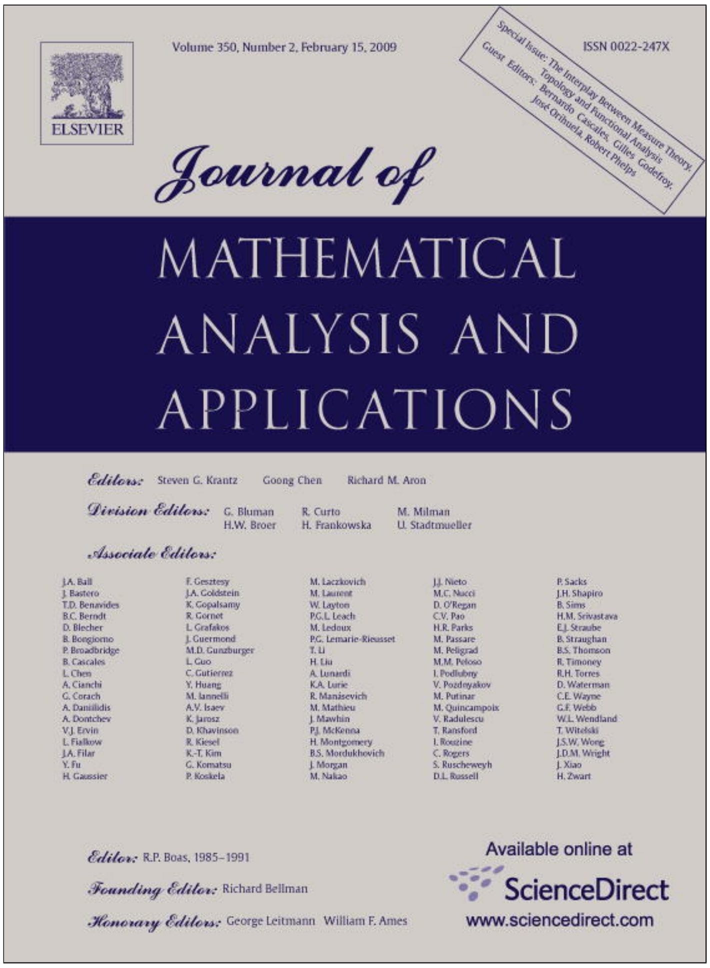
Volume 350, Number 2, February 15, 2009
ISSN 0022-247X
ELSEVIER
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J. Math. Anal. Appl. 350 (2009) 485–497
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
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Convex w^*-closures versus convex norm-closures in dual Banach spaces

A.S. Granero [a,*,1], M. Sánchez [b]

[a] *Universidad Complutense, Fac. de Matemáticas, Dept. Análisis Matemático, Pl. de Ciencias 3, 28040 Madrid, Spain*
[b] *Universidad Rey Juan Carlos, Fac. de CC. Experimentales y Tecnología, Dept. Matemática Aplicada, c/Tulipán s/n, 28933 Móstoles, Madrid, Spain*

Received 14 November 2007
Available online 26 February 2008
Submitted by B. Cascales

Dedicated to Professor I. Namioka on the occasion of his 80th birthday

Abstract

A subset Y of a dual Banach space X^* is said to *have the property* (P) if $\overline{\mathrm{co}}^{w^*}(H) = \overline{\mathrm{co}}(H)$ for every weak*-compact subset H of Y. The purpose of this paper is to give a characterization of the property (P) for subsets of a dual Banach space X^*, and to study the behavior of the property (P) with respect to additions, unions, products, whether the closed linear hull $\overline{[Y]}$ has the property (P) when Y does, etc. We show that the property (P) is stable under all these operations in the class of weak* $\mathcal{K}$-analytic subsets of X^*.

Keywords: Convex sets; Convex w^*-closure; Convex norm-closure; $\mathcal{K}$-analytic sets

1. Introduction

A subset Y of a dual Banach space X^* is said to *have the property* (P) if $\overline{\mathrm{co}}^{w^*}(H) = \overline{\mathrm{co}}(H)$ for every weak*-compact subset H of Y. The purpose of this paper is twofold: (i) first, to give a characterization of the property (P) for subsets of the dual Banach space X^*; (ii) second, to study the stability of the property (P), that is, its behavior with respect to additions, unions, products, whether the closed linear hull $\overline{[Y]}$ has the property (P) when Y does, etc.

In Section 2 we give a characterization of the property (P) for subsets of a dual Banach space X^*. Haydon [6] characterized the property (P) for a whole dual Banach space X^* as follows: X^* has the property (P) if and only if X fails to have a copy of ℓ_1 if and only if every $z \in X^{**}$ is universally measurable on (X^*, w^*). It happens that a dual Banach space X^* can have subsets with the property (P) (actually X^* always has such subsets), although X could contain a copy of ℓ_1. This fact suggests that (P) is a property dependent on subsets. So, it would be interesting to give an inner characterization of this property.

* Corresponding author.
E-mail addresses: as_granero@mat.ucm.es (A.S. Granero), marcos.sanchez@urjc.es (M. Sánchez).
1 Supported in part by grants DGICYT MTM2005-00082, UCM-910346 and UCM-BSCH PR34/07-1841.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2008.02.030

There are some interesting criteria for a weak*-compact subset K of a dual Banach space X^* to have the property (P). Indeed, Saab and Talagrand (see [11,16]) proved that, if K is weakly $\mathcal{K}$-analytic, then K has the property (P). Saab proved [12] that, if K is a weak* compact convex subset of X^* and every functional $x^{**} \in X^{**}$ is universally measurable on K, then K has the property (P). Cascales, Namioka, Orihuela and Vera (see [1–3]) have given different criteria for the property (P), for example, they proved that, if the weak* compact subset K is weakly Lindelöf, then K has the property (P).

The fragmentability is also a useful notion related with the property (P). Namioka proved [9, 2.3. Theorem] that a subset $Y \subset X^*$ has the property (P) whenever (Y, w^*) is norm-fragmented. So, norm-fragmentability implies the property (P). The converse is not true. Indeed, let X be the James Tree space JT (see [8]), which is a non-Asplund separable Banach space without a copy of ℓ_1. So, JT^* has the property (P) by [6] but the closed unit ball $B(JT^*)$ of JT^* is not norm-fragmentable, because the norm-fragmentability of $B(X^*)$ is equivalent to the Asplundness of X (see [9, 1.3. Theorem]).

We characterize the property (P) for arbitrary subsets $Y \subset X^*$ by means of a structure that we call a w^*–$\mathbb{N}$-family. This notion was introduced in [5, Definition 3.5], where we proved that, if a subset Y of a dual Banach space X^* fails to have a w^*–$\mathbb{N}$-family (in particular, if Y does not contain a copy of the basis of $\ell_1(\mathfrak{c})$), then $\overline{\mathrm{co}}^{w^*}(H) = \overline{\mathrm{co}}(H)$ for every weak*-compact subset H of Y, that is, the lack of a w^*–$\mathbb{N}$-family implies the property (P).

Section 3 is devoted to study the stability of the property (P) under unions, additions, products, closed linear hulls, etc. We prove that the property (P) is stable under all these operations in the class of $\mathcal{K}$-analytic subsets of (X^*, w^*). Moreover, we show that for this class of $\mathcal{K}$-analytic subsets of (X^*, w^*) (Proposition 3.8) the property (P) is equivalent to the lack of a w^*–$\mathbb{N}$-family. For non-$\mathcal{K}$-analytic subsets this equivalence can fail. Actually, we give examples of subsets that simultaneously have the property (P) and contain a w^*–$\mathbb{N}$-family.

Our notation is standard. If A and I are sets, $a \in A^I$ and $i \in I$, then a_i (or $a(i)$) denotes the ith coordinate of a and $\pi_i : A^I \to A$ the ith projection mapping such that $\pi_i(a) = a_i$. $|I|$ is the cardinality of I and $\mathfrak{c} := |\mathbb{R}|$. If B is a subset of I, ${}^cB := I \setminus B$ will denote the complement of B. A sequence $\{U_m, V_m\colon m \geqslant 1\}$ of subsets of I is said to be *independent* if $U_m \cap V_m = \emptyset$, $\forall m \geqslant 1$, and $(\bigcap_{m\in M} U_m) \cap (\bigcap_{n\in N} V_n) \neq \emptyset$ for every pair of disjoint finite subsets M, N of $\mathbb{N}$. βI denotes the Stone–Čech compactification of I (the I is endowed with the discrete topology) and $I^* := \beta I \setminus I$. The Cantor compact space $\{0, 1\}^{\mathbb{N}}$ is denoted by $\mathcal{C}$.

We shall consider only Banach spaces over the real field. If X is a Banach space, let $B(a; r) := \{x \in X\colon \|x - a\| \leqslant r\}$ be the closed ball with center at $a \in X$ and radius $r \geqslant 0$. $B(X)$ and $S(X)$ will be the closed unit ball and unit sphere of X, respectively, and X^* its topological dual. The weak*-topology of the dual Banach space X^* is denoted by w^* and the weak topology of X by w. If A is a subset of X, then $[A]$ and $\overline{[A]}$ denote the linear hull and the closed linear hull of A, respectively. If C is a convex subset of X^*, for $x^* \in X^*$ and $A \subset X^*$, let $d(x^*, C) = \inf\{\|x^* - c\|\colon c \in C\}$ be the distance from x^* to C and $\hat{d}(A, C) = \sup\{d(a, C)\colon a \in A\}$ the distance from A to C. $\mathrm{co}(A)$ denotes the convex closure of the set A, $\overline{\mathrm{co}}(A)$ is the $\|\cdot\|$-closure of $\mathrm{co}(A)$ and $\overline{\mathrm{co}}^{w^*}(A)$ the w^*-closure of $\mathrm{co}(A)$. Given $1 \leqslant M < \infty$, a convex subset C of X^* *is said to have M-control inside* X^* if $\hat{d}(\overline{\mathrm{co}}^{w^*}(K), C) \leqslant M\hat{d}(K, C)$ for every w^*-compact subset K of X^*.

If K is a w^*-compact subset of a dual Banach space X^* and μ a Radon Borel probability on K, then $r(\mu)$ will denote the barycenter of μ. Recall that:

(i) $r(\mu) \in \overline{\mathrm{co}}^{w^*}(K)$;
(ii) $x^* \in \overline{\mathrm{co}}^{w^*}(K)$ if and only if there exists a Radon Borel probability μ on K such that $r(\mu) = x^*$;
(iii) $r(\mu)(x) = \int_K x^*(x)\, d\mu(x^*)$ for all $x \in X$.

