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Corrigendum

Corrigendum to "Approximation on Nash sets with monomial singularities" [Adv. Math. 262 (2014) 59–114]

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A R T I C L E I N F O

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Even thought the statement of [1, Lemma 6.1] is correct, there is an imprecision in the proof that we fix in this short note. In any case, the results of the paper [1] are correct as originally stated. We keep the notations introduced in [1] and we only recall a few notations and concepts here. A $\mathcal{S}^\nu$ function on a semialgebraic set X is a $\mathcal{C}^\nu$ semialgebraic functions on X for $\nu \geq 1$. The ring of $\mathcal{S}^\nu$ functions on X is denoted with $\mathcal{S}^\nu(X)$. In addition, we denote with ${}^c\mathcal{S}^\nu(X)$ the ring of ${}^c\mathcal{S}^\nu$ functions on X, that

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is, those functions on X whose restrictions to each irreducible component of X are $\mathcal{S}^\nu$ functions.

The error in the original proof of [1, Lemma 6.1] concerns the way we use Łojasiewicz's inequality. Łojasiewicz's inequality as stated in [2, 2.6.2] holds in $\mathbb{R}^n$, but is not true for an arbitrary open semialgebraic set. Thus, that proof of [1, Lemma 6.1] must be replaced by the following one.

Lemma 6.1. *Let $X = L_1 \cup \cdots \cup L_s$ be a union of coordinate linear varieties in $\mathbb{R}^m$. Then there is a continuous extension linear map ${}^{\mathrm{c}}\theta : {}^{\mathrm{c}}\mathcal{S}^\nu(X) \to \mathcal{S}^\nu(\mathbb{R}^m)$.*

Proof. Fix a non-empty set of indices $I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, s\}$ and set $L_I = \bigcap_{i\in I} L_i$ and $X^I = \bigcup_{i\in I} L_i$. By [1, Prop. 4.C.1] every ${}^{\mathrm{c}}\mathcal{S}^\nu$ function $h : X^I \to \mathbb{R}$ has the following $\mathcal{S}^\nu$ extension to $\mathbb{R}^m$

$$H_I = \sum_{\varnothing \neq J \subset I} (-1)^{\#(I)+1} h \circ \pi_J.$$

Now consider the open semialgebraic set

$$\Omega_I = \{x \in \mathbb{R}^m : \mathrm{dist}(x, L_I) < 1\} \setminus \bigcup_{j \notin I} L_j,$$

which satisfies $X^I \cap \Omega_I = X \cap \Omega_I$. Consider a semialgebraic $\mathcal{S}^\nu$ partition of unity $\{\phi, \phi_I : I\}$ subordinated to $\{\mathbb{R}^m \setminus X, \Omega_I : I\}$, which is an open semialgebraic covering of $\mathbb{R}^m$. Define

$$\begin{aligned} {}^{\mathrm{c}}\theta_I &: {}^{\mathrm{c}}\mathcal{S}^\nu(X^I) \to \mathcal{S}^\nu(\mathbb{R}^m) : h \mapsto \phi_I \cdot H_I|_{\Omega_I}, \\ {}^{\mathrm{c}}\theta &: {}^{\mathrm{c}}\mathcal{S}^\nu(X) \to \mathcal{S}^\nu(\mathbb{R}^m) : h \mapsto \textstyle\sum_I {}^{\mathrm{c}}\theta_I(h|_{X^I}) \end{aligned}$$

where each $\phi_I \cdot H_I$ extends by zero off Ω_I. Since ϕ vanishes on X, $\sum_I \phi_I \equiv 1$ on X; hence, ${}^{\mathrm{c}}\theta(h)$ is a semialgebraic $\mathcal{S}^\nu$ extension of h. To prove that ${}^{\mathrm{c}}\theta$ is continuous it is enough to show that each ${}^{\mathrm{c}}\theta_I$ is continuous. And for the latter it suffices to prove that the map

$${}^{\mathrm{c}}\mathcal{S}^\nu(X^I) \to \mathcal{S}^\nu(\mathbb{R}^m) : h \mapsto \phi_I \cdot (h \circ \pi_J)|_{\Omega_I}$$

is continuous for each $\varnothing \neq J \subset I$. Recall that we consider the topology defined in ${}^{\mathrm{c}}\mathcal{S}^\nu(X^I)$ as subset of $\mathcal{S}^\nu(L_{i_1}) \times \cdots \times \mathcal{S}^\nu(L_{i_r})$ (this is the reason to keep referring to semialgebraic ${}^{\mathrm{c}}\mathcal{S}^\nu$ functions, although we already know they are all $\mathcal{S}^\nu$ functions [1, Prop. 4.C.1]).

