 0 *and let A be an ample line bundle. If $n \geq 2p+2$, then $K_X \otimes A^{\otimes n}$ satisfies property N_p.*

This result was known before for the special case of Abelian surfaces and due to Kempf (cf. [Ke]). Theorem 1.1.6 says in particular that Mukai's conjecture holds for minimal surfaces of Kodaira dimension 0 and $p = 0, 1$. This was also previously known in the special case of K3 surfaces, due to St. Donat (cf. [S-D]).

Finally we look at **surfaces with positive Kodaira dimension**. For minimal surfaces, we find explicit bounds in terms of intersection numbers of the ample line bundle A:

Theorem 1.1.7 (cf. [GP4], Corollaries 5.10 and 5.13). *Let S be a minimal surface of positive Kodaira dimension, let A be an ample line bundle and let $m = \left[\frac{(A\cdot(K_S+4A)+1)^2}{2A^2}\right]$. Let $L = K_S \otimes A^{\otimes n}$.*

1. *If $n \geq 2m$, then L satisfies property N_0. If $n \geq 3m$, then L satisfies property N_1.*
2. *If S is a regular surface of general type and $n \geq mp + m, p \geq 1$, then L satisfies property N_p.*

1.2. Looking for results for general line bundles.

The results regarding Mukai's conjecture we have just reviewed deal with line bundles of the form $K_X \otimes A^{\otimes n}$, where A is an ample line bundle on X. On curves the use of adjoint line bundles yielded an equivalent reformulation of the results of Castelnuovo, Green and others, which were first stated in terms of the degree of the line bundle inducing the embedding. The point was that on a curve any line bundle can be written as a power of an (ample) degree 1 line bundle. Therefore a line bundle of high degree can be written as $K_X \otimes A^{\otimes n}$, where n is also sufficiently high. For surfaces, this is no longer true. Take for instance the case of elliptic ruled surfaces, and to be even more concrete, consider the elliptic ruled surface of invariant $e = -1$ (cf. [H], V.2). Mukai's conjecture is true for $p = 0, 1$ (see 1.1.3 and 1.1.4) and is sharp (see [Ho1], [Ho2] and [GP1]). On the other hand, Homma characterized numerically (cf. [Ho1] and [Ho2]) those line bundles which satisfy property N_0. Likewise in [GP1] we characterized numerically those line bundles which satisfy property N_1. One can then easily see that there exist line bundles on X which cannot be written as $K_X \otimes A^{\otimes n}$ for $n \geq 4$ (resp. $n \geq 5$) and A ample line bundle, but satisfy property N_0 (resp. N_1) notwithstanding. Thus, a statement such as Mukai's cannot account for all line bundles on X satisfying property N_p.

Therefore the goal is to find other ways to state syzygy results for line bundles on surfaces. We observe that almost all the results we have mentioned fit into the following meta-principle:

1.2.1. *Let X be a surface. If L is the tensor product of $(p+1)$ ample and base-point-free line bundles on X (possibly different line bundles) satisfying "certain" cohomological conditions, then L satisfies the condition N_p.*

Indeed, if we review the results we have been talking about, we will see they fit into 1.2.1. Consider for example the result by Ein and Lazarsfeld previously stated as Theorem 1.1.1. If we specialize it to the case of a surface X, the theorem says that if A is very ample, $K_X \otimes A^{\otimes p+3}$ satisfies property N_p. On the other hand, $K_X \otimes A^{\otimes 3}$ is very ample, in particular base-point-free, by Reider's theorem, and so is A, so one can write $K_X \otimes A^{\otimes p+3}$ as tensor product of $p+1$ base-point-free line bundles on X. We take another example, namely Theorem 1.1.6. Observe that if X is a minimal surface of Kodaira dimension 0, it follows also from Reider's theorem that $A^{\otimes 2}$ is base-point-free for any ample line bundle A on X. One can similarly see that the same is true for the results on elliptic ruled surfaces, anticanonical rational surfaces, etc.

This suggests a way of stating results for line bundles which are not necessarily adjoint bundles. In the light of this philosophy we present now the more general results obtained for the different classes of surfaces:

For **surfaces with geometric genus** 0 (including ruled surfaces, Enriques surfaces, etc.) we showed the following:

Theorem 1.2.2 (cf. [GP2], Theorem 2.2). *Let X be a surface with geometric genus $p_g = 0$. Let B be a nonspecial, ample, and base-point-free line bundle. Then $B^{\otimes p+1}$ satisfies the property N_p, for all $p \geq 1$.*

In fact a more general version of this theorem holds (see [GP2], Lemma 2.8) in which different base-point-free line bundles are involved. The statement is somewhat more technical, and we will not display it here. We will state instead the following sequel of it for elliptic ruled surfaces:

Theorem 1.2.3 (cf. [GP1], Theorem 4.2 and [GP2], Theorem 6.1). *Let X be a surface geometrically ruled over a smooth elliptic curve and let $p \geq 1$. Let a, b be integers and let L be a line bundle in the numerical class of $aC_0 + bf$.*

1. *If $e = e(X) = -1$ and $a \geq p+1$, $a+b \geq 2p+2$ and $a+2b \geq 2p+2$, then L satisfies the property N_p.*
2. *If $e = e(X) \geq 0$ and $a \geq p+1$, $b - ae \geq 2p+2$, then L satisfies the property N_p.*

Moreover, if $p = 1$, the above sufficient conditions are also necessary for L to satisfy property N_p.

The statement of the previous theorem shows the fact that we are considering now line bundles of arbitrary "shape", and not just line bundles of the form $K_X \otimes A^{\otimes n}$, where A is ample and n is suitably large.

We make the following conjecture (which would imply Mukai's conjecture on elliptic surfaces) on the characterization of the N_p line bundles on an elliptic surface:

Conjecture 1.2.4 ([GP2], 7.3). *Let X be an elliptic ruled surface and let L be a line bundle on X in the numerical class $aC_0 + bf$.*

If $e(X) = -1$, L satisfies the property N_p iff $a \geq 1$, $a+b \geq p+3$, and $a + 2b \geq p+3$.