$\sum_{i\in I} \oplus_p X_i$ denotes the ℓ_p-sum of the family of Banach spaces $\{X_i\colon i \in I\}$ and π_i the canonical i-projection of $\sum_{i\in I} \oplus_p X_i$ onto X_i.

2. Characterizations of the property (P)

We begin this Section 2 with the definitions of w^*–$\mathbb{N}$-*family and Cantor skeleton*. The notion of w^*–$\mathbb{N}$-family was introduced in [5, Definition 3.5]. In this paper we work meanly with the notion of Cantor skeleton, which is similar to that of w^*–$\mathbb{N}$-family.

Definition 2.1. Let X be a Banach space.

(1) A subset $\mathcal{F}$ of X^* is said to be a w^*–$\mathbb{N}$-family of width $d > 0$ if $\mathcal{F}$ is bounded and has the form

$$\mathcal{F} = \{\eta_{M,N}\colon\ M, N \text{ disjoint subsets of } \mathbb{N}\},$$

and there exist two sequences $\{r_m\colon m \geqslant 1\} \subset \mathbb{R}$ and $\{x_m\colon m \geqslant 1\} \subset B(X)$ such that for every pair of disjoint subsets M, N of $\mathbb{N}$ we have

$$\eta_{M,N}(x_m) \geqslant r_m + d, \quad \forall m \in M, \quad \text{and} \quad \eta_{M,N}(x_n) \leqslant r_n, \quad \forall n \in N.$$

Moreover, if $r_m = r_0$, $\forall m \geqslant 1$, we say that $\mathcal{F}$ is a uniform w^*–$\mathbb{N}$-family in X^*.

(2) A subset $\mathcal{A}$ of X^* is said to be a Cantor skeleton of width $\delta > 0$ if $\mathcal{A}$ is a bounded set of the form $\mathcal{A} = \{k_\sigma\colon \sigma \in \mathcal{C}\}$ and there exist sequences $\{a_n\colon n \geqslant 1\} \subset \mathbb{R}$ and $\{x_m\colon m \geqslant 1\} \subset B(X)$ such that, for each $\sigma \in \{0,1\}^{\mathbb{N}}$ and for every $m \geqslant 1$, we have $\langle k_\sigma, x_m\rangle \leqslant a_m$, if $\sigma(m) = 0$, and $\langle k_\sigma, x_m\rangle \geqslant a_m + \delta$, if $\sigma(m) = 1$. Moreover, if $a_n = a$, $\forall n \geqslant 1$, we say that $\mathcal{A}$ is a uniform Cantor skeleton. A w^*-compact subset K of X^* is said to be endowed with a Cantor skeleton $\mathcal{K}$ if $\mathcal{K}$ is a Cantor skeleton and $\overline{\mathcal{K}}^{w^*} = K$.

Remark 2.2. (0) w^*–$\mathbb{N}$-families and Cantor skeletons are actually the same thing, but working with Cantor skeletons is easier. Let us explain this fact. Suppose that $\mathcal{F} := \{\eta_{M,N}\colon M, N \text{ disjoint subsets of } \mathbb{N}\}$ is a w^*–$\mathbb{N}$-family in X^* such that

$$\eta_{M,N}(x_m) \geqslant r_m + \delta, \quad \forall m \in M, \quad \text{and} \quad \eta_{M,N}(x_n) \leqslant r_n, \quad \forall n \in \mathbb{N}.$$

For each $\sigma \in \{0,1\}^{\mathbb{N}}$, let $M := \{n \in \mathbb{N}\colon \sigma(n) = 1\}$ and $N := \mathbb{N} \setminus M$, and define $h_\sigma := \eta_{M,N}$. Then, it is easy to see that $\mathcal{K} := \{h_\sigma\colon \sigma \in \{0,1\}^{\mathbb{N}}\}$ is a Cantor skeleton of width δ in X^*. Of course, $\mathcal{K}$ is uniform if $\mathcal{F}$ is uniform. The converse is also true: if $\{h_\sigma\colon \sigma \in \{0,1\}^{\mathbb{N}}\}$ is a Cantor skeleton of width $\delta > 0$ associated with the sequences $\{r_m\colon m \geqslant 1\} \subset \mathbb{R}$ and $\{x_m\colon m \geqslant 1\} \subset B(X)$, for each pair of disjoint subset M, N of $\mathbb{N}$ choose $\sigma_{M,N} \in \mathcal{C}$ such that $\sigma_{M,N}(m) = 1$, $\forall m \in M$ and $\sigma_{M,N}(n) = 0$, $\forall n \in N$. So, if for each pair of disjoint subset M, N of $\mathbb{N}$ we define $\eta_{M,N} = k_{\sigma_{M,N}}$, then $\{\eta_{M,N}\colon M, N \text{ disjoint subsets of } \mathbb{N}\}$ is a w^*–$\mathbb{N}$-family in X^*.

(1) Let K be a w^*-compact subset endowed with a Cantor skeleton $\mathcal{A} = \{k_\sigma\colon \sigma \in \mathcal{C}\}$ of width $\delta > 0$ associated with the sequences $\{r_m\colon m \geqslant 1\} \subset \mathbb{R}$ and $\{x_m\colon m \geqslant 1\} \subset B(X)$. Then we have:

(11) For every $k \in K$ and every $m \geqslant 1$ either $\langle k, x_m\rangle \leqslant a_m$ or $\langle k, x_m\rangle \geqslant a_m + \delta$. Moreover, if we define the mapping $\Phi : K \to \mathcal{C} = \{0,1\}^{\mathbb{N}}$ as

$$\forall k \in K,\ \forall m \geqslant 1, \quad \Phi(k)(m) = \begin{cases} 1 & \text{if } \langle k, x_m\rangle \geqslant a_m + \delta, \\ 0 & \text{if } \langle k, x_m\rangle \leqslant a_m, \end{cases}$$

we have that Φ is a continuous mapping that satisfies $\Phi(K) = \mathcal{C}$.

(12) In general, K may not be homeomorphic to $\mathcal{C}$, even K may not contain a subspace homeomorphic to $\mathcal{C}$. Indeed, pick the compact space $\beta\mathbb{N}$ considered homeomorphically embedded into $(B(C(\beta\mathbb{N})^*), w^*)$. It is clear that $\overline{\text{co}}(\beta\mathbb{N}) \subsetneq \overline{\text{co}}^{w^*}(\beta\mathbb{N})$ because $\overline{\text{co}}(\beta\mathbb{N})$ is the set of purely atomic probabilities on $\beta\mathbb{N}$ and $\overline{\text{co}}^{w^*}(\beta\mathbb{N})$ is the set of all Radon probabilities on $\beta\mathbb{N}$. This fact implies (by the next Proposition 2.5) that there exists a w^*-compact subset K of $\beta\mathbb{N}$ endowed with a uniform Cantor skeleton with respect to $C(\beta\mathbb{N})^*$. However, K cannot contain a homeomorphic copy of $\mathcal{C}$ because $\beta\mathbb{N}$ fails to contain non-trivial convergent sequences.

(13) For every $0 < \eta < \delta$ there exist an infinite subset $\mathbb{N}_\eta \subset \mathbb{N}$, a real number b_η and a subset $\mathcal{A}_\eta \subset \mathcal{A}$ such that $\mathcal{A}_\eta$ is a uniform Cantor skeleton of width η associated to the number b_η and the sequence $\{x_m\colon m \in \mathbb{N}_\eta\} \subset B(X)$. Indeed, since the family $\{a_n\colon n \geqslant 1\} \subset \mathbb{R}$ is bounded, there exists $b_\eta \in \mathbb{R}$ such that $\mathbb{N}_\eta := \{m \in \mathbb{N}\colon b_\eta + \eta - \delta \leqslant a_m \leqslant b_\eta\}$ is infinite. Let $\pi : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}_\eta}$ be the canonical projection and for each $\tau \in \{0,1\}^{\mathbb{N}_\eta}$ choose $\sigma(\tau) \in \pi^{-1}(\tau)$. Define $h_\tau := k_{\sigma(\tau)}$ for each $\tau \in \{0,1\}^{\mathbb{N}_\eta}$. Then it is easy to see that $\mathcal{A}_\eta := \{h_\tau\colon \tau \in \{0,1\}^{\mathbb{N}_\eta}\}$ is a uniform skeleton of width $\eta > 0$ associated with $b_\eta \in \mathbb{R}$ and the sequence $\{x_m\colon m \in \mathbb{N}_\eta\} \subset B(X)$.

In order to prove Proposition 2.5 we use the following lemmas.

Lemma 2.3. *Let $\mathcal{C} := \{0,1\}^{\mathbb{N}}$ be the Cantor compact set considered as a subset of the compact space $(B(\ell_\infty(\mathbb{N})), w^*)$. There exists a w^*-compact subset $D \subset \mathcal{C}$, homeomorphic to $\mathcal{C}$, such that $\overline{\text{co}}(D) \subsetneq \overline{\text{co}}^{w^*}(D)$. Actually, there exists $z_0 \in \overline{\text{co}}^{w^*}(D)$ such that $d(z_0, \overline{\text{co}}(D)) = 1 = \hat{d}(\overline{\text{co}}^{w^*}(D), \overline{\text{co}}(D))$.*

Proof. Consider the Cantor compact space $\mathcal{C} = \{0,1\}^{\mathbb{N}}$ and the set $\mathcal{S} := \{0,1\}^{<\mathbb{N}} = \{0,1\} \cup \{0,1\}^2 \cup \{0,1\}^3 \cup \cdots$. Let λ be the Haar probability on $\{0,1\}^{\mathbb{N}}$. If $\sigma = (\sigma_1, \sigma_2, \dots) \in \mathcal{C}$ and $n \in \mathbb{N}$, put $\sigma_{\restriction n} = (\sigma_1, \sigma_2, \dots, \sigma_n) \in \mathcal{S}$. If $A \subset \{0,1\}^n$, let $f_A : \mathcal{C} \to \{0,1\}$ be the continuous mapping defined by

$$\forall \sigma \in \mathcal{C}, \quad f_A(\sigma) = \begin{cases} 1 & \text{if } \sigma_{\restriction n} \in A, \\ 0 & \text{if } \sigma_{\restriction n} \notin A. \end{cases}$$

For each $n \in \mathbb{N}$ we define I_n as

$$I_n := \left\{ f_A \colon A \subset \{0,1\}^n \text{ with } |A| = 2^n - n \right\}.$$

Observe that I_n is finite and $\int_{\mathcal{C}} f_A \, d\lambda = 1 - n2^{-n}$ for each $f_A \in I_n$. Let $I := \bigcup_{n \geqslant 1} I_n$. Clearly, $|I| = \aleph_0$ and so we can put $I = \{f_{A_m} \colon m \geqslant 1\}$. We shall identify I with $\mathbb{N}$ by means of the identification of m and f_{A_m}. So, instead of $\ell_\infty(\mathbb{N})$ we also write $\ell_\infty(I)$. Observe that:

(1) The family I separates points in $\mathcal{C}$.
(2) For every $k \in \mathbb{N}$, the subset $\{f_A \in I \colon \int_{\mathcal{C}} f_A \, d\lambda \leqslant 1 - \frac{1}{k}\}$ is finite. So, $\lim_{m\to\infty} \int_{\mathcal{C}} f_{A_m} \, d\lambda \to 1$.
(3) Let $\{\sigma_j \colon j = 1, \dots, k\}$ be a finite subset of $\mathcal{C}$. Then for each $n \geqslant k$, there is $f_A \in I_n$ such that $f_A(\sigma_j) = 0$ for each $j = 1, \dots, k$.
(4) For every $f_A \in I$ there exists $\sigma \in \mathcal{C}$ such that $f_A(\sigma) = 1$.