Clearly, it is enough to see that if all restrictions $h|_{L_i}$, $i \in I$, are close enough to zero, then $\phi_I \cdot (h \circ \pi_J)|_{\Omega_I}$ is arbitrarily close to zero. Thus, pick any positive continuous semialgebraic function $\varepsilon : \mathbb{R}^m \to \mathbb{R}$. We know from Łojasiewicz's inequality [2, 2.6.2] that there are a constant $C > 0$ and an integer p large enough so that

$$\Big|\frac{\partial^{|\gamma|}\phi_I}{\partial x^\gamma}(x)\Big| < (C+\|x\|^2)^p \quad \text{and} \quad \frac{1}{(C+\|x\|^2)^p} < \varepsilon(x) \quad \text{for every } x \in \mathbb{R}^m \text{ and } |\gamma| \le \nu.$$

Recall that $|\gamma| = \gamma_1 + \cdots + \gamma_m$. In addition, $\frac{\partial^{|\gamma|}\phi_I}{\partial x^\gamma}$ is identically zero outside Ω_I for $|\gamma| \le \nu$.

Let $x \in \Omega_I$ and $J \subset I$; in particular, $L_I \subset L_J$. Then for the orthogonal projection $\pi_J : \mathbb{R}^m \to L_J$ we have

$$\|x\|^2 = \operatorname{dist}(x, L_J)^2 + \|\pi_J(x)\|^2 \le \operatorname{dist}(x, L_I)^2 + \|\pi_J(x)\|^2 < 1 + \|\pi_J(x)\|^2$$

and so

$$\frac{1}{(C+1+\|\pi_J(x)\|^2)^p} < \frac{1}{(C+\|x\|^2)^p} < \varepsilon(x).$$

Denote $\delta(x) = \frac{1}{(\nu+1)^m(C+1+\|x\|^2)^{2p}}$ and suppose that all restrictions $h|_{L_i}$, $i \in I$, are δ close to zero in the $\mathcal{S}^\nu$ topology. Let us check that $\phi_I \cdot (h \circ \pi_J)$ is ε close to zero. Look at any partial derivative

$$\frac{\partial^{|\alpha|}(h \circ \pi_J)}{\partial x^\alpha}(x) = \frac{\partial^{|\alpha|}(h \circ \pi_J)}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}}(x), \quad \text{where } |\alpha| = \alpha_1 + \cdots + \alpha_m \le \nu,$$

at a point $x \in \Omega_I$. Since composition with π_J is substituting zero for the coordinates in L_J, we see that $h \circ \pi_J$ does not depend on those coordinates, which implies that the above partial derivative is zero whenever such a coordinate appears in the derivative. Thus we look at derivatives that do not include them. But for those, we have

$$\Big|\frac{\partial^{|\alpha|}(h \circ \pi_J)}{\partial x^\alpha}(x)\Big| = \Big|\frac{\partial^{|\alpha|}h}{\partial x^\alpha}(\pi_J(x))\Big| < \delta(\pi_J(x))$$

because $\pi_J(x) \in L_J \subset L_i$ for some $i \in I$. Consequently, if $x \in \Omega_I$,

$$\begin{aligned}\Big|\frac{\partial^{|\alpha|}\phi_I(x)(h \circ \pi_J)(x)}{\partial x^\alpha}\Big| &\le \sum_{\beta+\gamma=\alpha} \Big|\frac{\partial^{|\beta|}\phi_I}{\partial x^\beta}(x)\Big|\Big|\frac{\partial^{|\gamma|}(h \circ \pi_J)}{\partial x^\gamma}(x)\Big| \\ &< (\nu+1)^m(C+\|x\|^2)^p\delta(\pi_J(x)) \\ &< (\nu+1)^m(C+1+\|\pi_J(x)\|^2)^p\delta(\pi_J(x)) \\ &= \frac{1}{(C+1+\|\pi_J(x)\|^2)^p} < \varepsilon(x).\end{aligned}$$

As outside Ω_I each $\mathcal{S}^\nu$ function $\phi_I \cdot (h \circ \pi_J)$ is identically zero, we conclude that $\phi_I \cdot (h \circ \pi_J)$ is ε close to zero. Hence the map $h \mapsto \phi_I \cdot (h \circ \pi_J)|_{\Omega_I}$ is continuous for each $\varnothing \ne J \subset I$, as required. □

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