MATRICES COMMUTING WITH A GIVEN NORMAL TROPICAL MATRIX

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Tropical matrices over $(\mathbb{R} \cup \{-\infty\}, \oplus, \odot)$

$\oplus = \max$ is tropical sum, $\odot = +$ is tropical product

$A = (a_{ij})$, $B = (b_{ij})$ square, $A \leq B$ means $a_{ij} \leq b_{ij}$, all i, j

$$I = \begin{bmatrix} 0 & -\infty & \cdots & -\infty \\ -\infty & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\infty \\ -\infty & \cdots & -\infty & 0 \end{bmatrix}, \; 0 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix},$$

identity m.
zero matrix

DFN.: $A = (a_{ij})$ is normal if

$$I \leq A \leq 0$$

A normal $\Longrightarrow$

$$I \leq A \leq A^2 \leq A^3 \leq \cdots \leq 0$$
$$A^0 = I \leq A \leq A^2 \leq A^3 \leq \cdots \leq A^{n-1} = A^n = A^{n+1} := A^* \leq 0$$

A^* is Kleene star of A, (n is size of A)

The set $\Omega(A)$ in $(M_n^{nor}, \oplus, \odot)$

Given $A \in M_n^{nor}$ real

Question: study the set

$$\Omega(A) := \{X \in M_n^{nor} : AX = XA\}$$

If $n = 2 \Longrightarrow AB = A \oplus B = BA \quad \forall B \Longrightarrow \Omega(A) = M_2^{nor}$

Assume $n \geq 3$, then

- $\{A^0 := I, A, A^2, \ldots, A^{n-1} = A^*, 0\} \subset \Omega(A)$, (proper subset)
- $\exists \epsilon > 0$, $\exists i \neq j$ s.t. $E_{ij}(\epsilon) \in \Omega(A)$, $E_{ij}(\epsilon)$ perturbation of matrix 0

  $$E_{ij}(\epsilon)_{kl} = \begin{cases} -\epsilon, & \text{if } (i,j) = (k,l) \\ 0, & \text{otherwise} \end{cases} \text{ (not power of } A\text{)}$$

$\Omega(A)$ closed under small perturbations? Wait until end of talk!

Bijection: $M_n^{nor} \longleftrightarrow [-\infty, 0]^{n^2-n}$ has product interval TOPOLOGY

Off–diagonal entries of same size

If $\quad 2a \leq b \leq a \leq 0 \quad$ say $a, b \in \mathbb{R}$ are of the same size

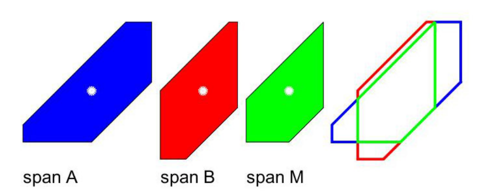
THM: For $n \in \mathbb{N}$, $r \in (-\infty, 0]$, $A, B \in M_n^{nor}$, $M := A \oplus B$

$$\text{if } 2r \leq a_{ij}, b_{ij} \leq r, \text{ all } i \neq j \Longrightarrow AB = M = BA \Longrightarrow B \in \Omega(A)$$

EXAMPLE: $r = -4$

$$A = \begin{bmatrix} 0 & -4 & -8 \\ -6 & 0 & -6 \\ -7 & -8 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & -5 & -5 \\ -6 & 0 & -8 \\ -4 & -8 & 0 \end{bmatrix}$$



$$M = \begin{bmatrix} 0 & -4 & -5 \\ -6 & 0 & -6 \\ -4 & -8 & 0 \end{bmatrix} = AB = BA$$

Geom: here $A = A^2$, $B = B^2$, $M = M^2$ are Kleene stars $\Longrightarrow$ span A, span B, span M convex and intersection given by $AB = M = BA$. Not always so easy!

w matrix of winning positions

EXAMPLE:

$$A = \begin{bmatrix} 0 & -4 & -6 & -3 \\ -6 & 0 & -4 & -3 \\ -3 & -6 & 0 & -3 \\ -6 & -3 & -3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & -4 & -4 & -3 \\ -2 & 0 & -3 & -3 \\ -3 & -6 & 0 & -3 \\ -5 & -3 & -2 & 0 \end{bmatrix} = AB = BA$$

$$B = \begin{bmatrix} 0 & -4 & -4 & -6 \\ -2 & 0 & -3 & -4 \\ -5 & -6 & 0 & -5 \\ -6 & -5 & -2 & 0 \end{bmatrix} \qquad \begin{bmatrix} & (1,1) & (1,3) & \color{red}{(4,1)} \\ (2,1) & & (2,3) & (4,2) \\ (1,3) & (2,2) & & (4,3) \\ (2,3) & (2,4) & (4,3) & \end{bmatrix} = w$$

$$(AB)_{14} = \max\{-6, -8, -11, \underline{-3}\} = (BA)_{14} = \max\{\underline{-3}, -7, -7, -6\}$$

$$\Longrightarrow w_{14} = \color{red}{(4,1)} \Longrightarrow B \in \Omega_w(A)$$

$$\text{DFN: } \Omega_w(A) := \{X \in \Omega(A) :$$
$$(AX)_{ij} = a_{i,w_{ij1}} + x_{w_{ij1},j} = (XA)_{ij} = x_{i,w_{ij2}} + a_{w_{ij2},j}, \forall i \neq j\}$$

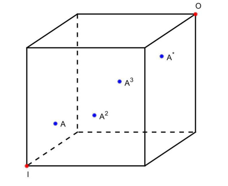
$$\text{THM: } \Omega(A) = \bigcup_{w \in W_n} \Omega_w(A) \text{ (is finite union)}$$

