Volume of Alcoved Polyhedra and Mahler Conjecture

María J. de la Puente
Universidad Complutense
Facultad de Mátemáticas
Madrid, Spain
mpuente@ucm.es

Pedro L. Clavería
Universidad de Zaragoza
Zaragoza, Spain
plcv@unizar.es

ABSTRACT

The facet equations of a 3–dimensional alcoved polyhedron $\mathcal{P}$ are only of two types ($x_i = cnst$ and $x_i - x_j = cnst$) and the f–vector of $\mathcal{P}$ is bounded above by $(20, 30, 12)$. In general, $\mathcal{P}$ is a dodecahedron with 20 vertices and 30 edges. We represent an alcoved polyhedron by a real square matrix A of order 4 and we compute the exact volume of $\mathcal{P}$: it is a polynomial expression in the a_{ij}, homogeneous of degree 3 with rational coefficients. Then we compute the volume of the polar $\mathcal{P}^\circ$, when $\mathcal{P}$ is centrally symmetric. Last, we show that Mahler conjecture holds in this case: the product of the volumes of $\mathcal{P}$ and $\mathcal{P}^\circ$ is no less that $4^3/3!$, with equality only for boxes. Our proof reduces to computing a certificate of non–negativeness of a certain polynomial (in 3 variables, of degree 6, non homogeneous) on a certain simplex.

CCS CONCEPTS

• **Computing methodologies → Representation of mathematical objects**;

KEYWORDS

volume, alcoved polyhedron, tropical semiring, idempotent semiring, dioid, normal matrix, idempotent matrix, symmetric matrix, perturbation, Mahler conjecture, polynomial, certificate of non–negativeness.

ACM Reference Format:
María J. de la Puente and Pedro L. Clavería. 2018. Volume of Alcoved Polyhedra and Mahler Conjecture. In *ISSAC '18: 2018 ACM International Symposium on Symbolic and Algebraic Computation, July 16–19, 2018, New York, NY, USA.* ACM, New York, NY, USA, 8 pages. https://doi.org/10.1145/3208976.3208990

1 INTRODUCTION

The theory of polytopes is a meeting point for optimization, convex geometry, lattice geometry and geometry of numbers. There are deep results, computationally difficult problems and open conjectures concerning polytopes, even in three–dimensional space, as the collective book [22] beautifully shows. The computation of the volume is $\sharp$-P, as M.E. Dyer and A.M. Frieze (together with Valiant's result on permanents) and, independently, L. Khachiyan showed in the 1980's. The difficulty is present, no matter whether the description is by given the list of vertices or the facet equations. The Mahler conjecture (an easy–to–state question on the product of the volumes of a symmetric polytope and its polar) remains open since 1938, although progress has been reported [11, 12, 16, 24].

Volumes can be used to compute intersection numbers in algebraic geometry. More general than volume is the concept of mixed volume, introduced by H. Minkowski. In the survey paper [8], the authors ask for families of polytopes for which volume and mixed volumes can be computed. We propose alcoved polytopes (as described in this paper) for that role, as well as more general alcoved polytopes (as in [18, 19, 25]). Volumes of matroid polytopes are introduced in [2]. A lack of concrete non–trivial worked examples in these papers is filled in by the present paper.

A frequent problem in *real algebraic geometry* is checking whether a given polynomial is positive on a given semi–algebraic set. This and related questions were studied by D. Hilbert and H. Minkowski. The foundational statement is *Hilbert 17th problem.* An algebraic method to give an answer to such queries is to show a *certificate*, i.e., to express the given polynomial with a formula for which the positiveness is self–evident.

In classical linear algebra one works with matrices. A linear map between finite–dimensional spaces is represented by a matrix, after having fixed bases. Here we do something similar: an alcoved polyhedra is represented by a matrix. Then we develop a *dictionary*: properties of alcoved polyhedra are converted into properties of matrices, and viceversa. Our matrix–based technique has a big potential and could be extended to higher dimensions and to alcoved based on different root systems.