Let $\psi : \mathcal{C} \to \{0,1\}^I \subset B(\ell_\infty(I))$ be the mapping such that

$$\forall i = f_A \in I, \ \forall \sigma \in \mathcal{C}, \quad \psi(\sigma)(i) = f_A(\sigma).$$

Clearly, ψ is a continuous injective mapping, when we consider in $\{0,1\}^I$ the w^*-topology of $\ell_\infty(I)$, that coincides with the product topology of $\{0,1\}^I$. Thus $D := \psi(\mathcal{C}) \subset \{0,1\}^I$ is a compact subset, homeomorphic to $\mathcal{C}$. Let $\mu := \psi(\lambda)$ be the Radon Borel probability on D image of the Haar probability λ under the continuous mapping ψ, and let $r(\mu) =: z_0 \in \overline{\mathrm{co}}^{w^*}(D)$ be the barycenter of μ. Clearly, $z_0 \in [0,1]^I$ and so $d(z_0, \overline{\mathrm{co}}(D)) \leqslant 1$. Note that for each $i = f_A \in I_n$ we have

$$z_0(f_A) = \pi_i(z_0) = \int_D \pi_i \, d\mu = \int_{\mathcal{C}} \pi_i\big(\psi(\sigma)\big) \, d\lambda(\sigma) = \int_{\mathcal{C}} f_A \, d\lambda = 1 - n2^{-n}. \tag{2.1}$$

In order to show that $d(z_0, \overline{\mathrm{co}}(D)) = 1$, it is enough to show that $\|z_0 - p\| = 1$ for each $p \in \mathrm{co}(D)$. Let $p = \sum_{j=1}^k t_j \psi(\sigma_j)$, where $t_j \in [0,1]$, $\sum_{j=1}^k t_j = 1$ and $\sigma_j \in \mathcal{C}$ for each j. Then by (3) for each $n \geqslant k$, one can choose an $f_A \in I_n$ with the property stated there. Therefore using Eq. (2.1)

$$1 \geqslant \|z_0 - p\| \geqslant z_0(f_A) - \sum_{j=1}^k t_j \psi(\sigma_j)(f_A) = z_0(f_A) = 1 - n2^{-n}.$$

Since $n \geqslant k$ is arbitrary, $\|z_0 - p\| = 1$. $\square$

Lemma 2.4. *Let K be a w^*-compact subset of a dual Banach space X^* such that K contains a Cantor skeleton of width $\delta > 0$. Then there exists a w^*-compact subset H of K such that $\hat{d}(\overline{\mathrm{co}}^{w^*}(H), \overline{\mathrm{co}}(H)) \geqslant \delta$.*

Proof. Let $\mathcal{A} := \{k_\sigma \colon \sigma \in \mathcal{C}\}$ be a Cantor skeleton of width $\delta > 0$ inside K. Without loss of generality, we suppose that $K = \overline{\mathcal{A}}^{w^*}$.

(A) First, we assume that K is a w^*-compact subset of ℓ_∞ and $\mathcal{A}$ a uniform Cantor skeleton of width $\delta = 1$ of K so that, for each $\sigma \in \{0,1\}^{\mathbb{N}}$ and for every $m \geqslant 1$, we have $\pi_m(k_\sigma) \leqslant 0$, if $\sigma(m) = 0$, and $\pi_m(k_\sigma) \geqslant 1$, if $\sigma(m) = 1$. Consider the continuous mapping $\Phi : K \to \mathcal{C}$ such that , $\forall k \in K$, $\Phi(k)(m) = 1$, if $k_m \geqslant 1$, and $\Phi(k)(m) = 0$, if $k_m \leqslant 0$. Clearly, $\Phi(K) = \mathcal{C}$. By the proof of Lemma 2.3 there exist a w^*-compact subset $D \subset \mathcal{C} \subset \ell_\infty$ and a Radon probability μ on D so that $\mu = \psi\lambda$, where λ is the Haar probability on $\mathcal{C}$ and $\psi : \mathcal{C} \to \{0,1\}^I$ is the mapping such that $\psi(\sigma)(i) = f_A(\sigma)$, $\forall i = f_A \in I$. Let $z_0 = r(\mu)$ be the barycenter of μ, that satisfies $z_0 \in \overline{\mathrm{co}}^{w^*}(D) \setminus \overline{\mathrm{co}}(D)$. Let

$$D_m^1 = \left\{ d \in D \colon \pi_m(d) = 1 \right\} \quad \text{and} \quad D_m^0 = \left\{ d \in D \colon \pi_m(d) = 0 \right\}, \quad m \geqslant 1.$$

Claim 1. *$\mu(D_m^1) \to 1$ and so $\mu(D_m^0) = \mu(D \setminus D_m^1) \to 0$ for $m \to \infty$.*

Indeed, in Lemma 2.3 we have identified $\mathbb{N}$ with the set $I = \{f_A\colon A \subset \{0,1\}^n$ with $|A| = 2^n - n$ and $n \in \mathbb{N}\}$. So, with the notation of Lemma 2.3, if $f_{A_m} \in I$ is the element of I corresponding to $m \in \mathbb{N}$, we have

$$\mu\big(D_m^1\big) = \int\limits_D \pi_m(x)\,d\mu(x) = \int\limits_{\mathcal{C}} \pi_m \circ \psi(\sigma)\,d\lambda(\sigma) = \int\limits_{\mathcal{C}} \psi(\sigma)(f_{A_m})\,d\lambda(\sigma) = \int\limits_{\mathcal{C}} f_{A_m}(\sigma)\,d\lambda(\sigma).$$

Now apply that $\lim_{m\to\infty} \int_{\mathcal{C}} f_{A_m}\,d\lambda = 1$ by (2) in the proof of Lemma 2.3.

Claim 2. *If $\Phi^{-1}(D) =: H \subset K$, then there exists $u_0 \in \overline{\mathrm{co}}^{w^*}(H)$ such that $d(u_0, \overline{\mathrm{co}}(H)) \geqslant 1$.*

Indeed, since $\Phi(H) = D$ and Φ is w^*–w^*-continuous, there exists a Radon Borel probability ν on H such that $\Phi\nu = \mu$. Let $u_0 := r(\nu)$ be the barycenter of ν, that satisfies $u_0 \in \overline{\mathrm{co}}^{w^*}(H)$.

Sub-Claim. *Given $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that $\pi_m(u_0) \geqslant 1 - \epsilon$, $\forall m \geqslant n_\epsilon$.*

Indeed, observe that $\pi_m(u_0) = \pi_m(r(\nu)) = \int_H \pi_m(h)\,d\nu(h)$, $\forall m \geqslant 1$. Let $0 \leqslant M < \infty$ be such that $\|h\| \leqslant M$, $\forall h \in H$, and choose $\eta > 0$ with $\epsilon \geqslant \eta(1+M)$. Now we choose $n_\epsilon \in \mathbb{N}$ such that $\mu(D_m^1) \geqslant 1 - \eta$, $\forall m \geqslant n_\epsilon$ (and $\mu(D_m^0) \leqslant \eta$). Then for $m \geqslant n_\epsilon$ we have

$$\begin{aligned}\int\limits_H \pi_m(h)\,d\nu(h) &= \int\limits_{\Phi^{-1}(D_m^1)} \pi_m(h)\,d\nu(h) + \int\limits_{\Phi^{-1}(D_m^0)} \pi_m(h)\,d\nu(h) \geqslant \int\limits_{\Phi^{-1}(D_m^1)} 1\,d\nu(h) + \int\limits_{\Phi^{-1}(D_m^0)} (-M)\,d\nu(h)\\ &= \nu\big(\Phi^{-1}\big(D_m^1\big)\big) - M\nu\big(\Phi^{-1}\big(D_m^0\big)\big) = \mu\big(D_m^1\big) - M\mu\big(D_m^0\big) \geqslant 1 - \eta - M\eta \geqslant 1 - \epsilon.\end{aligned}$$

In order to show that $d(u_0, \overline{\mathrm{co}}(H)) \geqslant 1$, it is sufficient to show that $\|u_0 - p\| \geqslant 1$ for each $p \in \mathrm{co}(H)$. Let $p = \sum_{j=1}^k t_j h_j$, where $t_j \in [0,1]$, $\sum_{j=1}^k t_j = 1$, $h_j \in H$ and $\Phi(h_j) =: d_j \in D$ for each j. By (3) of the proof of Lemma 2.3 there exists a sequence of integers $m_1 < m_2 < \cdots$ such that $\pi_{m_r}(d_j) = 0$ for $r \geqslant 1$ and $j = 1, \ldots, k$. So, by the definition of Φ we have $\pi_{m_r}(h_j) \leqslant 0$ for $r \geqslant 1$ and $j = 1, \ldots, k$, that is, $\pi_{m_r}(p) \leqslant 0$ for $r \geqslant 1$. Thus from the Sub-Claim we obtain $\|u_0 - p\| \geqslant 1$. So, this proves Claim 2 and completes the proof of the statement in this case (A).