If $e(X) \geq 0$, L satisfies the property N_p iff $a \geq 1$ and $b - ae \geq p+3$.

We look now at **surfaces with Kodaira dimension** 0. In [GP4] and [GP5] we proved the following theorem which fits also in the philosophy of 1.2.1:

Theorem 1.2.5 (cf. [GP4], Theorem 0.2). *Let X be a minimal surface with Kodaira dimension 0 and let $B_1, \ldots, B_n$ be numerically equivalent, ample and base-point-free line bundles. Assume that the sectional genus of B_i is greater than or equal to 4 if X is an Enriques, Abelian or bielliptic surface and greater than or equal to 3 if X is a K3 surface. Then $B_1 \otimes \cdots \otimes B_n$ satisfies N_p for all $n \geq p+1$ and $p \geq 1$.*

Note again that this theorem addresses line bundles more general than those adjunction bundles related to Mukai's conjecture. For instance, Theorem 1.2.5 implies that on an Abelian surface a line bundle of type $(p+1, 3p+3)$ satisfies property N_p. This fact does not follow from results such as Kempf's.

As in section 1.1, we end our tour by looking at **surfaces with positive Kodaira dimension**:

Theorem 1.2.6 (cf. [GP4], Theorem 5.1). *Let S be a regular surface of positive Kodaira dimension and geometric genus $p_g \geq 4$. Let B be an ample and base-point-free line bundle such that $H^1(B) = 0$.*

(1) *If $\kappa(S) = 1$ and $B^2 > K_S \cdot B$, then $K_S \otimes B^{\otimes n}$ satisfies property N_1, for all $n \geq 2$.*
(2) *If $\kappa(S) = 2$ and $B^2 \geq K_S \cdot B$, then $K_S \otimes B^{\otimes n}$ satisfies property N_0, for all $n \geq 2$.*
(3) *If $\kappa(S) = 2$ and $B^2 \geq 2K_S \cdot B$, then $K_S \otimes B^{\otimes n}$ and $B^{\otimes m}$ satisfy property N_1, for all $n, m \geq 2$.*
(4) *If $\kappa(S) = 2$ and $B^2 \geq 2K_S \cdot B$, then $B^{\otimes m}$ satisfies property N_p, for all $m \geq p + 1$, $p \geq 1$.*

Theorem 1.2.7 (cf. [GP4], Theorem 5.8). *Let S be an irregular surface of positive Kodaira dimension. Let B be an ample line bundle such that $B^2 \geq 5$ and B' is base-point-free and $H^1(B') = 0$ for all B' homologous to B (respectively numerically equivalent). Let L homologous to $K \otimes B^{\otimes n}$ (respectively numerically equivalent).*

(1) *If $\kappa(S) = 1$ and $B^2 > K_S \cdot B$, then L satisfies property N_0 if $n \geq 2$, and L satisfies property N_1 if $n \geq 3$.*
(2) *If $\kappa(S) = 2$ and $B^2 \geq 2K_S \cdot B$, then L satisfies property N_0 if $n \geq 2$;*
(3) *if $\kappa(S) = 2$ and $B^2 \geq K_S \cdot B$, then L satisfies property N_1 if $n \geq 3$.*

As a consequence of Theorems 1.2.6 and 1.2.7 we obtain the following result on pluricanonical embeddings of surfaces of general type:

Theorem 1.2.8 (cf. [GP4], Corollaries 5.9, 5.11, Theorems 5.12, 5.16). *Let S be a surface of general type with ample canonical bundle.*

a) Assume that either
  1. $K_S^2 \geq 5$ or
  2. $K_S^2 \geq 2$ and $p_g \geq 1$, but it does not happen that $q = p_g = 1$ and $K_S^2 = 3$ or 4.

  If $n \geq 2p + 4$, then $K_S^{\otimes n}$ satisfies property N_p.

b) Assume that S is irregular and $K_S^2 \geq 5$. If $n \geq 5$, then $K_S^{\otimes n}$ satisfies property N_0.

c) Assume that S is regular and $p_g \geq 3$. If $n \geq 2p + 2$ and $p \geq 1$, then $K_S^{\otimes n}$ satisfies property N_p.

d) Assume that S is regular, K_S is base-point-free and $p_g \geq 4$. If $n \geq p + 2$, $p \geq 1$, then $K_S^{\otimes n}$ satisfies property N_p.

1.3. Reider-type results.

We consider now a different approach to this quest for results for general line bundles: to prove results in terms of the intersection number of L with the irreducible curves on the surface X, in the flavor of Reider's theorem for base-point-freeness and very ampleness. One of the versions of Reider's theorem (which, as noted in the introduction, implies Fujita's conjecture for base-point-freeness and very ampleness on surfaces) is the following:

Theorem 1.3.1 (cf. [R], Theorem 1). *Let S a surface.*

1. *If $L^2 \geq 5$ and $L \cdot C \geq 2$ for every irreducible curve C on S, then $K_S \otimes L$ is base-point-free.*
2. *If $L^2 \geq 10$ and $L \cdot C \geq 3$ for every irreducible curve C on S, then $K_S \otimes L$ is very ample.*

For anticanonical surfaces and property N_p we obtain this precise analogue of Reider's Theorem for very ampleness:

Theorem 1.3.2 (cf. [GP6]). *Let X be an anticanonical surface, let L be an ample line bundle. If $L^2 \geq (p+3)^2+1$ and $L \cdot C \geq p+3$ for any irreducible curve C, then $K_X \otimes L$ satisfies N_p.*

This is the first result of its kind (for higher syzygies) for any surface. It is a very interesting question to ask in which generality this kind of result is true. In this sense we make the following conjecture:

Conjecture 1.3.3 (cf. [GP6]). *Let X be a regular surface and L an ample line bundle on X. If $L^2 \geq (p+3)^2+1$ and $L \cdot C \geq p+3$ for any irreducible curve C, then $K_X \otimes L$ satisfies N_p.*

As noted in Section 1.1, Theorem 1.3.2 yields as a corollary Mukai's conjecture for anticanonical surfaces and 1.3.3 would yield Mukai's conjecture for regular surfaces.