Our main result is the exact volume formula in theorem 4.1. As application, we show that *Mahler conjecture* for alcoved centrally symmetric polyhedra holds true. In this case, the conjecture is equivalent to the following assertion: the polynomial *MC* displayed in p. 7 is non–negative on a certain simplex $\mathcal{S} \subseteq \mathbb{R}^3$. We show that this is true, by finding a certificate of non–negativeness of *MC* over $\mathcal{S}$, i.e., we express *MC* as a linear combination, with non–negative coefficients, of products of polynomials w_1, w_2, w_3, w_4 such that w_j is non–negative on $\mathcal{S}$, all j. The w_j provide equations for the facets of $\mathcal{S}$.

We work with *normal* matrices operated tropically. We further need to impose *tropical idempotency* on matrices in order to achieve convexity (alcoved polytopes, in our setting). Without idempotency, we do not get one convex set, but a complex of such sets. Among NI matrices, we concentrate on two matrix families: visualized and symmetric. *Visualized matrices* are easy to work with because in them we clearly distinguish *two parts*: a *box matrix* and a *perturbation matrix*. Accordingly, the alcoved polyhedron is simply a *perturbed box*, where we are able to read off the edge–lengths.

ISSAC '18, July 16–19, 2018, New York, NY, USA

ACM ISBN 978-1-4503-5550-6/18/07…$15.00
https://doi.org/10.1145/3208976.3208990

Symmetric matrices are useful because, as our dictionary shows, an alcoved polyhedron is symmetric with respect to the origin if and only if its defining matrix is symmetric.

The oldest known volume formula is Egyptian, found in Problem 14 in the Moscow Mathematical Papyrus (ca. 1850 BC). It expresses the volume of a frustum of a square pyramid. Later, Piero della Francesca (s. XVI) and Carnot (s. XVIII) found polynomial expressions with rational coefficients for the square of the volume of a tetrahedron in terms of the edge–lengths. Both expressions can be rewritten, obtaining the Cayley–Menger determinantal formula.

Our volume formula has the same flavor, for two reasons. First, it is a polynomial formula, with rational coeffcients. Second, any alcoved polyhedron *is a tetrahedron*, in the sense that it is spanned (tropically) by four points.

Tropical mathematics (also called Idempotent mathematics or Max–plus mathematics) is the setting of this paper. There are several books and paper collections on the subject (with grown branches in analysis, geometry, algebra, etc.) For algebra, we refer the reader to [1, 4, 20]. Normal matrices can be traced back to a paper of M. Yoeli in the 1960's (under a different name). Lately, they have been used by Butcovič [4] and thoroughly studied in the thesis [21] (under a different name). Idempotent tropical matrices have been used in [14]. Visualized tropical matrices have been used in [23].

Volumes for non–square classical (i.e., non–tropical) matrices have been introduced in [3].

2 MATRICES WITH TROPICAL OPERATIONS

For $n \in \mathbb{N}$, let $[n]$ denote the set $\{1, 2, \ldots, n\}$. Consider the *tropical semiring* (also called *max–plus semiring*) $(\overline{\mathbb{R}}, \oplus, \odot)$, where $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$ is the extended real numbers and $a \oplus b = \max\{a, b\}$ and $a \odot b = a + b$, $a, b \in \overline{\mathbb{R}}$. The neutral element with respect to *tropical addition* $\oplus$ is $-\infty$, and the neutral element with respect to *tropical multiplication* $\odot$ is 0. Addition is idempotent, because $a \oplus a = \max\{a, a\} = a$, so that $(\overline{\mathbb{R}}, \oplus, \odot)$ is an *idempotent semiring* or *dioid*. Let M_n be the set of order n square matrices over $\overline{\mathbb{R}}$. The tropical operations are extended to matrices in the standard way. $(M_n, \oplus, \odot)$ is a semiring. We also add matrices classically, but we *never* multiply them classically. Therefore, we omit the symbol $\odot$ between matrices (for simplicity). For example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ -\infty & 5 \end{bmatrix}, \quad A + B = \begin{bmatrix} 0 & 2 \\ -\infty & 9 \end{bmatrix}$$

$$A \oplus B = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}, AB = \begin{bmatrix} \max\{0, -\infty\} & \max\{1, 7\} \\ \max\{2, -\infty\} & \max\{3, 9\} \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 2 & 9 \end{bmatrix}.$$

Definition 2.1. A matrix $A = [a_{ij}] \in M_n$ is *normal* if $a_{ii} = 0$ and $a_{ij} \leq 0$, all $i, j \in [n]$.