(B) Now, we suppose that K is a w^*-compact subset of ℓ_∞ endowed with a Cantor skeleton $\mathcal{A} := \{k_\sigma\colon \sigma \in \mathcal{C}\}$ of width $\delta > 0$ associated with the numbers $(a_n)_{n\geqslant 1} \in \ell_\infty$ and the sequence of canonical projections $\{\pi_m\colon m \geqslant 1\}$, where $\pi_m(k) = k_m$, $\forall k \in \ell_\infty$. Let $T : \ell_\infty \to \ell_\infty$ be the mapping such that $T(x)(n) = (x_n - a_n)/\delta$, $\forall n \in \mathbb{N}$. Then T is an affine mapping which is w^*–w^*-continuous and $\|\cdot\|$-continuous. If $L = T(K)$, then L is a w^*-compact subset endowed with a uniform Cantor skeleton $T(\mathcal{A})$, which satisfies the requirements of case (A). So, there exists a w^*-compact subset $W \subset L$ and a point $w_0 \in \overline{\mathrm{co}}^{w^*}(W)$ such that $d(w_0, \overline{\mathrm{co}}(W)) \geqslant 1$. Let $H := T^{-1}(W)$. Clearly, H is a w^*-compact subset of K such that $T(H) = W$, $T(\overline{\mathrm{co}}(H)) \subset \overline{\mathrm{co}}(W)$ and $T(\overline{\mathrm{co}}^{w^*}(H)) = \overline{\mathrm{co}}^{w^*}(W)$. Thus, if $u_0 \in \overline{\mathrm{co}}^{w^*}(H)$ satisfies $T(u_0) = w_0$, then $d(u_0, \overline{\mathrm{co}}(H)) \geqslant \delta$, by the form of the mapping T.

(C) Finally, we suppose that K is a w^*-compact subset of an arbitrary dual Banach space X^* endowed with a Cantor skeleton $\mathcal{A} := \{k_\sigma\colon \sigma \in \mathcal{C}\}$ of width $\delta > 0$ associated with the numbers $(a_n)_{n\geqslant 1} \in \ell_\infty$ and the sequence $\{x_n\colon n \geqslant 1\} \subset B(X)$. Consider the continuous operator $T : \ell_1 \to X$ such that, $\forall (\lambda_n)_{n\geqslant 1} \in \ell_1$, $T((\lambda_n)_{n\geqslant 1}) = \sum_{n\geqslant 1} \lambda_n x_n \in X$. Observe that $\|T\| \leqslant 1$. Then, $T^*(K)$ is a w^*-compact subset of ℓ_∞ and $\{T^*(k_\sigma)\colon \sigma \in \mathcal{C}\}$ is a Cantor skeleton of $T^*(K)$ of width $\delta > 0$, that satisfies the requirements of case (B). So, there exists a w^*-compact subset $W \subset T^*(K)$ and a point $w_0 \in \overline{\mathrm{co}}^{w^*}(W)$ such that $d(w_0, \overline{\mathrm{co}}(W)) \geqslant \delta$. Let $H := T^{*-1}(W) \cap K$. Then H is a w^*-compact subset of K such that $T^*(H) = W$ and $T^*(\overline{\mathrm{co}}^{w^*}(H)) = \overline{\mathrm{co}}^{w^*}(W)$. Let $u_0 \in \overline{\mathrm{co}}^{w^*}(H)$ be such that $T^*(u_0) = w_0$. Taking into account the fact that $\|T^*\| \leqslant 1$ and that $\mathrm{co}(W) \subset T^*(\overline{\mathrm{co}}(H)) \subset \overline{\mathrm{co}}(W)$, we get $d(u_0, \overline{\mathrm{co}}(H)) \geqslant d(T^*(u_0), T^*(\overline{\mathrm{co}}(H))) = d(w_0, \overline{\mathrm{co}}(W)) \geqslant \delta$ and this completes the proof of the lemma. □

Let (X, τ) be a Hausdorff topological space, Y a subset of X and μ a finite positive Borel Radon measure on X. $\mathcal{B}_0(X)$ will denote the σ-algebra of Borel subsets of X. The positive Radon measure μ *is carried by* Y if there exists a sequence of compact subsets $\{K_n\colon n \geqslant 1\}$ of Y such that $K_n \subset K_{n+1}$ and $\mu(K_n) \uparrow \mu(X)$. Y is said to be a *universally measurable* subset of X if Y is μ-measurable for every finite positive Borel Radon measure μ on X. A mapping $f : X \to \mathbb{R}$ is said to be *μ-measurable* if $f^{-1}(G)$ is μ-measurable for all open subset G of $\mathbb{R}$. If (Z, T) is a topological space, a mapping $f : X \to Z$ is said to be *Lusin μ-measurable* if for each $\epsilon > 0$ there exists a compact subset K of X such that $\mu(X \setminus K) \leqslant \epsilon$ and $f \restriction K$ is continuous. Recall that by Lusin's Theorem a mapping $f : X \to \mathbb{R}$ is μ-measurable if and only f is Lusin μ-measurable. A mapping $f : X \to Z$ is said to be *universally measurable on Y* if and only if f is Lusin μ-measurable for every positive finite Radon Borel measure μ carried by Y, which is equivalent to say that, for every compact subset $K \subset Y$ and for every Radon Borel probability μ on K, f is Lusin μ-measurable.

Proposition 2.5. *Let X be a Banach space and Y a subset of X^*. The following statements are equivalent*:

(1) *Y does not have the property (P).*
(2) *There exist a w^*-compact subset H of Y and two real numbers $a < b$ such that for every finite family $\mathcal{F}$ of w^*-open subsets of X^* with $V \cap H \neq \emptyset$, $\forall V \in \mathcal{F}$, there exists $x_{\mathcal{F}} \in B(X)$ fulfilling that*
$$\inf\langle V \cap H, x_{\mathcal{F}}\rangle < a < b < \sup\langle V \cap H, x_{\mathcal{F}}\rangle, \quad \forall V \in \mathcal{F}.$$
(3) *There exists a w^*-compact subset K of Y endowed with a uniform Cantor skeleton.*
(4) *There exist a functional $\psi \in X^{**}$ which is not universally measurable on Y.*
(5) *There exists a w^*-compact subset H of Y which is uniformly non-fragmentable, that is, there exists $\delta > 0$ such that for every finite family $\mathcal{F}$ of w^*-open subsets of X^* with $V \cap H \neq \emptyset$, $\forall V \in \mathcal{F}$, there exist $x_{\mathcal{F}} \in B(X)$ and $r_{\mathcal{F}} \in \mathbb{R}$ such that*
$$\inf\langle V \cap H, x_{\mathcal{F}}\rangle < r_{\mathcal{F}} < r_{\mathcal{F}} + \delta < \sup\langle V \cap H, x_{\mathcal{F}}\rangle, \quad \forall V \in \mathcal{F}.$$
(6) *There exists a w^*-compact subset H of Y that contains a w^*–$\mathbb{N}$-family.*

Proof. (1) $\Rightarrow$ (2). Since Y does not have the property (P), there exists a w^*-compact subset $K \subset Y$ such that $\hat{d}(\overline{\mathrm{co}}^{w^*}(K), \overline{\mathrm{co}}(K)) > d > 0$. By [5, Lemma 3.2] (see also the proof of [6, 3.1. Proposition]) there exist $r_0 \in \mathbb{R}$, $\psi \in S(X^{**})$ and a w^*-compact subset $H \subset K$ such that: (i) $\psi(k) < r_0$, $\forall k \in K$; (ii) for every w^*-open subset V of X^* with $V \cap H \neq \emptyset$ there exists $\xi \in \overline{\mathrm{co}}^{w^*}(V \cap H)$ such that $\psi(\xi) > r_0 + d$. Therefore, if $\mathcal{F}$ is a finite family of w^*-open subsets of X^* such that $V \cap H \neq \emptyset$, $\forall V \in \mathcal{F}$, there exist $k_V \in V \cap H$ and $\xi_V \in \overline{\mathrm{co}}^{w^*}(V \cap H)$ so that $\psi(k_V) < r_0$ and $\psi(\xi_V) > r_0 + d$ for every $V \in \mathcal{F}$. Thus, as $B(X)$ is w^*-dense in $B(X^{**})$, we can find a vector $x_{\mathcal{F}} \in B(X)$ such that
$$\inf\langle V \cap H, x_{\mathcal{F}}\rangle < r_0 < r_0 + d < \sup\big\langle \overline{\mathrm{co}}^{w^*}(V \cap H), x_{\mathcal{F}}\big\rangle, \quad \forall V \in \mathcal{F}.$$
Since $x_{\mathcal{F}} \in X$, then $\sup\langle \overline{\mathrm{co}}^{w^*}(V \cap H), x_{\mathcal{F}}\rangle = \sup\langle V \cap H, x_{\mathcal{F}}\rangle$ and so (2) holds with $a := r_0$ and $b := r_0 + d$.

(2) $\Rightarrow$ (3). Let H be a w^*-compact subset of Y fulfilling (2). First, we construct an independent sequence $\{(A_m, B_m)\colon m \geqslant 1\}$ in H.

Step 1. By (2) there exists $x_1 \in B(X)$ such that
$$\inf\langle H, x_1\rangle < a < b < \sup\langle H, x_1\rangle.$$
Define $V_{11} = \{h \in X^*\colon \langle h, x_1\rangle < a\}$ and $V_{12} = \{h \in X^*\colon \langle h, x_1\rangle > b\}$. Observe that $V_{1i} \cap H \neq \emptyset$, $i = 1, 2$.

Step 2. By (2) there exists $x_2 \in B(X)$ such that
$$\inf\langle V_{1i} \cap H, x_2\rangle < a < b < \sup\langle V_{1i} \cap H, x_2\rangle, \quad i = 1, 2.$$
Let $V_{21} = \{h \in X^*\colon \langle h, x_2\rangle < a\}$ and $V_{22} = \{h \in X^*\colon \langle h, x_2\rangle > b\}$. Observe that $V_{1i} \cap V_{2j} \cap H \neq \emptyset$, $i, j = 1, 2$.