2. Free resolutions and Koszul cohomology

In the next two sections we introduce some of the techniques and ideas used to approach the questions and to prove the results presented in Sections 0 and 1.

In this section we explain why the vanishing of cohomology groups of certain vector bundles on X imply results about the linearity of the free resolution of a given embedding of X.

The general set-up is this: Let X be a variety embedded in $\mathbf{P}^r$ by the complete series of a line bundle L (in particular, the image of X is not contained in a hyperplane). We would like to relate the vanishings of graded Betti numbers of the minimal free resolution of the homogeneous coordinate ring of X with the vanishing of cohomology groups of certain vector bundles on X. We assume that L satisfies property N_0 (later we will also formulate this property in cohomology terms, as the other properties N_p). The fact that L satisfies N_0 means that $R(L) = \bigoplus_{m=0}^{\infty} \mathrm{H}^0(L^{\otimes m})$ is the homogeneous coordinate ring of X. We will denote this ring by R and we will denote the homogeneous coordinate ring $\mathbf{k}[x_0, \dots, x_r]$ of $\mathbf{P}^r$ by S. Let

$$0 \to E_{r-1} \to E_{r-2} \to \cdots \to E_1 \to E_0 = S \to R \to 0 \qquad (2.1)$$

be the minimal free resolution of R over S.

The minimality of (2.1) means that the entries of the matrices corresponding to each homomorphism in (2.1) belong to the irrelevant ideal. Thus each $E_i = S^{a_{i,i+1}}(-i-1) \oplus S^{a_{i,i+2}}(-i-2) \oplus \cdots \oplus S^{a_{i,i+l_i}}(-i-l_i)$. Recall that the property N_p means that the resolution is linear until the pth stage, i.e., that $E_i = S^{a_{i,i+1}}(-i-1)$ for some $a_{i,i+1} \geq 1$ and all $1 \leq i \leq p$.

Therefore L satisfies the property N_p iff $a_{i,i+k} = 0$ for all $1 \leq i \leq p$ and all $k \geq 2$. We want to relate the vanishings of the graded Betti numbers to the vanishings of certain cohomology groups. First note that $a_{i,i+k} = \dim \operatorname{Tor}_i^S(R, \mathbf{k})_{i+k}$. Indeed, since (2.1) is a minimal resolution, if we tensor it by $\mathbf{k}$ we obtain a complex all whose boundary maps are zero. Hence $\operatorname{Tor}_i^S(R, \mathbf{k})$ is nothing but the graded $\mathbf{k}$-algebra $E_i \otimes \mathbf{k}$ and $\operatorname{Tor}_i^S(R, \mathbf{k})_{i+k} = \mathbf{k}^{a_{i,i+k}}$, so the claim is clear.

We have just computed $\operatorname{Tor}_i^S(R, \mathbf{k})_{i+k}$ using a free resolution of R. Now we will use a free resolution of $\mathbf{k}$, namely, the Koszul resolution:

$$0 \to \bigwedge^{r+1} V \otimes S(-r-1) \to \cdots \to \bigwedge^{2} V \otimes S(-2) \to V \otimes S(-1) \to S \to \mathbf{k} \to 0 \quad (2.2)$$

where $V = \mathrm{H}^0(X, L)$ and $\mathbf{P}^r = \mathbf{P}(V)$. If we tensor (2.2) by the coordinate ring $R = \bigoplus_{m=0}^{\infty} \mathrm{H}^0(L^{\otimes m})$, we obtain that $\operatorname{Tor}_i^S(R, \mathbf{k})_{i+k}$ is equal to the homology at the middle of the sequence

$$\bigwedge^{i+1} \mathrm{H}^0(L) \otimes \mathrm{H}^0(L^{\otimes k-1}) \to \bigwedge^{i} \mathrm{H}^0(L) \otimes \mathrm{H}^0(L^{\otimes k}) \to \bigwedge^{i-1} \mathrm{H}^0(L) \otimes \mathrm{H}^0(L^{\otimes k+1}) \,. \quad (2.3)$$

To finish our task we introduce the vector bundle M_L, which is defined by the following exact sequence:

$$0 \to M_L \to \mathrm{H}^0(L) \otimes \mathcal{O}_X \to L \to 0 \,. \qquad (2.4)$$

(The exact sequence (2.4) makes sense for any variety X and any vector bundle L as long as L is globally generated).

From (2.4) we obtain this diagram:

$$
\begin{array}{ccccc}
0 & & & & \\
\downarrow & & & & \\
\bigwedge^{i+1} M_L \otimes L^{\otimes k-1} & & & & \\
\downarrow & & & & \\
\bigwedge^{i+1} \mathrm{H}^0(L) \otimes L^{\otimes k-1} & & & & 0 \\
\downarrow & \searrow & & & \downarrow \\
\bigwedge^{i} M_L \otimes L^{\otimes k} & \rightarrow & \bigwedge^{i} \mathrm{H}^0(L) \otimes L^{\otimes k} & \rightarrow & \bigwedge^{i-1} M_L \otimes L^{\otimes k+1} \\
\downarrow & & & \searrow & \downarrow \\
0 & & & & \bigwedge^{i-1} \mathrm{H}^0(L) \otimes L^{\otimes k+1} \ .
\end{array}
$$

After taking global sections we see that the exactness of (2.3) at the middle (and thus the vanishing of $\mathrm{Tor}_i^S(R, \mathbf{k})_{i+k}$) is equivalent to the injectivity of

$$H^1(\bigwedge^{i+1} M_L \otimes L^{\otimes k-1}) \longrightarrow \bigwedge^{i+1} \mathrm{H}^0(L) \otimes H^1(L^{\otimes k-1}) \ .$$