Why normal? The matrix $[a_{ij}] \in M_n$ with $a_{ii} = 0$ and $a_{ij} = -\infty$, for $i \neq j$ is denoted I_n. The all–zero matrix is denoted Z_n.

A normal matrix A satisfies $AZ_n = Z_n = Z_n A$ (but a general matrix does not). A matrix A is normal if and only if $I_n \leq A \leq Z_n$ if and only if $I_n \leq A \leq A^2 \leq A^3 \leq \cdots \leq Z_n$ (since matrix tropical multiplication is monotonic).

In the set M_n^N of normal matrices, notice that I_n is the identity for both tropical addition and multiplication in M_n^N (but not in M_n).

Restricting to work with normal matrices is advantageous and no serious limitation. Indeed, every matrix can be normalized (in a non unique way, using the *Hungarian algorithm* [17]). Normalization consists of a translation and a rearrangement of columns. This is explained below.

Definition 2.2. In M_n,

(1) a matrix $D = [d_{ij}] \in M_n$ is *diagonal* if d_{ii} is real and $d_{ij} = -\infty$, if $i \neq j$. The *inverse* of the diagonal matrix D, denoted D^{-1}, is $[a_{ij}]$ such that $a_{ii} = -d_{ii}$ and $a_{ij} = -\infty$, if $i \neq j$.
(2) For a permutation $\sigma \in S_n$, the permutation matrix $P^\sigma = [a_{ij}]$ is defined by $a_{i\sigma(i)} = 0$, $a_{ij} = -\infty$, otherwise.

Definition 2.3. For a matrix $A \in M_n$, we define

(1) $\operatorname{row}(A, j)$ to be the j–th row of A, and $\operatorname{col}(A, j)$ to be the j–th column of A, all $j \in [n]$,
(2) if $\operatorname{row}(A, n)$ is real, then the *visualization* of A is the matrix $A_0 := AD^{-1}$, where D is the diagonal $D = \operatorname{diag}(\operatorname{row}(A, n))$. Obviously, $\operatorname{row}(A_0, n)$ is all–zero.

Visualization has a meaning. Think of the columns of a matrix $A \in M_n$ as n points in n–dimensional space. Consider n parallel lines passing through these points with directional vector $(1, \ldots, 1)$. Intersect these n lines with the hyperplane $\{x_n = 0\}$ and write the coordinates of the resulting points as the columns of a matrix. This matrix is A_0. When the matrix is visualized, we have $A = A_0$.

What do we want to see? The set $\operatorname{span} A$ gathers of all tropical linear combinations of columns of $A \in M_n$

$$\operatorname{span} A := \{\lambda_1 \operatorname{col}(A, 1) \oplus \cdots \oplus \lambda_n \operatorname{col}(A, n) : \lambda_1, \ldots, \lambda_n \in \overline{\mathbb{R}}\}. \quad (1)$$

Clearly $\operatorname{span} A = \operatorname{span} A_0$ (because columns of A and A_0 are proportional).

Normality has a geometric meaning: A is normal if and only if

$$\operatorname{col}(A_0, j) \in R_j, \text{ all } j \in [n]. \quad (2)$$

where the regions

$$R_j := \{x \in \mathbb{R}^{n-1} : 0 \leq x_j \text{ and } x_k \leq x_j, \ k \in [n-1]\}, \quad j \in [n-1],$$

$$R_n := \{x \in \mathbb{R}^{n-1} : x_k \leq 0, \ k \in [n-1]\}.$$

provide a closed covering of $\mathbb{R}^{n-1} = \cup_{j=1}^n R_j$ and $\mathbb{R}^{n-1}$ is identified with $\{x_n = 0\}$ inside $\mathbb{R}^n$ (see figure 1).

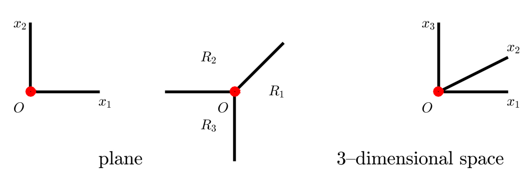

Figure 1: Coordinate axes in the plane (left), closed regions R_1, R_2, R_3 in the plane (center) and coordinate axes in 3–dimensional space (right).