Further, we proceed by iteration. We obtain a sequence $\{V_{n1}, V_{n2}\colon n \geqslant 1\}$ of w^*-open subsets of X^* such that $V_{1i_1} \cap \cdots \cap V_{ni_n} \cap H \neq \emptyset$, $i_j \in \{1, 2\}$, $n \geqslant 1$. Thus, if we define
$$A_m = \big\{h \in H\colon \langle h, x_m\rangle \geqslant b\big\} \quad \text{and} \quad B_m = \big\{h \in H\colon \langle h, x_m\rangle \leqslant a\big\}, \quad m \geqslant 1,$$
then it is easy to verify that $\{(A_m, B_m)\colon m \geqslant 1\}$ is an independent sequence of w^*-closed subsets of H. Now, for each $\sigma \in \{0,1\}^{\mathbb{N}}$ and each $n \in \mathbb{N}$, let $C_{(\sigma,n)} = A_n$, if $\sigma(n) = 1$, and $C_{(\sigma,n)} = B_n$, if $\sigma(n) = 0$. By compactness, it is clear that $\bigcap_{n\geqslant 1} C_{(\sigma,n)} \neq \emptyset$, $\forall \sigma \in \{0,1\}^{\mathbb{N}}$. So, we can choose $h_\sigma \in \bigcap_{n\geqslant 1} C_{(\sigma,n)}$, $\forall \sigma \in \{0,1\}^{\mathbb{N}}$. Let $K := \overline{\{h_\sigma\colon \sigma \in \{0,1\}^{\mathbb{N}}\}}^{w^*}$. It is easy to see that $\{h_\sigma\colon \sigma \in \{0,1\}^{\mathbb{N}}\}$ is a uniform Cantor skeleton of K of width $b - a$.

(3) $\Rightarrow$ (4). Let K be a w^*-compact subset of Y endowed with a uniform Cantor skeleton $\{h_\sigma\colon \sigma \in \{0,1\}^{\mathbb{N}}\}$ of width $\delta > 0$ associated with the number $r_0 \in \mathbb{R}$ and the sequence $\{x_m\colon m \geqslant 1\} \subset B(X)$. So, $K = \overline{\{h_\sigma\colon \sigma \in \{0,1\}^{\mathbb{N}}\}}^{w^*}$. Let $T : \ell_1 \to X$ be the continuous operator such that $T(e_n) = x_n$, $\forall n \geqslant 1$, $\{e_n\colon n \geqslant 1\}$ being the canonical basis of ℓ_1. So, its adjoint $T^* : X^* \to \ell_\infty$ fulfills $T^*(x^*) = (x^*(x_m))_m$, $\forall x^* \in X^*$. Define the mapping $\Phi : \ell_\infty \to \ell_\infty$ as follows

$$\forall (a_n)_n \in \ell_\infty, \quad \Phi\big((a_n)_n\big) = \frac{1}{\delta}\big(\big((a_n - r_0) \vee 0\big) \wedge \delta\big)_n.$$

The mapping Φ is w^*–w^*-continuous and satisfies $\Phi \circ T^*(K) = \{0,1\}^{\mathbb{N}} = \mathcal{C}$. Let λ be the Haar probability on $\mathcal{C}$ and μ a Radon probability on K such that $\Phi \circ T^*(\mu) = \lambda$, that is, λ is the image of μ under the w^*–w^*-continuous mapping $\Phi \circ T^*$. By a well-known Sierpinski's argument ([15], [14, 14.5.1]), for every $p \in \beta\mathbb{N} \setminus \mathbb{N}$ the point mass $\delta_p \in S(\ell_\infty^*)$ is not λ-measurable. By [13, Theorem 9, p. 35] the mapping $\delta_p \circ \Phi \circ T^* : K \to \mathbb{R}$ is not μ-measurable on K, which actually means that $\{x^* \in K\colon \delta_p \circ \Phi \circ T^*(x^*) \geqslant 1\}$ is not μ-measurable (because for every $c \in \mathcal{C}$ either $\delta_p(c) = 1$ or $\delta_p(c) = 0$). As

$$\big\{x^* \in K\colon \delta_p \circ \Phi \circ T^*\big(x^*\big) \geqslant 1\big\} = \big\{x^* \in K\colon \delta_p \circ T^*\big(x^*\big) \geqslant r_0 + \delta\big\},$$

we conclude that $\delta_p \circ T^* \in X^{**}$ is not μ-measurable. So, $\delta_p \circ T^* \in X^{**}$ is a functional which is not universally measurable on Y.

(4) $\Rightarrow$ (5). Let K be a w^*-compact subset of Y and μ a Radon Borel probability on K such that there exists a functional $\psi \in X^{**}$ which fails to be μ-measurable on K. For every subset $A \subset K$ we define the "inner measure $\mu_*(A)$" as follows

$$\mu_*(A) = \sup\big\{\mu(L)\colon L \text{ a } w^*\text{-Borel subset of } K \text{ with } L \subset A\big\}.$$

It is easy to see that: (i) μ_* is monotone and $0 \leqslant \mu_*(A) \leqslant 1$, $\forall A \subset K$; (ii) if $A \subset K$, there exists a Borel subset $L \subset A$ such that $\mu(L) = \mu_*(A)$; (iii) if $\{A_n\colon n \geqslant 1\}$ is a sequence of subsets of K with $A_{n+1} \subset A_n$, then $\mu_*(\bigcap_{n \geqslant 1} A_n) = \inf_{n \geqslant 1} \mu_*(A_n)$; (iv) a subset $A \subset K$ is not μ-measurable if and only if $\mu_*(A) + \mu_*(K \setminus A) < 1$. For every $r \in \mathbb{R}$ we define

$$A_r = \big\{\xi \in K\colon \psi(\xi) > r\big\} \quad \text{and} \quad B_r = \big\{\xi \in K\colon \psi(\xi) < r\big\}.$$

Since ψ fails to be μ-measurable, there exists $r_0 \in \mathbb{R}$ such that A_{r_0} is not μ-measurable, that is, $\mu_*(A_{r_0}) + \mu_*(K \setminus A_{r_0}) < 1$. As $K \setminus A_{r_0} = \bigcap_{n \geqslant 1} B_{r_0 + \frac{1}{n}}$, we get $\mu_*(K \setminus A_{r_0}) = \inf_{n \geqslant 1} \mu_*(B_{r_0 + \frac{1}{n}})$ and so there is some $\delta_0 > 0$ such that $\mu_*(A_{r_0}) + \mu_*(B_{r_0 + \delta_0}) < 1$.

Claim. *There exists a non-empty w^*-compact subset $H \subset K$ such that, if V is a w^*-open subset of X^* with $V \cap H \neq \emptyset$, then $V \cap H$ intersects simultaneously $K \setminus A_{r_0}$ and $K \setminus B_{r_0+\delta_0}$.*

Indeed, let $L \subset A_{r_0}$ and $M \subset B_{r_0+\delta_0}$ be Borel subsets such that $\mu(L) = \mu_*(A_{r_0})$ and $\mu(M) = \mu_*(B_{r_0+\delta_0})$. Clearly, $\mu(L \cup M) \leqslant \mu(L) + \mu(M) = \mu_*(A_{r_0}) + \mu_*(B_{r_0+\delta_0}) < 1$, whence $\mu(K \setminus (L \cup M)) > 0$. Let $H \subset K \setminus (L \cup M)$ be any w^*-compact subset such that, if $\nu := \mu \restriction H$, then $\nu > 0$ and $\operatorname{supp}(\nu) = H$. Let V be a w^*-open subset with $V \cap H \neq \emptyset$. Then $\mu(V \cap H) > 0$. Assume that $V \cap H \subset A_{r_0}$. Put $L' = L \cup (V \cap H)$. Clearly, $\mu_*(A_{r_0}) \geqslant \mu(L') = \mu(L) + \mu(V \cap H) > \mu_*(A_{r_0})$, a contradiction that proves that $(K \setminus A_{r_0}) \cap (V \cap H) \neq \emptyset$. In a similar way one can prove that $(K \setminus B_{r_0+\delta_0}) \cap (V \cap H) \neq \emptyset$.

Let $\mathrm{e} > 0$ be such that $r_0 + \mathrm{e} < r_0 + \delta_0 - \mathrm{e}$ and define $r_1 := r_0 + \mathrm{e}$ and $\delta := \delta_0 - 2\mathrm{e}$. Then $\delta > 0$. By the claim, if $\mathcal{F}$ is a finite family of w^*-open subsets of X^* such that $V \cap H \neq \emptyset$, $\forall V \in \mathcal{F}$, for each $V \in \mathcal{F}$ we can find vectors $\xi_V, \eta_V \in V \cap H$ so that

$$\psi(\eta_V) < r_1 < r_1 + \delta < \psi(\xi_V).$$

Since $B(X)$ is w^*-dense in $B(X^{**})$, we can find a vector $x_{\mathcal{F}} \in B(X)$ such that

$$\langle \eta_V, x_{\mathcal{F}} \rangle < r_1 < r_1 + \delta < \langle \xi_V, x_{\mathcal{F}} \rangle, \quad \forall V \in \mathcal{F}.$$

(5) $\Rightarrow$ (6). Let H be a w^*-compact subset of Y, which is uniformly non-fragmentable for some $\delta > 0$. By using an argument similar to the one of the implication (2) $\Rightarrow$ (3), we find two sequences $\{r_m\colon m \geqslant 1\} \subset \mathbb{R}$ and $\{x_m\colon m \geqslant 1\} \subset B(X)$ such that, if

$$A_m = \big\{h \in H\colon \langle h, x_m \rangle \geqslant r_m + \delta\big\} \quad \text{and} \quad B_m = \big\{h \in H\colon \langle h, x_m \rangle \leqslant r_m\big\}, \quad m \geqslant 1,$$

then $\{(A_m, B_m)\colon m \geqslant 1\}$ is an independent sequence of w^*-closed subsets of H. By an argument of compactness, for each pair of disjoint subsets M, N of $\mathbb{N}$ we have $(\bigcap_{m\in M} A_m) \cap (\bigcap_{n\in N} B_n) \neq \emptyset$. So, we can choose $\eta_{M,N} \in (\bigcap_{m\in M} A_m) \cap (\bigcap_{n\in N} B_n)$. Clearly, $\{\eta_{M,N}\colon M, N$ disjoint subsets of $\mathbb{N}\}$ is a w^*–$\mathbb{N}$-family in H such that

$$\eta_{M,N}(x_m) \geqslant r_m + \delta, \quad \forall m \in M, \quad \text{and} \quad \eta_{M,N}(x_n) \leqslant r_n, \quad \forall n \in \mathbb{N}.$$

(6) $\Rightarrow$ (1). Let $\{\eta_{M,N}\colon M, N$ disjoint subsets of $\mathbb{N}\}$ be a w^*–$\mathbb{N}$-family in some w^*-compact subset H of Y. For each $\sigma \in \{0,1\}^{\mathbb{N}}$, let $M := \{n \in \mathbb{N}\colon \sigma(n) = 1\}$ and $N := \mathbb{N} \setminus M$, and define $h_\sigma := \eta_{M,N}$. Then, it is easy to see that $\{h_\sigma\colon \sigma \in \{0,1\}^{\mathbb{N}}\}$ is a Cantor skeleton of the w^*-compact subset $\overline{\{h_\sigma\colon \sigma \in \{0,1\}^{\mathbb{N}}\}}^{w^*} =: K \subset H$. Now it is enough to apply Lemma 2.4. $\square$

Remark 2.6. By Proposition 2.5, if Y is a w^*-compact subset of a dual Banach space X^*, then Y fulfills the property (P) if and only if Y does not contain a Cantor skeleton. Actually, this equivalence holds for the class of $\mathcal{K}$-analytic subsets of (X^*, w^*) (see Proposition 3.8). On the other hand, there exist subsets Y (non-w^*–$\mathcal{K}$-analytic) of X^* that simultaneously have the property (P) and contain a Cantor skeleton. Let us see an example. In [4, Proposition 5] we have proved the following fact: if Z is a Banach space with a copy of $\ell_1(\mathfrak{c})$, there exists a dual Banach space X^* with an isomorphic copy Y of Z such that Y has the property (P), but Y fails to have 3-control inside X^*. Thus, by [5, Proposition 3.5] Y contains a w^*–$\mathbb{N}$-family and so a Cantor skeleton.