Recall that L is assumed to satisfy property N_0. This property can also be characterized using M_L. Tensoring (2.4) with $L^{\otimes s}$ we obtain

$$0 \longrightarrow M_L \otimes L^{\otimes s} \longrightarrow H^0(L) \otimes L^{\otimes s} \longrightarrow L^{\otimes s+1} \longrightarrow 0 \qquad (2.5)$$

(in fact, this is the left hand side vertical exact sequence of the diagram above, when $i = 0$). The line bundle L satisfies property N_0 if and only if it is ample and the map

$$H^0(L) \otimes H^0(L^{\otimes s}) \longrightarrow H^0(L^{\otimes s+1})$$

surjects for all $s \geq 1$. Taking cohomology in (2.5) we realize that L satisfies property N_0 if and only if the map

$$0 \longrightarrow H^1(M_L \otimes L^{\otimes s}) \longrightarrow H^0(L) \otimes H^1(L^{\otimes s})$$

is injective for all $s \geq 1$.

Thus we can summarize all the above in the following:

Theorem 2.6 (Green, cf. [G]). *Let X be a variety and let L be an ample line bundle. The line bundle L satisfies property N_p if and only if*

$$H^1(\bigwedge^{i+1} M_L \otimes L^{\otimes s}) \longrightarrow \bigwedge^{i+1} H^0(L) \otimes H^1(L^{\otimes s})$$

is injective for all $0 \leq i \leq p$ and all $s \geq 1$.

In particular, L satisfies property N_p if $H^1(\bigwedge^{i+1} M_L \otimes L^{\otimes s}) = 0$ for all $0 \leq i \leq p$ and all $s \geq 1$.

If more information about X and L is available, the previous theorem can be strengthen. For instance, if $L^{\otimes s}$ is non special for all $s \geq 1$, then the vanishing of $H^1(\bigwedge^{i+1} M_L \otimes L^{\otimes s})$ is also a sufficient condition for L to satisfy property N_p. If we know in advance about the regularity of the resolution of R, we already know the vanishings of some of the graded Betti numbers of the resolution. For example, if the resolution is 3-regular, (which happens if $H^i(L^{\otimes 2-i}) = 0$ for all $i \geq 1$, cf. [Mu1]), to see if property N_p holds one only has to check that $a_{i,i+2} = 0$ for all $1 \leq i \leq p$, (hence it suffices to check the vanishing of $H^1(\bigwedge^{i+1} M_L \otimes L)$ for all $i \geq 0$). Another reduction that can be made is the following: if L satisfies property N_0 and p is less than or equal to the codimension of X inside $\mathbf{P}^r = \mathbf{P}(\mathrm{H}^0(L))$, then for property N_p it is only necessary to check that $a_{p,p+k} = 0$ for all $k \geq 2$, hence it would be enough to check that $H^1(\bigwedge^{i+1} M_L \otimes L^{\otimes s}) = 0$ for all $s \geq 1$. The reason for this is that, if m is the codimension of X in $\mathbf{P}^r$, we can dualize the minimal resolution of R obtaining a complex which is exact until the m-th step. This is a resolution of some module, and it is obviously again minimal. This means that in the resolution of R not only the minimum of the degrees of the generators of the modules E_i increases strictly in every step, but also the maximum of the degrees of the generators increases strictly, until the m-th step. Although it is interesting to have in mind these simplifications, which may save some work when trying to prove the property N_p, the key idea is the one stated in Theorem 2.6, which showed how to translate property N_p in terms of the cohomology of vector bundles.

3. Computing Koszul cohomology groups of surfaces

As we have just seen we can reduce the study of the free resolution of an embedded variety to the computation of the cohomology groups of certain vector bundles. Thus we are interested in finding ways to compute these cohomology groups on surfaces. Let X be a surface. We are concerned with line bundles L

fitting in 1.2.1, i.e., line bundles which are tensor product of $p+1$ ample and base-point-free line bundles on X. For most of this exposition we will assume that the ample and base-point-free line bundles are all equal to B. We want to show that L (is very ample and) satisfies property N_p. Then according to the arguments in Chapter 2, it suffices to prove the vanishings of

$$H^1(\bigwedge^i M_L \otimes L^{\otimes s})$$

for all $1 \leq i \leq p+1$ and all $s \geq 1$. As we also observed in Chapter 2, under many circumstances it is not necessary to check all the vanishings but just a few of them. At any event, since in this chapter we are not interested in giving complete and slick proofs of any particular result, but rather an idea of how to prove the vanishing of cohomology groups of this kind, we will focus our attention on obtaining the vanishing of

$$H^1(\bigwedge^{p+1} M_L \otimes L^{\otimes s}) \ .$$

In fact, as property N_p is defined inductively, for the purpose of this exposition we will assume that

$$H^1(\bigwedge^i M_L \otimes L^{\otimes s})$$

vanishes for all $1 \leq i \leq p$.

One way of attacking the problem is to obtain the vanishing of

$$H^1(M_L^{\otimes p+1} \otimes L^{\otimes s})$$

instead of computing directly the vanishing of the twists of the exterior product of M_L. This works in general if the characteristic of the field is greater than p (if we are working in order to prove property N_p) and, certainly, if the characteristic of the field is 0.