Definition 2.4. A matrix $A \in M_n$ is *visualized* if $A = A_0$.

Definition 2.5. Given matrices $A, D, P^\sigma \in M_n$ with D diagonal and P^σ permutation matrix,

(1) the *translate* of A by D is the matrix DA,
(2) the *conjugate* of A by D is the matrix ${}^D A := DAD^{-1}$,

(3) the matrix AP^σ is the result of relabeling the columns of A according to the permutation σ.

In the paragraph prior to definition 2.2, we said that every matrix $M \in M_n$ can be normalized. This means that a there exists a translate DM of M, and a relabeling of columns DMP of DM, such that $A = DMP$ satisfies (2).

Definition 2.6. A matrix $A = [a_{ij}] \in M_n$ is *idempotent* if $A^2 = A$.

Thus, a matrix A is normal idempotent if and only if $I_n \leq A^2 \leq A \leq Z_n$ if and only if

$$a_{ii} = 0 \text{ and } a_{ij} + a_{jk} \leq a_{ik} \leq 0, \text{ all } i,j,k \in [n]. \tag{3}$$

Let M_n^{NI} the set of normal idempotent matrices of order n.

Corollary 2.7. *Any conjugate* ${}^DA$ *is normal (resp. idempotent), whenever* A *is.*

A matrix is *symmetric* if $A = A^T$, where A^T is the classical *transposed* matrix. Let M_n^{SNI} the set of symmetric normal idempotent matrices and M_n^{VNI} be the set of visualized normal idempotent matrices. The only symmetric and visualized normal idempotent matrix is the zero matrix Z_n.

In the rest of paper we work with normal idempotent matrices, the reason being that they represent alcoved polyhedra.

3 ALCOVED POLYHEDRA FROM NORMAL IDEMPOTENT MATRICES

In order to study the set $\operatorname{span} A \subset \mathbb{R}^n$ defined in (1), it is enough to study its intersection with the hyperplane $\{x_n = 0\}$ (identified with $\mathbb{R}^{n-1}$), because both sets determine each other

$$\mathcal{P}(A) := \operatorname{span} A \cap \{x_n = 0\}. \tag{4}$$

For a general matrix $A \in M_n$, the set $\mathcal{P}(A) \subset \mathbb{R}^{n-1}$ is a polytopal complex of impure dimension no bigger that $n-1$. However, if A is **normal idempotent**, then $\mathcal{P}(A)$ reduces to just one convex polytope and this polytope is **alcoved** (see [5, 6, 13, 15, 18, 19, 23, 25]). In other words, normality and idempotency prevents the existence of lower dimensional parts in $\mathcal{P}(A)$.

Definition 3.1. An **alcoved polytope** in $\mathbb{R}^{n-1}$ is a bounded polytope $\mathcal{P}$ whose facet equations are of type $x_i = cnst$, and $x_i - x_j = cnst$, $i,j \in [n-1]$. A **box** in $\mathbb{R}^{n-1}$ is a bounded polytope $\mathcal{B}$ whose facet equations are of type $x_i = cnst$, $i \in [n-1]$. A **cube** is a box of equal edge–lengths.

Clearly, the property of being alcoved is preserved by translation. The simplest alcoved polytopes are boxes. Every alcoved polytope is a perturbed box, more precisely, it is a **canted box**. The verb to cant means to bevel, to form an oblique surface upon something. We cant edges of boxes, always at an angle of $\pi/4$ radians (45 degrees). To cant an edge in a box means to create a new facet. 45 degrees is the angle determined by the intersecting planes $x_i = cnst$ and $x_i - x_j = cnst$. Note that Assume $n = 4$, so our box is in $\mathbb{R}^3$. Only six edges of a box are cantable: front top, top left, left back, back bottom, bottom right and right front. Note that the cantable edges are arranged in a *cycle*: $\ell_1, \ell_2, \ldots, \ell_6$ (see figure 2).