3. Stability of the property (P)

This section is devoted to the questions: (i) Is the property (P) stable for unions, additions and products? (ii) If Y is a subset of X^* with the property (P), does the closed linear span $\overline{[Y]}$ have the property (P)? We obtain in the sequel positive answers when Y is $\mathcal{K}$-analytic in (X^*, w^*). The good behavior of the class of $\mathcal{K}$-analytic subsets is due to the following fact [13, Theorem 12, p. 126]: if X, Z are topological spaces, Y a $\mathcal{K}$-analytic subset of X and $\phi : X \to Z$ a continuous mapping, then, for every Radon Borel probability μ carried by $\phi(Y)$, there exists a Radon Borel probability ν carried by Y such that $\phi\nu = \mu$; so, by [13, Theorem 9, p. 35] a mapping $f : \phi(Y) \to \mathbb{R}$ is universally measurable on $\phi(Y)$ iff $f \circ \phi$ is universally measurable on Y.

Let us recall some topological notions. If (X, τ) is a topological space, a subset $Y \subset X$ is said to be *$\mathcal{K}$-analytic* in (X, τ) if there is an upper-semicontinuous compact set-valued map $\phi : \mathbb{N}^{\mathbb{N}} \to 2^X$ such that $\phi(\sigma)$ is compact, for every $\sigma \in \mathbb{N}^{\mathbb{N}}$, and $Y = \bigcup_{\sigma\in\mathbb{N}^{\mathbb{N}}} \phi(\sigma)$ (see [10, p. 11]). Recall that the set-valued map ϕ is said to be *upper-semicontinuous* if for each $\sigma \in \mathbb{N}^{\mathbb{N}}$ and for an open subset U of X such that $\phi(\sigma) \subset U$ there exists a neighborhood G of σ with $\phi(G) \subset U$. If (X, τ) is Hausdorff, every $\mathcal{K}$-analytic subset of X is universally measurable in X [10, pp. 42 and 346]. The union, intersection and product of a countable family of $\mathcal{K}$-analytic subsets as well as closed subsets and continuous images of $\mathcal{K}$-analytic subsets are $\mathcal{K}$-analytic.

A subset $Y \subset X^*$ of a dual Banach space X^* is said to be $w^*\mathcal{KA}$ if it is $\mathcal{K}$-analytic in (X^*, w^*).

Lemma 3.1. *Let X_i, Z_i be Hausdorff topological spaces and let $\psi_i : X_i \to Z_i$ be a universally measurable mapping for $i = 1, 2$. The mapping $\psi : X_1 \times X_2 \to Z_1 \times Z_2$ such that $\psi(x_1, x_2) = (\psi_1(x_1), \psi_2(x_2))$, $\forall (x_1, x_2) \in X_1 \times X_2$, is universally measurable.*

Proof. Let μ be a Radon Borel probability on $X_1 \times X_2$ and $\epsilon > 0$. We show that there exists a compact subset $K_\epsilon \subset X_1 \times X_2$ such that $\psi \restriction K_\epsilon$ is continuous and $\mu(K_\epsilon) \geqslant 1 - \epsilon$. Let $\mu_i := \pi_i(\mu)$, where $\pi_i : X_1 \times X_2 \to X_i$ is the canonical i-projection for $i = 1, 2$. Recall that μ_i is a Radon Borel probability on X_i, $i = 1, 2$. Moreover, for $B_i \in \mathcal{B}_0(X_i)$, $i = 1, 2$, we have $\mu_1(B_1) = \mu(B_1 \times X_2)$ and $\mu_2(B_2) = \mu(X_1 \times B_2)$. So, as ψ_i is universally measurable on X_i, there exists a compact subset $K_i \subset X_i$ such that $\psi_i \restriction K_i$ is continuous and $\mu_i(K_i) \geqslant 1 - \frac{1}{2}\epsilon$. Let $K_\epsilon := K_1 \times K_2$. Then trivially $\psi \restriction K_\epsilon$ is continuous. Moreover as ${}^cK_\epsilon = ({}^cK_1 \times X_2) \cup (X_1 \times {}^cK_2)$ we have

$$1 - \mu(K_\epsilon) = \mu\big({}^cK_\epsilon\big) \leqslant \mu\big({}^cK_1 \times X_2\big) + \mu\big(X_1 \times {}^cK_2\big) = \mu_1\big({}^cK_1\big) + \mu_2\big({}^cK_2\big) \leqslant \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.$$

Thus ψ is universally measurable. $\square$

Lemma 3.2. *Let X_1, X_2 be Banach spaces, $X = X_1 \oplus_1 X_2$ and $Y_i \subset X_i^*$ a subset fulfilling the property (P) for $i = 1, 2$. Then $Y := Y_1 \oplus Y_2 \subset X^*$ has the property (P).*

Proof. By hypothesis $X^* = X_1^* \oplus_\infty X_2^*$ and $X^{**} = X_1^{**} \oplus_1 X_2^{**}$. Let $u \in X^{**}$. Then $u = u_1 \oplus u_2$ with $u_1 \in X_i^{**}$, $u_2 \in X_2^{**}$ and $u(x_1^* \oplus x_2^*) = u_1(x_1^*) + u_2(x_2^*)$ for every $x_1^* \oplus x_2^* \in X^*$. By Proposition 2.5 u_i is universally measurable on Y_i, $i = 1, 2$. Thus the mapping $\Phi : Y_1 \oplus Y_2 \to \mathbb{R} \oplus_\infty \mathbb{R}$ such that $\Phi(y_1 \oplus y_2) = (u_1(y_1) \oplus u_2(y_2))$, $\forall y_1 \oplus y_2 \in Y_1 \oplus Y_2$, is universally measurable by Lemma 3.1. As the mapping $S : \mathbb{R} \oplus_\infty \mathbb{R} \to \mathbb{R}$ such that $S(t, s) = t + s$ is continuous, we conclude that the mapping $S \circ \Phi : Y_1 \oplus Y_2 \to \mathbb{R}$ is universally measurable. So u is universally measurable on $Y_1 \oplus Y_2$ because $u = S \circ \Phi$. Thus $Y_1 \oplus Y_2$ has the property (P) by Proposition 2.5. □

Lemma 3.3. *Let X, Z be Banach spaces, Y be a $w^*\mathcal{KA}$ subset of X^* with the property (P) and $\varphi : X^* \to Z^*$ be a w^*–w^*-continuous affine mapping. Then $\varphi(Y)$ is a $w^*\mathcal{KA}$ subset of Z^* with the property (P).*

Proof. First, it is trivial that $\varphi(Y)$ is $w^*\mathcal{KA}$. Moreover, $\psi := \varphi - \varphi(0) : X^* \to Z^*$ is a linear norm-continuous mapping. Let μ be a Radon Borel probability on $\varphi(Y)$ and $u \in Z^{**}$ a functional. We shall prove that u is μ-measurable. Since Y satisfies the property (P) and $u \circ \psi \in X^{**}$, then $u \circ \psi$ is universally measurable on Y by Proposition 2.5. Thus $u \circ \varphi = u \circ \psi + u(\varphi(0))$ is also universally measurable on Y. By [13, Theorem 12, p. 126] there exists a Radon Borel probability ν on Y such that $\varphi\nu = \mu$. Thus u is μ-measurable by [13, Theorem 9, p. 35] and so $\varphi(Y)$ has the property (P) by Proposition 2.5. □

Lemma 3.4. *Let X be a Banach space.*

(A) *If $\{U_n\colon n \geqslant 1\}$ is a sequence of universally measurable subsets of (X^*, w^*) such that $U_n \subset U_{n+1}$ and each U_n has the property (P), then $\bigcup_{n\geqslant 1} U_n$ has the property (P).*

(B) *If Y is a $w^*\mathcal{KA}$ subset of X^*, the following statements are equivalent*:

 (1) *Y has the property (P)*;

 (2) *$\mathbb{R}Y := \{ty\colon t \in \mathbb{R},\ y \in Y\}$ has the property (P).*

(C) *If $\{Y_n\colon n \geqslant 1\}$ is a sequence of $w^*\mathcal{KA}$ subsets of X^* each fulfilling the property (P), then $\bigcup_{n\geqslant 1} Y_n$ has the property (P).*

Proof. (A) Let μ be a Radon Borel probability carried by $\bigcup_{n\geqslant 1} U_n$ and $u \in X^{**}$. We want to prove that u is μ-measurable. Fix $\epsilon > 0$. Since $U_n \uparrow \bigcup_{n\geqslant 1} U_n$, there exists $p \in \mathbb{N}$ such that $\mu(U_p) > 1 - \epsilon$. Let $\nu := \mu \restriction U_p$. Clearly ν is a positive finite Radon Borel measure carried by U_p, which has the property (P). Thus u is ν-measurable and so there exists a w^*-compact subset $K \subset U_p$ such that $\mu(K) = \nu(K) > 1 - \epsilon$ and $u \restriction K$ continuous. Therefore u is Lusin μ-measurable and this proves the statement.