Now $H^1(M_L^{\otimes p+1} \otimes L^{\otimes s})$ can be seen in our situation naturally as a cokernel of a multiplication map α of global sections:

$$0 \longrightarrow H^0(M_L^{\otimes p+1} \otimes L^{\otimes s}) \longrightarrow H^0(M_L^{\otimes p} \otimes L^{\otimes s}) \otimes H^0(L) \xrightarrow{\alpha} H^0(M_L^{\otimes p} \otimes L^{\otimes s+1}) \longrightarrow H^1(M_L^{\otimes p+1} \otimes L^{\otimes s}) \longrightarrow 0 \ .$$

Indeed, to obtain the above sequence we tensor (2.4) with $M_L^{\otimes p} \otimes L^{\otimes s}$, take global sections and consider the long exact sequence of cohomology. The term that follows $H^1(M_L^{\otimes p+1} \otimes L^{\otimes s})$ is in fact $H^1(M_L^{\otimes p} \otimes L^{\otimes s}) \otimes H^0(L)$, but as explained before, we assume the vanishings required to prove property N_{p-1} and in particular, the vanishing of $H^1(M_L^{\otimes p} \otimes L^{\otimes s})$.

Thus our purpose now is to prove the surjectivity of α. Recall that $L = B^{\otimes p+1}$ with B base-point-free line bundle. To see the surjectivity of α it suffices to see the surjectivity of

$$H^0(M_L^{\otimes p} \otimes B^{\otimes s}) \otimes H^0(B) \xrightarrow{\beta} H^0(M_L^{\otimes p} \otimes B^{\otimes s+1})$$

for all $s \geq 1$. Now we perform a sort of induction on the number $p+1$ of copies of B forming L. For that we use the following theorem by Castelnuovo, improved by Mumford:

Theorem 3.1(cf. [Mu2]). *Let L be a base-point-free line bundle on a variety X and let $\mathcal{F}$ be a coherent sheaf on X. If $H^i(\mathcal{F} \otimes L^{-i}) = 0$ for all $i \geq 1$, then the multiplication map*

$$H^0(\mathcal{F} \otimes L^{\otimes i}) \otimes H^0(L) \to H^0(\mathcal{F} \otimes L^{\otimes i+1})$$

is surjective for all $i \geq 0$.

Therefore we reduce the surjectivity of α to seeing that $H^1(M_L^{\otimes p} \otimes B^{\otimes s-1})$ and $H^2(M_L^{\otimes p} \otimes B^{\otimes s-2})$ vanish. Using (2.4) the last vanishing can be reduced to checking vanishings of the type $H^1(M_L^{\otimes i} \otimes B^{\otimes t})$, with $i \leq p-1$. Now to complete the argument by induction on p we need to prove a first step for this induction. In the proofs of our results this step is typically $H^1(M_L^{\otimes 2} \otimes B^{\otimes s}) = 0$ for $s \geq 2$. Until now we have used general, known methods. At this point, the geometry of the surface X starts playing its role and the real work begins. Because of the different approaches we use, we distinguish two classes of surfaces: the surfaces with irregularity 0 and the surfaces with positive irregularity.

3.1. Surfaces with positive irregularity

We consider surfaces where the irregularity $h^1(O_X)$ is positive. The first approach we present consists in finding a suitable decomposition of L as tensor product of base-point-free line bundles. This decomposition will allow us to use

Castelnuovo–Mumford arguments to obtain the surjectivity of multiplication maps of vector bundles on the surface.

To explain this strategy we consider a particular example, namely, the case of an Abelian surface X and a line bundle $L = B^{\otimes 2}$, where B is ample and base-point-free. The line bundle L satisfies property N_1, although for simplicity sake, in this article we will only outline how to see that L satisfies property N_0 (for the complete proof and details see [GP4], Theorem 4.1). We need to see that

$$H^0(L^{\otimes n}) \otimes H^0(L) \longrightarrow H^0(L^{\otimes n+1})$$

surjects for all $n \geq 1$. The most difficult point is to see that

$$H^0(L) \otimes H^0(L) \longrightarrow H^0(L^{\otimes 2}).$$

For this we will write L as tensor product of suitable ample and base-point-free line bundles B_1 and B_2. Precisely we want to find B_1 and B_2 so that

$$\begin{gathered} H^0(L) \otimes H^0(B_1) \longrightarrow H^0(L \otimes B_1) \\ H^0(L \otimes B_1) \otimes H^0(B_2) \longrightarrow H^0(L^{\otimes 2}) \end{gathered}$$

surject, and we want to prove this surjectivity using Theorem 3.1. Then we need $H^1(B_2)$, $H^1(B_1^{\otimes 2})$, $H^2(B_1^{\otimes 2} \otimes B_2^*)$ and $H^2(B_1 \otimes B_2^*)$ all vanish. Since $L = B^{\otimes 2}$ the first choice that comes in mind is $B_1 = B_2 = B$ but this does not work, because $H^2(\mathcal{O}_X) \neq 0$ if X is an Abelian surface. What we do then is to alter B by tensoring by a suitable $\mathfrak{d} \in \operatorname{Pic}^0(X)$ (recall that $\operatorname{Pic}^0(X)$ is a complex torus of dimension 2). Precisely we set $B_1 = B \otimes \mathfrak{d}$ and $B_2 = B \otimes \mathfrak{d}^*$ with $\mathfrak{d}$ not of 2-torsion in $\operatorname{Pic}^0(X)$. At this point we need to make an extra assumption on B, namely, we want $B^2 \geq 5$. Then by Reider theorem both $B \otimes \mathfrak{d}$ and $B \otimes \mathfrak{d}^*$ are base-point-free. By our choice, $H^2(B_1 \otimes B_2^*) = H^0(\mathfrak{d}^{-2})^* = 0$ and $H^1(B_2) = H^1(B_1^{\otimes 2}) = H^2(B_1^{\otimes 2} \otimes B_2^*) = 0$ by Kodaira vanishing.

The key point of the above argument was the fact that $\operatorname{Pic}^0(X)$ is "large" enough. This allowed us to play with the decomposition of L as tensor product of ample and base-point-free line bundles. As our results show, this approach also works for other irregular surfaces. In other words, a thorough study of the cohomology of the line bundles of an irregular surface X together with a way to find smart decompositions for the line bundles on X can yield general syzygy results on X. This was done for elliptic ruled surfaces in [GP1], Theorem 4.2 and [GP2], Theorem 6.1.