For each alcoved polytope $\mathcal{P} \subset \mathbb{R}^{n-1}$ we have a *preferred translation*; it is $v_{\mathcal{P}} : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that the origin O of $\mathbb{R}^{n-1}$ satisfies $O = \max v_{\mathcal{P}}(\mathcal{P})$. This is easily achieved matrix–wise, as the next lemma shows.

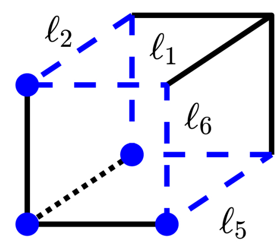

Figure 2: Cantable edges (in dashed blue lines) in a cube make a cycle: ℓ_1 is front top, ℓ_2 is top left, ℓ_3 is left back, ℓ_4 is back bottom, ℓ_5 is bottom right and ℓ_6 is right front. Generators are marked with big blue dots and labeled 1, 2, 3, 4, indicating the column in A_0 each of them is: 1 is bottom right, 2 is bottom back, 3 is top left, 4 is bottom left.

Lemma 3.2. *For* $A \in M_n^{NI}$, *the matrix* A *is visualized if and only if* $O = \max \mathcal{P}(A)$.

Proof. We know that $\mathcal{P}(A)$ is alcoved. Assume $A = A_0$ is NI. By (2) we have $\operatorname{col}(A_0, j) \in R_j \cap R_n$, all j, whence $O \in \mathcal{P}(A)$ and $\mathcal{P}(A)$ is contained in the non–positive octant R_n. Thus $O = \max \mathcal{P}(A)$. The converse is similar. □

Mahler conjecture deals with centrally symmetric bodies. For a centrally symmetric alcoved polytope $\mathcal{P} \subset \mathbb{R}^{n-1}$, a *preferred translation* is $s_{\mathcal{P}} : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that O is the center of symmetry of $s_{\mathcal{P}}(\mathcal{P})$. This is easily achieved matrix–wise, as the next lemma shows.

Lemma 3.3 (Lemma 3 in [13]). *For* $A \in M_n^{NI}$, *the matrix* A *is symmetric if and only if* $-\mathcal{P}(A) = \mathcal{P}(A)$ *(i.e.,* O *is the center of symmetry of* $\mathcal{P}(A)$*).*

Example 3.4. Given a column vector $t \in \mathbb{R}^4_{\leq 0}$ with coordinates $(t_1, t_2, t_3, 0)^T$, the following matrix (easily checked to be idempotent) is called **visualized box matrix**

$$VB(t) := \begin{bmatrix} 0 & t_1 & t_1 & t_1 \\ t_2 & 0 & t_2 & t_2 \\ t_3 & t_3 & 0 & t_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in M_4^{VNI}. \tag{5}$$

A conjugate of $VB(t)$ is the matrix $SB(t) = {}^DVB(t)$, with $D = \operatorname{diag}(-t/2)$). We have $SB(t) = (tt^T)/2 \oplus I_4 = [b_{ij}]$, with $b_{ii} = 0$ and $b_{ij} = (t_i + t_j)/2$, if $i \neq j$. The matrix $SB(t)$ is called **symmetric box matrix** and we easily get

$$SB(t)_0 = \begin{bmatrix} -t_1/2 & t_1/2 & t_1/2 & t_1/2 \\ t_2/2 & -t_2/2 & t_2/2 & t_2/2 \\ t_3/2 & t_3/2 & -t_3/2 & t_3/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{6}$$

The polyhedron $\mathcal{P}(SB(t))$ is a box in $\mathbb{R}^3$, centered at O, whose edge–lengths are $|t_1|, |t_2|, |t_3|$. The polyhedron $\mathcal{P}(VB(t))$ is a translate of $\mathcal{P}(SB(t))$, satisfying $O = \max \mathcal{P}(VB(t))$. Note that $\min \mathcal{P}(VB(t))$ is the vector t.

These boxes are cubes if $t_i = t_j$, $i \neq j$.

We will work in $\mathbb{R}^3$, from now on.

Definition 3.5 (Perturbation). For a visualized normal idempotent matrix $V = [v_{ij}] \in M_4^{VNI}$, we write $B := VB(\mathrm{col}(V,4))$ (as in (5)) and $E := B - V$. The matrix E is called **perturbation matrix** of V.