(B) As (2) ⇒ (1) is trivial, let us prove (1) ⇒ (2). Let $u \in X^{**}$ and let μ be a Radon Borel probability carried by $\mathbb{R}Y$. We want to prove that u is μ-measurable. Let $\Phi : \mathbb{R} \oplus_\infty X^* \to X^*$ be such that $\Phi(t \oplus x^*) = tx^*$, $\forall t \oplus x^* \in \mathbb{R} \oplus_\infty X^*$. Clearly Φ is a w^*–w^*-continuous mapping and $\Phi(\mathbb{R} \oplus Y) = \mathbb{R}Y$. As $\mathbb{R}$ is a $\mathcal{K}_\sigma$ set, $\mathbb{R} \oplus Y$ is a $w^*\mathcal{KA}$ subset of $\mathbb{R} \oplus_\infty X^* = (\mathbb{R} \oplus_1 X)^*$ and so by [13, Theorem 12, p. 126] there exists a Radon Borel probability ν carried by $\mathbb{R} \oplus Y$ such that $\Phi\nu = \mu$.

Claim. *$u \circ \Phi$ is ν-measurable.*

Indeed, as Y has the property (P), u is universally measurable on Y by Proposition 2.5 and so the mapping $\Psi : \mathbb{R} \oplus_\infty X^* \to \mathbb{R} \oplus_\infty \mathbb{R}$ such that $\Psi(t \oplus x^*) = t \oplus u(x^*)$, $\forall t \oplus x^* \in \mathbb{R} \oplus_\infty X^*$, is universally measurable on $\mathbb{R} \oplus Y$ by Lemma 3.1. As the mapping $Q : \mathbb{R} \oplus_\infty \mathbb{R} \to \mathbb{R}$ such that $Q(t \oplus s) = ts$ is continuous, we conclude that $Q \circ \Psi$ is universally measurable on $\mathbb{R} \oplus Y$ and so Lusin ν-measurable. On the other hand, $u \circ \Phi = Q \circ \Psi$. Hence $u \circ \Phi$ is ν-measurable.

Therefore u is μ-measurable by [13, Theorem 9, p. 35] and this proves that $\mathbb{R}Y$ has the property (P) by Proposition 2.5.

(C) First, $\mathbb{R}Y_1 \oplus \mathbb{R}Y_2$ is $w^*\mathcal{KA}$ and has the property (P) inside $X^* \oplus_\infty X^*$ by (B) and Lemma 3.2. As $S : X^* \oplus_\infty X^* \to X^*$ such that $S(x^* \oplus y^*) = x^* + y^*$ is a linear w^*–w^*-continuous map, then $\mathbb{R}Y_1 + \mathbb{R}Y_2$ has the property (P) by Lemma 3.3, whence we deduce that $Y_1 \cup Y_2$ has the property (P) because $Y_1 \cup Y_2 \subset \mathbb{R}Y_1 + \mathbb{R}Y_2$. So, if $U_n := \bigcup_{i=1}^n Y_i$, we get by induction that each U_n has the property (P) and is $w^*\mathcal{KA}$ and so universally measurable. Now we apply (A). □

Lemma 3.5. *Let Y be a $w^*\mathcal{KA}$ subset of ℓ_∞ such that Y contains a Cantor skeleton $\{k_\sigma\colon \sigma \in \mathcal{C}\}$ satisfying $k_\sigma(m) \leqslant 0$, if $\sigma(m) = 0$, and $k_\sigma(m) \geqslant 1$, if $\sigma(m) = 1$. Then Y fails to have the property (P).*

Proof. Assume that Y has the property (P). Let us recall the notation of the proofs of Lemmas 2.3 and 2.4, that is, $I = \{f_A\colon A \subset \{0,1\}^n$ with $|A| = 2^n - n$ and $n \in \mathbb{N}\}$, ψ, λ, $\mu = \psi(\lambda)$, $D := \psi(\mathcal{C}) \subset \{0,1\}^I$, D_m^1, D_m^0, etc. As $|I| = \aleph_0$, we may put $I := \{f_{A_m}\colon m \geqslant 1\}$ and we identify I with $\mathbb{N}$ (and so $\ell_\infty(I)$ with $\ell_\infty(\mathbb{N})$) by means of the identification of m with f_{A_m}. Let $\Phi : \ell_\infty(I) \to \ell_\infty(I)$ be such that $\Phi((x_n)_n) = ((x_n \vee 0) \wedge 1)_n$ for every $(x_n)_n \in \ell_\infty(I)$. Observe that Φ is a w^*–w^*-continuous mapping. Define $H := \overline{\{k_\sigma\colon \sigma \in \mathcal{C}\}}^{w^*}$, $L := H \cap Y$ and $L_0 := L \cap \Phi^{-1}(D)$. Clearly L and L_0 are $w^*\mathcal{KA}$ bounded subsets of X^* such that $\{k_\sigma\colon \sigma \in \mathcal{C}\} \subset L$, $\Phi(L) = \{0,1\}^I$ and $\Phi(L_0) = D$. Suppose that $\|x\| \leqslant 1 + a$, $\forall x \in L_0$, for some $a \geqslant 0$. Since μ is carried by D, by [13, Lemma 19 and Theorem 12, p. 126] there exist a Radon Borel probability ρ on L_0 and a sequence $\{L_n\colon n \geqslant 1\}$ of w^*-compact subsets of L_0 such that:

(a) $\Phi\rho = \mu$;
(b) $L_n \subset L_{n+1}$ for $n \geqslant 1$;
(c) $\rho(L_n) \uparrow 1$.

Let $u_0 \in (1+a)B(\ell_\infty(I))$ be the barycenter $r(\rho)$ of ρ.

Claim 1. $1 \leqslant \liminf_{m\to\infty} u_0(m) \leqslant 1 + a$.

Indeed, if $m \in I$, then $u_0(m) = \pi_m(u_0) = \int_{L_0} \pi_m(x^*)\,d\rho(x^*)$. So, as $u_0 \in (1+a)B(\ell_\infty(I))$ and $L_0 = (L_0 \cap \Phi^{-1}(D_m^1)) \uplus (L_0 \cap \Phi^{-1}(D_m^0))$ ($\uplus$ means disjoint union), we have

$$1 + a \geqslant \int_{L_0} \pi_m\big(x^*\big)\,d\rho\big(x^*\big) = \int_{L_0\cap\Phi^{-1}(D_m^1)} x_m^*\,d\rho\big(x^*\big) + \int_{L_0\cap\Phi^{-1}(D_m^0)} x_m^*\,d\rho\big(x^*\big).$$

As $\pi_m(x^*) = x_m^* \geqslant \pi_m(\Phi(x^*)) = 1$ on $L_0 \cap \Phi^{-1}(D_m^1)$ we have

$$\int_{L_0\cap\Phi^{-1}(D_m^1)} x_m^*\,d\rho\big(x^*\big) \geqslant \int_{L_0\cap\Phi^{-1}(D_m^1)} \pi_m\big(\Phi\big(x^*\big)\big)\,d\rho\big(x^*\big) = \int_{D_m^1} \pi_m\big(y^*\big)\,d\mu\big(y^*\big) = \mu\big(D_m^1\big) \underset{m\to\infty}{\longrightarrow} 1.$$

On the other hand, as $|x_m^*| \leqslant 1 + a$ for every $x^* \in L_0$, we have

$$\left|\int_{L_0\cap\Phi^{-1}(D_m^0)} x_m^*\,d\rho\big(x^*\big)\right| \leqslant \int_{L_0\cap\Phi^{-1}(D_m^0)} (1+a)\,d\rho\big(x^*\big) = (1+a)\int_{D_m^0} d\mu = (1+a)\mu\big(D_m^0\big) \underset{m\to\infty}{\longrightarrow} 0.$$

Thus

$$1 \leqslant \liminf_{m\to\infty} \int_{L_0} \pi_m\big(x^*\big)\,d\rho\big(x^*\big) \leqslant 1 + a.$$

Claim 2. $\liminf_{m\to\infty} u_0(m) \leqslant 0$.

Indeed, let $\rho_n := \rho \restriction L_n$ denote the restriction of ρ to L_n. Clearly $\rho_n(L_n) \uparrow 1$ when $n \to \infty$. We consider two cases.

Case 1. $\rho = \rho_q$ for some $q \in \mathbb{N}$.

In this case ρ is carried by the w^*-compact subset L_q and so $u_0 = r(\rho) \in \overline{\mathrm{co}}^{w^*}(L_q)$. Since $L_q \subset Y$ and Y fulfills the property (P), we have $u_0 \in \overline{\mathrm{co}}(L_q)$. Thus, in order to show that $\liminf_{m\to\infty} u_0(m) \leqslant 0$, it is sufficient to show that $\liminf_{m\to\infty} p(m) \leqslant 0$ for every $p \in \mathrm{co}(L_q)$. Let $p = \sum_{j=1}^k t_j l_j$, where $t_j \in [0,1]$, $\sum_{j=1}^k t_j = 1$, $l_j \in L_q$ and $\Phi(l_j) =: d_j \in D$ for $j = 1, \dots, k$. By (3) of the proof of Lemma 2.3 there exists a sequence of integers $m_1 < m_2 < \cdots$

such that $\pi_{m_r}(d_j) = 0$ for $r \geqslant 1$ and $j = 1, \ldots, k$. So, by the definition of Φ we have $\pi_{m_r}(l_j) \leqslant 0$ for $r \geqslant 1$ and $j = 1, \ldots, k$, that is, $p(m_r) \leqslant 0$ for $r \geqslant 1$ and this proves that $\liminf_{m\to\infty} p(m) \leqslant 0$.

Case 2. $\rho(L_n) < 1$ for every $n \in \mathbb{N}$.