3.2. Surfaces with irregularity zero

The strategy displayed in the previous section breaks down if $h^1(\mathcal{O}_X) = 0$ (think for instance of a $K3$ surface of Picard number 1 to consider a very extreme case). We therefore use a different approach whose general outline is this: we reduce the problem of showing the surjectivity of certain multiplication map of global sections of vector bundles on the surface X to the problem of showing the surjectivity of a multiplication map of semistable vector bundles on a suitable divisor $C \in |B|$. One can regard this process as some sort of induction on the dimension of the variety. Once we are arguing on a curve, we can use modern results regarding surjectivity of multiplication maps of semistable bundles on curves such as Butler's (cf. [B]), Eisenbud-Koh-Stillman's (cf. [EKS]) or Pareschi's (cf. [P]), or classical results such as Noether's or Enriques-Petri's for the canonical curve, which can also be reformulated in terms of surjectivity of multiplication maps (cf. [GL1], [PP]).

To illustrate all the above we will focus on a particular case, the case in which X is a K3 surface. We will show this

Theorem 3.1.1. *Let X be a K3 surface. Let B be a base-point-free line bundle such that $B^2 \geq 4$. Then $H^1(M_{B^{\otimes r}}^{\otimes 2} \otimes B^{\otimes s}) = 0$ for all $r, s \geq 2$.*

Outline of proof. With the same arguments as before, we conclude that it is sufficient to see the surjectivity of

$$H^0(M_{B^{\otimes r}} \otimes B^{\otimes s}) \otimes H^0(B) \xrightarrow{\beta} H^0(M_{B^{\otimes r}} \otimes B^{\otimes s+1}), \text{ for all } r, s \geq 2 .$$

This map β is the map of multiplication of sections of vector bundles on the surface X to which we referred before. Now we choose the suitable divisor C needed to perform the argument by "induction on the dimension". We choose C to be a smooth member in $|B|$. Consider the commutative diagram

$$\begin{array}{ccccc} H^0(N) \otimes H^0(\mathcal{O}_X) & \hookrightarrow & H^0(N) \otimes H^0(B) & \twoheadrightarrow & H^0(N) \otimes H^0(B_C) \\ \downarrow & & \downarrow & & \downarrow \\ H^0(N) & \hookrightarrow & H^0(N \otimes B) & \twoheadrightarrow & H^0(N \otimes B_C), \end{array}$$

where $N = M_{B^{\otimes r}} \otimes B^{\otimes s}$ and for a sheaf F on X we write $F_C = F \otimes \mathcal{O}_C$. From this we see that for β to surject it is sufficient to show that

$$H^0(M_{B^{\otimes r}} \otimes B^{\otimes s} \otimes \mathcal{O}_C) \otimes H^0(B_C) \xrightarrow{\gamma} H^0(M_{B^{\otimes r}} \otimes B^{\otimes s+1} \otimes \mathcal{O}_C)$$

surjects.

This map is difficult to handle because $M_{B^{\otimes r}} \otimes B^{\otimes s} \otimes \mathcal{O}_C$ is not semistable. Then we construct this commutative diagram to filtrate $M_{B^{\otimes r}} \otimes \mathcal{O}_C$:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & H^0(B^{\otimes r-1}) \otimes \mathcal{O}_C & \to & M_{B^{\otimes r}} \otimes \mathcal{O}_C & \to & M_{B_C^{\otimes r}} & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & H^0(B^{\otimes r-1}) \otimes \mathcal{O}_C & \to & H^0(B^{\otimes r}) \otimes \mathcal{O}_C & \to & H^0(B_C^{\otimes r}) \otimes \mathcal{O}_C & \to 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & \to & B_C^{\otimes r} & \to & B_C^{\otimes r} & \to 0 \\
 & & & \downarrow & & \downarrow & \\
 & & & 0 & & 0 & .
\end{array}$$

From the above diagram we obtain that $M_{B^{\otimes r}} \otimes \mathcal{O}_C$ is the direct sum of $H^0(B^{\otimes r-1}) \otimes \mathcal{O}_C$ and $M_{B_C^{\otimes r}}$. Then to show that γ surjects it suffices to show that

$$H^0(M_{B_C^{\otimes r}} \otimes B_C^{\otimes s}) \otimes H^0(B_C) \xrightarrow{\delta} H^0(M_{B_C^{\otimes r}} \otimes B_C^{\otimes s+1}) \text{ and}$$
$$H^0(B_C^{\otimes s}) \otimes H^0(B_C) \xrightarrow{\eta} H^0(B_C^{\otimes s+1})$$

surject, for all $r, s \geq 2$. The surjectivity of η follows from the fact that the canonical ring of C is generated in degree 2. Now we look at δ. Note that, in contrast with $M_{B^{\otimes r}} \otimes \mathcal{O}_C$, $M_{B_C^{\otimes r}}$ is semistable. Indeed, since $r \geq 2$, $\deg(B_C^{\otimes r}) \geq 2g(C)$ as long as $B^2 \geq 2$; then, by [B], Theorem 1.2, $M_{B_C^{\otimes r}}$ is semistable. In fact

$$0 \to H^0(B^{\otimes r-1}) \otimes \mathcal{O}_C \to M_{B^{\otimes r}} \otimes \mathcal{O}_C \to M_{B_C^{\otimes r}} \to 0$$

is the Harder-Narasimhan filtration of $M_{B^{\otimes r}} \otimes \mathcal{O}_C$. Therefore we have been able to reduce the problem of seeing the surjectivity of β to seeing the surjectivity of the map δ, which is a multiplication map of global sections of semistable vector bundles on the curve C on X.