By definition, B is a visualized matrix box. We say that V is the result of **perturbing** B by E. Similarly, we say that $\mathcal{P}(B)$ is the **bounding box** of $\mathcal{P}(V)$ (see figure 3).

Notice that a perturbation matrix E is normal visualized, but not idempotent, in general.

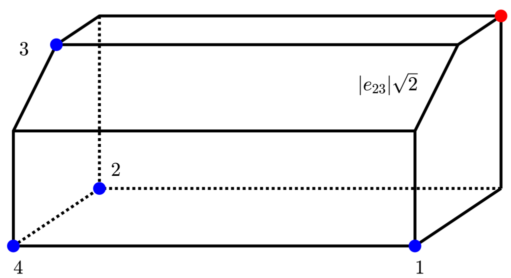

Figure 3: One cant performed on a box. The box is contained in the non–positive orthant R_3. The origin O is marked in red. Generators, i.e., columns of the defining matrix $A = A_0$ are marked in blue big dots and labeled 1,2,3,4. We have $O = \max \mathcal{P}(A)$ with $A = B - E$ and $e_{ij} = 0$ unless $(i,j) = (2,3)$. Notice the edge–length $|e_{23}|\sqrt{2}$.

Definition 3.6 (Cant tuple). For $V = B - E \in M_4^{VNI}$ as in definition 3.5, the **cant tuple** of V is $c = (c_j) \in \mathbb{R}^6_{\leq 0}$, with $c_1 := e_{23}$, $c_2 := e_{13}$, $c_3 := e_{12}$, $c_4 := e_{32}$, $c_5 := e_{31}$ and $c_6 := e_{21}$. Set $m_j := \min\{|c_j|, |c_{j+1}|\}$, $M_j := \max\{|c_j|, |c_{j+1}|\}$, $j \in [6]$ and $c_7 = c_1$. Also set $\ell_i := |v_{i4}|$, $i \in [3]$.

In summary, every alcoved polytope is a perturbed box. A box is rather degenerate (i.e., non–maximal) alcoved polyhedron.

4 VOLUME OF AN ALCOVED POLYHEDRON

In this section we show that the volume of an alcoved convex polyhedron $\mathcal{P}$ is a *cubic homogeneous polynomial in the entries v_{ij} of a defining matrix V for $\mathcal{P}$, with rational coefficients.* Multiplying by 3! we get integral coefficients.

The volume is a *valuation*, i.e., $\mathrm{vol}(\mathcal{P}_1) + \mathrm{vol}(\mathcal{P}_2) = \mathrm{vol}(\mathcal{P}_1 \cup \mathcal{P}_1) + \mathrm{vol}(\mathcal{P}_1 \cap \mathcal{P}_2)$. Further, the volume of $\mathcal{P}$ is preserved under translation, so we can assume that a defining matrix for $\mathcal{P}$ is VNI.

Theorem 4.1. *For $V = [v_{ij}] \in M_4^{VNI}$, take c_j, m_j, M_j, ℓ_j as in definition 3.6. Then the volume of the alcoved polytope $\mathcal{P}(V)$ is*

$$\mathrm{vol}\,\mathcal{P}(V) = \ell_1\ell_2\ell_3 + \sum_{j=1}^{6} \frac{m_j^2 M_j}{2} - \frac{m_j^3}{6} - \frac{c_j^2 \ell_j}{2}. \tag{7}$$

Proof. Write $V = B - E$. The volume of the bounding box is $\ell_1\ell_2\ell_3$. From this box we remove six right prisms $\mathcal{P}_j$, $j \in [6]$. The base of prism $\mathcal{P}_j$ is a right isosceles triangle legged $|c_j|$. The prisms are organized in a cycle. The intersection of two consecutive prisms has been removed twice, so it must be added once. We get

$$\mathrm{vol}\,\mathcal{P}(V) = \mathrm{vol}\,\mathcal{P}(B) - \sum_{j=1}^{6} \mathrm{vol}\,\mathcal{P}_j + \sum_{j \in [6] \mod 6} \mathrm{vol}(\mathcal{P}_j \cap \mathcal{P}_{j+1}). \tag{8}$$

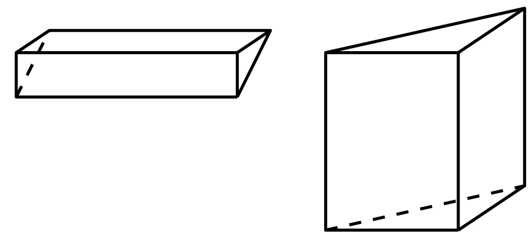

Figure 4: Two prisms $\mathcal{P}_1$ (left) and $\mathcal{P}_6$ (right) to be intersected.