In this case, if $\tau_n = \rho - \rho_n$, then τ_n is a positive finite Radon measure such that $\|\tau_n\| > 0$, $\forall n \geqslant 1$. Without loss of generality assume that $\|\rho_n\| > 0$, $\forall n \geqslant 1$. Then

$$u_0 = r(\rho) = \|\rho_n\| r\left(\frac{\rho_n}{\|\rho_n\|}\right) + \|\tau_n\| r\left(\frac{\tau_n}{\|\tau_n\|}\right)$$

and so

$$u_0(m) = r(\rho)(m) = \|\rho_n\| r\left(\frac{\rho_n}{\|\rho_n\|}\right)(m) + \|\tau_n\| r\left(\frac{\tau_n}{\|\tau_n\|}\right)(m)$$

for every $m \in I$. As $\rho_n/\|\rho_n\|$ is a Radon probability carried by the w^*-compact subset L_n and L_n fulfills the property (P), we have $r(\rho_n/\|\rho_n\|) \in \overline{\mathrm{co}}^{w^*}(L_n) = \overline{\mathrm{co}}(L_n)$. Hence $\liminf_{m\to\infty} r(\rho_n/\|\rho_n\|)(m) \leqslant 0$ as in the proof of Case 1. On the other hand, $r(\tau_n/\|\tau_n\|) \in (1+a)B(\ell_\infty(I))$ because $\tau_n/\|\tau_n\|$ is a Radon probability on L_0 and $L_0 \subset (1+a)B(\ell_\infty(I))$. So $|r(\tau_n/\|\tau_n\|)(m)| \leqslant 1 + a$ for every $m \in I$. Since $\|\rho_n\| \uparrow 1$ and $\|\tau_n\| \downarrow 0$ for $n \to \infty$, we get $\liminf_{m\to\infty} u_0(m) \leqslant 0$.

So we obtain a contradiction which proves the lemma. □

Lemma 3.6. *Let X be a Banach space and Y be a $w^*\mathcal{KA}$ subset of X^* fulfilling the property (P). Then $\overline{Y}$ does not contain a Cantor skeleton and so $\overline{Y}$ fulfills the property (P).*

Proof. Assume that $\overline{Y}$ contains a Cantor skeleton $\mathcal{K} := \{k_\sigma\colon \sigma \in \mathcal{C}\}$. By Remark 2.2 we may assume that $\mathcal{K}$ is a uniform Cantor skeleton, we say, for some sequence $\{x_m\colon m \geqslant 1\} \subset B(X)$ and $a_0, \epsilon \in \mathbb{R}$ with $\epsilon > 0$, we have $k_\sigma(x_m) \leqslant a_0$, if $\sigma(m) = 0$, and $k_\sigma(x_m) \geqslant a_0 + \epsilon$, if $\sigma(m) = 1$. Now we perturb $\mathcal{K}$ in order to obtain a uniform Cantor skeleton inside Y. Indeed, for each $\sigma \in \mathcal{C}$ choose $h_\sigma \in Y$ such that $\|h_\sigma - k_\sigma\| \leqslant \epsilon/4$. Then $\{h_\sigma\colon \sigma \in \mathcal{C}\}$ is a bounded subset of Y such that $h_\sigma(x_m) \leqslant a_0 + \frac{1}{4}\epsilon$, if $\sigma(m) = 0$, and $h_\sigma(x_m) \geqslant a_0 + \frac{1}{4}\epsilon + \frac{2}{4}\epsilon$, if $\sigma(m) = 1$. Define the mapping $T : X^* \to \ell_\infty(\mathbb{N})$ as follows

$$\forall x^* \in X^*, \quad T\left(x^*\right) = \left(\frac{x^*(x_m) - a_0 - \frac{1}{4}\epsilon}{\frac{2}{4}\epsilon}\right)_m.$$

Clearly the mapping T is affine norm-continuous and w^*–w^*-continuous. Observe that $\{T(h_\sigma)\colon \sigma \in \mathcal{C}\}$ is a uniform Cantor skeleton inside $T(Y)$ such that $T(h_\sigma)(m) \leqslant 0$, if $\sigma(m) = 0$, and $T(h_\sigma)(m) \geqslant 1$, if $\sigma(m) = 1$. On the other hand, $T(Y)$ is a $w^*\mathcal{KA}$ subset of $\ell_\infty(\mathbb{N})$ with the property (P) by Lemma 3.3. Thus by Lemma 3.5 we get a contradiction, which proves that $\overline{Y}$ fails to contain a Cantor skeleton. Finally $\overline{Y}$ has the property (P) by Proposition 2.5. □

Lemma 3.7. *Let X be a Banach space and Y a subset of X^*. Then*:

(1) *If Y is $w^*\mathcal{KA}$ in X^*, $[Y]$ and $\overline{[Y]}$ are $w^*\mathcal{KA}$ in X^*.*
(2) *If Y is $w^*\mathcal{KA}$ in X^* and has the property (P), then $\overline{[Y]}$ has the property (P).*

Proof. (1) As $\mathbb{R}$ is a $\mathcal{K}_\sigma$ set, then $\mathbb{R} \oplus Y$ and $\mathbb{R}Y$ are $w^*\mathcal{KA}$ in $\mathbb{R} \oplus_\infty X^*$ and X^*, respectively. Thus $\mathbb{R}Y \oplus \overset{n}{\smile} \oplus \mathbb{R}Y$ is $w^*\mathcal{KA}$ in $X^* \oplus_\infty \overset{n}{\smile} \oplus_\infty X^*$ because countable products of $\mathcal{K}$-analytic sets are $\mathcal{K}$-analytic. Since $\Phi_n : X^* \oplus_\infty \overset{n}{\smile} \oplus_\infty X^* \to X^*$ such that $\Phi_n(x_1^* \oplus \cdots \oplus x_n^*) = \sum_{i=1}^n x_i^*$ is a w^*–w^*-continuous linear mapping, then $W_n := \Phi(\mathbb{R}Y \oplus \overset{n}{\smile} \oplus \mathbb{R}Y)$ is $w^*\mathcal{KA}$ in X^*. On the other hand, $[Y] = \bigcup_{n\geqslant 1} W_n$ and $\overline{[Y]} = \bigcap_{k\geqslant 1}([Y] + \frac{1}{k}B(X^*))$. So, $[Y]$ and $\overline{[Y]}$ are $w^*\mathcal{KA}$ in X^* because finite additions as well as countable unions and intersections of $\mathcal{K}$-analytic sets are $\mathcal{K}$-analytic.

(2) With the notation of (1), each subset W_n is $w^*\mathcal{KA}$ in X^* and has the property (P) by Lemmas 3.2–3.4. By Lemma 3.4 $[Y] = \bigcup_{n\geqslant 1} W_n$ is a $w^*\mathcal{KA}$ subset in X^* with the property (P). Finally $\overline{[Y]}$ has the property (P) by Lemma 3.6. □

Proposition 3.8. *Let X be a Banach space and Y a $w^*\mathcal{KA}$ subset of X^*. The following statements are equivalent*:

(1) *Y has the property (P).*
(1′) *Y does not contain a Cantor skeleton.*
(1″) *Y does not contain a w^*–$\mathbb{N}$-family.*
(2) *$\overline{[Y]}$ has the property (P).*
(2′) *$\overline{[Y]}$ does not contain a Cantor skeleton.*
(2″) *$\overline{[Y]}$ does not contain a w^*–$\mathbb{N}$-family.*
(3) *Every convex subset of $\overline{[Y]}$ has 3-control inside X^*.*

Proof. The implications $(2) \Rightarrow (1)$, $(2') \Rightarrow (2'') \Rightarrow (1'')$ and $(2') \Rightarrow (1') \Rightarrow (1'')$ are trivial.

$(1'') \Rightarrow (1)$ follows from Proposition 2.5.

$(3) \Rightarrow (2)$. Let $K \subset \overline{[Y]}$ be a w^*-compact subset. Thus $\overline{\mathrm{co}}(K)$ is a convex subset of $\overline{[Y]}$ and so has 3-control inside X^*. Hence

$$\hat{d}\big(\overline{\mathrm{co}}^{w^*}(K), \overline{\mathrm{co}}(K)\big) \leqslant 3\hat{d}\big(K, \overline{\mathrm{co}}(K)\big) = 0.$$

Therefore $\overline{\mathrm{co}}^{w^*}(K) = \overline{\mathrm{co}}(K)$ and this proves that $\overline{[Y]}$ has the property (P).

$(1) \Rightarrow (2')$ follows from Lemmas 3.7 and 3.6.

Finally $(2'') \Rightarrow (3)$ follows from [5, Proposition 3.5]. $\square$

Remark 3.9. If Y is a subset of a dual Banach space X^* and Y has the property (P) but Y is not $w^*\mathcal{KA}$, $[Y]$ and $\overline{[Y]}$ could fail to have the property (P). Let us give a counterexample. Take $X := \ell_1$. Then $X^* = \ell_\infty$ does not have the property (P). As the unit sphere $S(\ell_\infty)$ of ℓ_∞ satisfies $|S(\ell_\infty)| = \mathfrak{c}$, we put $S(\ell_\infty) = \{s_\alpha\colon \alpha < \mathfrak{c}\}$. Let $R_\alpha := (0, 1]s_\alpha$, $\forall\alpha < \mathfrak{c}$, and observe that $|R_\alpha| = \mathfrak{c}$. Since $(B(\ell_\infty), w^*)$ is a polish space, if K is a w^*-compact subset of $B(\ell_\infty)$, either $|K| \leqslant \aleph_0$ or $|K| = \mathfrak{c}$ (see [7, 6.5. Corollary, p. 32]). Let $\mathcal{W}$ denote the family of w^*-compact subsets W of $B(\ell_\infty)$ such that $|W| = \mathfrak{c}$. It is an easy exercise to see that $|\mathcal{W}| = \mathfrak{c}$. Thus we may put $\mathcal{W} = \{W_\alpha\colon \alpha < \mathfrak{c}\}$. Now by induction we choose in $B(\ell_\infty)$ two sequences $\{r_\alpha\colon \alpha < \mathfrak{c}\}$ and $\{w_\alpha\colon \alpha < \mathfrak{c}\}$ such that $r_\alpha \in R_\alpha$, $w_\alpha \in W_\alpha$, $\forall\alpha < \mathfrak{c}$, and $\{r_\alpha\colon \alpha < \mathfrak{c}\} \cap \{w_\alpha\colon \alpha < \mathfrak{c}\} = \emptyset$. Let $Y := \{r_\alpha\colon \alpha < \mathfrak{c}\}$. Clearly $[Y] = \overline{[Y]} = \ell_\infty$. We claim that Y has the property (P). Indeed, let $K \subset Y$ be a w^*-compact subset. By construction $K \notin \mathcal{W}$. Thus $|K| \leqslant \aleph_0$, whence $\overline{[K]}$ is separable. By [5, Proposition 4.3] we have $\overline{\mathrm{co}}^{w^*}(K) = \overline{\mathrm{co}}(K)$ and so Y has the property (P). Finally Y fails to be $w^*\mathcal{KA}$ in ℓ_∞ by Proposition 3.8, although Y is weak* countably K-determined because it is metric separable.

Acknowledgment

The authors are grateful to the referee who made numerous remarks and suggestions which helped to improve this paper.

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