Then to finish the argument one can use results about surjectivity of multiplication maps on curves such as [B], Theorem 2.2 or [P], Corollary 4, or we can directly observe that $H^1(M_{B_C^{\otimes r}} \otimes M_{B_C} \otimes B_C^{\otimes s}) = 0$. To see the latter, recall that $M_{B_C^{\otimes r}}$ is semistable. On the other hand $B_C = K_C$, thus by [PR] or by a result of T.R. Ramadas and the second author in [GP5] , Theorem

1.7, M_{B_C} is also semistable. Finally, by [Mi], Corollary 3.7, the vector bundle $E = M_{B_C^{\otimes r}} \otimes M_{B_C} \otimes B_C^{\otimes s}$ is semistable, so it is enough to see that its slope is strictly bigger than $2g(C) - 2$. We will first see that $\mu(M_{B_C^{\otimes r}}) > -2$. Since $r \geq 2$,

$$\mu(M_{B_C^{\otimes r}}) = \frac{-\deg(B_C^{\otimes r})}{\deg(B_C^{\otimes r}) - g(C)} ,$$

for the rank of M_{rB_C} equals $h^0(B_C^{\otimes r}) - 1 = \deg(B_C^{\otimes r}) - g(C)$ and $\deg(M_{B_C^{\otimes r}}) = -\deg(B_C^{\otimes r})$. Therefore the desired inequality will follow if we show that $r(2g(C)-2) > 2g(C)$. This holds as long as $r \geq 2$ and $B^2 \geq 4$. On the other hand $\mu(M_{B_C}) = -2$, hence $\mu(E) > 2g(C) - 2$, since $s \geq 2$ and $B^2 \geq 4$. □

The previous example illustrates some methods to handle multiplication maps on a regular surface X by studying analogous maps on suitable curves on X. The choice of the divisor to which one reduces the multiplication map is not always as canonical as above, but the example captures some of the spirit of our methods.

3.3. Extremal curves with respect to the N_p properties

The ideas introduced in the previous section are also useful when trying to prove the sharpness of a syzygy result for surfaces. The philosophy is that the failure of a line bundle L on a variety X to satisfy property N_p can be traced to the existence of an "extremal" curve C on X, such that the restriction L to C does not satisfy property N_p. We mention three examples of this. The first is the case of elliptic ruled surfaces and property N_0 and N_1 (cf. [Ho1], [Ho2] and [GP1]). We focus on the elliptic ruled surface X of invariant -1. In [GP1] we proved the following result:

Theorem 3.3.1 ([GP1], Theorem 4.2). *Let X be an elliptic ruled surface with invariant $e(X) = -1$. Let C_0 be a minimal section of X and let f be a fiber of X. Let L be a line bundle on X numerically equivalent to $aC_0 + bf$. The line bundle L satisfies property N_1 if and only if $a \geq 1$, $a + b \geq 4$ and $a + 2b \geq 4$.*

As we mentioned at the end of Section 3.1, one of the implications, namely, that L satisfies property N_1 if $a \geq 1$, $a + b \geq 4$ and $a + 2b \geq 4$ is proved by decomposing in a smart way any line bundle fulfilling these numerical conditions and using then the ideas exposed in Section 3.1. We give now the idea of the proof of the other implication. Consider for example a line bundle L numerically equivalent $aC_0 + bf$, such that $a + b = 3$. We want to show that it cannot satisfy property N_1. If such a line bundle L satisfied property N_1, then so would its restriction to C_0. Thus we assume L satisfies property N_1 and we find a contradiction. Using constructions similar to those showed in the proof of Theorem 3.1.1 , we obtain that

$$\mathrm{H}^1(\bigwedge^2 M_{L\otimes\mathcal{O}_{C_0}} \otimes L) = 0 \ .$$

On the other hand the line bundle $L \otimes \mathcal{O}_{C_0}$ on C_0 satisfies property N_0 because C_0 is an elliptic curve and deg $(L \otimes \mathcal{O}_{C_0}) = 3$. Thus Theorem 2.6 implies that $L \otimes \mathcal{O}_{C_0}$ satisfies property N_1. This is obviously false since the complete linear series of $L \otimes \mathcal{O}_{C_0}$ embeds C_0 as a plane cubic.

The other relevant boundary of the convex set of N_1 line bundles on the elliptic ruled surface X is also justified by the existence of "extremal" curves. In this occasion they are elliptic curves E in the numerically equivalence class of $-K_X = 2C_0 - f$. Indeed, a line bundle L numerically equivalent to $aC_0 + bf$ and such that $a + 2b = 3$, restricts to E as a line bundle of degree 3, and if it satisfied property N_1, by the same arguments as above, so would its restriction to E.

Another case in which we can trace the failure of properties N_p to the existence of extremal curves is Calabi-Yau threefolds. Recall that a Calabi-Yau threefold X is a regular threefold for which $K_X = \mathcal{O}_X$.

For Calabi-Yau threefolds we have proved the following result:

Theorem 3.3.2 (cf. [GP3], Theorem 2.4.). *Let X be a Calabi-Yau threefold. Let B be an ample and base-point-free divisor with $h^0(B) \geq 5$. Let $L = B^{\otimes p+2+k}$. If $k \geq 0$ and $p \geq 1$, L satisfies property N_p.*

We consider a Calabi-Yau threefold X (cf. [GP3], Example 2.5), which is a cyclic triple cover of $\mathbf{P}^3$, ramified along a smooth sextic surface. Let B be the pullback of $\mathcal{O}_{\mathbf{P}^3}(1)$ to X. Therefore $h^0(B) = 4$, so $L = B^{\otimes 3}$ does not satisfy the hypothesis of Theorem 3.3.2. Actually, L does not satisfy property N_1. We will outline the proof of this fact by showing the "extremal" curve which prevents L from satisfying property N_1. Let $S \in |B|$ and let C be a smooth curve in $|B \otimes \mathcal{O}_C|$. The curve C has genus $g(C) = 4$ and $L \otimes \mathcal{O}_C$ has degree $2g(C) + 1 = 9$, thus, by the result of Castelnuovo, L satisfies property N_0. On the other hand Green and Lazarsfeld (cf. [GL2]), proved that a line bundle does not satisfy property N_1 if it is the tensor product of the canonical bundle on C and an effective line bundle of degree 3. This is the case of $L \otimes \mathcal{O}_C$. However, arguing as in the case of elliptic ruled surfaces, if L satisfied property N_1, so would $L \otimes \mathcal{O}_C$ and we conclude that L cannot satisfy property N_1.