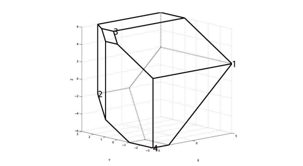
The structure of $\Omega(A) \subseteq M_n^{nor}$

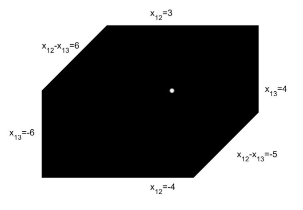
$X = (x_{ij}) \in M_n^{nor} \simeq [-\infty, 0]^{n^2-n}$ product of closed half–lines (a box)

$$\Omega_w(A) := \{X : (AX)_{ij} = a_{i,w_{ij_1}} + x_{w_{ij_1},j} = (XA)_{ij} = x_{i,w_{ij_2}} + a_{w_{ij_2},j}, \forall i \neq j\}$$

THM: $\Omega_w(A)$ is alcoved polytope, (i.e., defined by linear equations $x_{ij} = c_{ij}$ and $x_{ij} - x_{kl} = c_{ijkl}, \quad i, j, k, l \in [n], \quad i \neq j,\ k \neq l$) (thus, convex)







COR: $\Omega(A) = \bigcup_{w \in W_n} \Omega_w(A)$ is finite union of alcoved polytopes

$\Omega^S(A)$ and $\Omega'(A)$

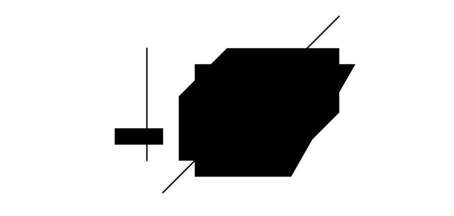
Related subsets:

$$\Omega^S(A) := \{X \in \Omega(A) : XA = AX = S\}, \text{ for } S \in M_n^{nor}$$

$$\Omega'(A) := \{X \in \Omega(A) : XA = AX = X\}$$

EXAMPLES: $A^*A = AA^* = A^*$, $0A = A0 = 0 \Longrightarrow A^*, 0 \in \Omega'(A)$
$A \in \Omega^{A^2}(A)$

THM: $\Omega^A(A)$ and $\Omega'(A)$ are finite unions of alcoved polytopes



Neighborhoods

Given $A \in M_n^{nor}$ real

DFN: $m(A) := \min_{i \neq j} a_{ij}, \qquad M(A) := \max_{i \neq j} a_{ij}$

Notation: $K(r)_{ij} = \begin{cases} 0 & \text{if } i = j \\ r & \text{otherwise} \end{cases}$ normal constant matrix r, $r \in [-\infty, 0]$

THM: $B \in \Omega^A(A)$ i.e., $BA = AB = A$ whenever $I \leq B \leq K(m(A))$. So $\Omega^A(A)$ is neighborhood of matrix I.

DFN: A strictly normal if $a_{ij} < 0 \quad \forall i \neq j \quad$ (i.e. $M(A) < 0$)

THM: Suppose A strictly normal. Then $B \in \Omega'(A)$ i.e., $BA = AB = B$ whenever $K(M(A)) \leq B \leq 0$. So $\Omega'(A)$ is neighborhood of matrix 0.

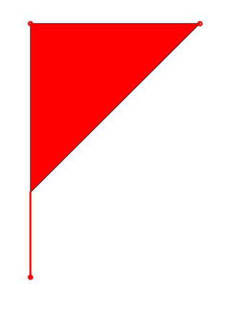
THM: Given $r \in (-\infty, 0)$, $A \in M_n^{nor}$,
if $2r < a_{ij} < r \ \forall i \neq j \Longrightarrow \Omega(A)$ is neighborhood of A

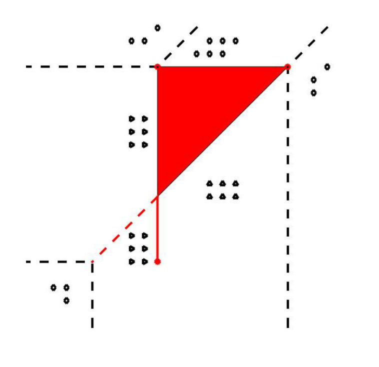
Geometry and commutativity

$A \in M_n^{nor}$ real

- span A is polyhedral complex in $\mathbb{R}^{n-1}$ non–pure dimensional
- map $f_A : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$

f_A *wraps* $\mathbb{R}^{n-1}$ around span A, *projecting* onto span A in n coord. directions

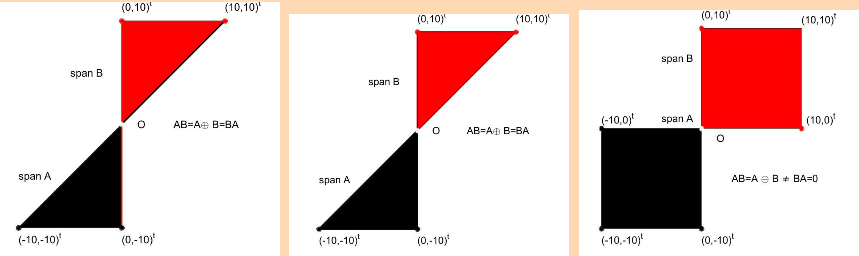




Geometry and commutativity

Properties:

- $\text{span}(AB) \subseteq \text{span}\, A$
- $\text{span}(BA) \subseteq \text{span}\, B$
- $AB = BA \Longrightarrow \text{span}(AB) \subseteq \text{span}\, A \cap \text{span}\, B$ (not equal, in general)



Example on the right: $AB \neq BA$ and B is small perturbation of B in central example $\Longrightarrow \Omega(A)$ not closed under small perturbations.

Thank you for your attention!

Большое спасибо