The last example of this phenomenon we show here is the Veronese embedding of $\mathbf{P}^n$. Ottaviani and Paoletti showed the following:

Theorem 3.3.3 ([OP], Theorem 2.1). *The line bundle $L = \mathcal{O}_{\mathbf{P}^n}(d)$ does not satisfy property N_p if $p \geq 3d-2$, $d \geq 3$.*

The bound on d they give for property N_p to fail suggests the existence of an extremal curve in the sense discussed in this section. Indeed, $d \leq \lfloor \frac{p+2}{3} \rfloor$ and if one considers a plane cubic C, the restriction L to C has degree less than or equal to $p+2$. However on an elliptic curve a line bundle satisfies property N_p if and only if its degree is greater than or equal to $p+3$.

References

[ACGH] E. Arbarello, M. Cornalba, P.A. Griffiths & J. Harris, Geometry of Algebraic Curves, Volume I, Springer-Verlag 1985.

[B] D. Butler, *Normal generation of vector bundles over a curve*, J. Differential Geometry **39** (1994) 1-34.

[EKS] D. Eisenbud, J. Koh & M. Stillman, *Determinantal equations for curves of high degree*, Amer. J. Math. **110** (1989), 513-540.

[EL1] L. Ein & R. Lazarsfeld, *Koszul cohomology and syzygies of projective varieties*, Inv. Math. **111** (1993), 51-67.

[EL2] ——, *Global generation of pluricanonical and adjoint linear series on smooth projective threefolds*, J. Amer. Math. Soc. **6** (1993), 875-903.

[F] T. Fujita, *Remarks on Ein-Lazarsfeld criterion of spannedness of adjoint bundles on polarized threefolds*, preprint.

[GP1] F.J. Gallego & B.P. Purnaprajna *Normal presentation on elliptic ruled surfaces*, J. Algebra **186**, (1996), 597-625.

[GP2] ——, *Higher syzygies of elliptic ruled surfaces*, J. Algebra **186**, (1996), 626-659.

[GP3] ——, *Very ampleness and higher syzygies for Calabi-Yau threefolds*, Math. Ann. **312** (1998), 133-149.

[GP4] ——, *Projective normality and syzygies of algebraic surfaces*, J. reine angew. Math. **506** (1999), 145-180.

[GP5] ——, *Vanishing theorems and syzygies for K3 surfaces and Fano varieties*, to appear in J. Pure App. Alg.

[GP6] ——, *Some results on rational surfaces and Fano varities*, preprint.

[G] M. Green, *Koszul cohomology and the geometry of projective varieties*, J. Differential Geometry **19** (1984) 125-171.

[GL1] M. Green & R. Lazarsfeld, *a simple proof of Petri's Theorem on canonical curves*, Progress in Mathematics **60**: Geometry of Today, Birkhäuser, 1985, 129-142.

[GL2] ——, *Some results on the syzygies of finite sets and algebraic curves*, Compositio Math. **67** (1989) 301-314.

[H] R. Hartshorne, Algebraic Geometry, Springer–Verlag 1977.

[Ho1] Y. Homma, *Projective normality and the defining equations of ample invertible sheaves on elliptic ruled surfaces with* $e \geq 0$, Natural Science Report, Ochanomizu Univ. **31** (1980) 61-73.

[Ho2] ——, *Projective normality and the defining equations of an elliptic ruled surface with negative invariant*, Natural Science Report, Ochanomizu Univ. **33** (1982) 17-26.

[I] S.P. Inamdar, *On Syzygies of Projective Varieties* , Pacific. J. Math. **177** (1997), 71-76.

[Ka] Y. Kawamata, *On Fujita's freeness conjecture for* 3*-folds and* 4*-folds*, Math. Ann. **308** (1997), 491-505.

[Ke] G. Kempf, *Projective coordinate rings of Abelian varieties*, Algebraic Analysis, Geometry and Number Theory, the Johns Hopkins Univ. Press, 1989, 225-235.

[L] R. Lazarsfeld, *A sampling of vector bundle techniques in the study of linear series*, Lectures on Riemann Surfaces, World Scientific Press, Singapore, 1989, 500-559.

[Mi] Y. Miyaoka, *The Chern class and Kodaira dimension of a minimal variety*, Algebraic Geometry –Sendai 1985, Advanced Studies in Pure Math., Vol. 10, North-Holland, Amsterdam, 449-476.

[Mu1] D. Mumford, Lectures on Curves on an Algebraic Surface, Princeton University Press, 1966.

[Mu2] ——, *Varieties defined by quadratic equations*, Corso CIME in Questions on Algebraic Varieties, Rome, 1970, 30-100.

[N] M. V. Nori, *Algebraic cycles and Hodge theory connectivity*, Invent. math. **111** (1993), 349-373.

[OP] G. Ottaviani & R. Paoletti, *Syzygies of Veronese embeddings*, preprint.

[PR] K. Paranjape & S. Ramanan, *On the canonical ring of a curve*, Algebraic Geometry and Commutative Algebra in honor of Nagata, Vol 2, 503-516.

[P] G. Pareschi, *Gaussian maps and multiplication maps on certain projective varieties*, Compositio Math. **98** (1995), 219-268.

[PP] G. Pareschi & B.P. Purnaprajna, *Canonical ring of a curve is Koszul: A simple proof*, Illinois J. Math. **41** (1997).

[R] I. Reider, *Vector bundles of rank* 2 *and linear systems on an algebraic surface*, Ann. of Math. (2) **127** (1988) 309-316.

[S-D] B. Saint-Donat, *On projective models of K3 surfaces*, Amer. J. Math. **96**, (1974), 602-639.