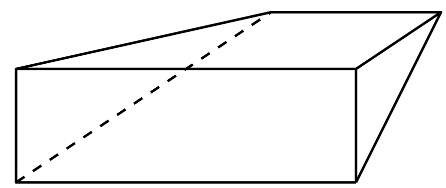

Figure 5: The wedge $\mathcal{P}_6 \cap \mathcal{P}_1$.

Clearly we have $\mathrm{vol}\,\mathcal{P}_j = \frac{c_j^2 \ell_j}{2}$, where $\ell_i = \ell_{i-3}$, $i = 4,5,6$. In addition, the intersection $\mathcal{P}_j \cap \mathcal{P}_{j+1}$ is a wedge, (i.e., a prism $\mathcal{P}_j$ from which a tetrahedron $\mathcal{T}$ has been removed) (see figures 4 and 5). Thus $\mathrm{vol}(\mathcal{P}_j \cap \mathcal{P}_{j+1}) = \mathrm{vol}\,\mathcal{P}_j - \mathrm{vol}\,\mathcal{T} = \frac{m_j^2 M_j}{2} - \frac{m_j^3}{6}$. □

After a symmetry of $\mathbb{R}^3$ (i.e., a map preserving distances and angles), each prism $\mathcal{P}_j$ is an alcoved polyhedron. The same holds for the tetrahedron $\mathcal{T}$ appearing in the former proof. Tetrahedra as such have been studied by M. Fiedler, and called *Schläfli simplex* by this author. $\mathcal{T}$ is a right simplex whose tree of legs is a path (see [7]).

Example 4.2. We want to compute the volume of the polyhedron $\mathcal{P}$ defined by the inequalities $-7 \leq x_1 \leq 1$, $-6 \leq x_2 \leq 1$, $-5 \leq x_3 \leq -3$, $-6 \leq x_1 - x_2 \leq 1$, $-8 \leq x_2 - x_3 \leq 3$, $-1 \leq x_3 - x_1 \leq 8$. A defining matrix is $A = \begin{bmatrix} 0 & -6 & -8 & -7 \\ -1 & 0 & -8 & -6 \\ -1 & -3 & 0 & -5 \\ -1 & -2 & -3 & 0 \end{bmatrix}$, which is normal and idempotent. The conjugate matrix of A by $D = \mathrm{diag}(\mathrm{row}(A,4))$ is $V = V_0 = \begin{bmatrix} 0 & -5 & -6 & -8 \\ -2 & 0 & -7 & -8 \\ -3 & -4 & 0 & -8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\mathcal{P}(V)$ is a translate of $\mathcal{P}$. To V we apply theorem 4.1: we let $V = B - E$ with perturbation matrix $E = \begin{bmatrix} 0 & -3 & -2 & 0 \\ -6 & 0 & -1 & 0 \\ -5 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and cubic box of edge–length 8. The

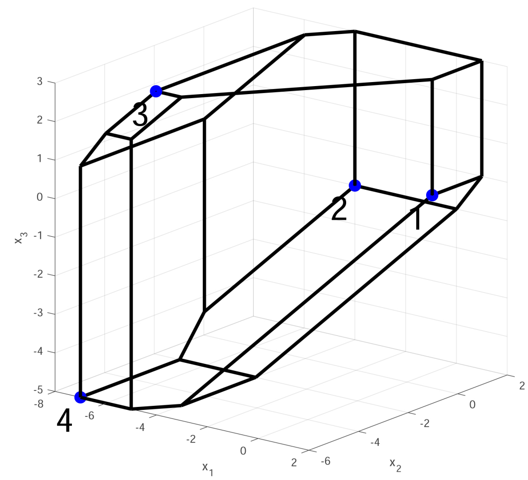

Figure 6: The polyhedron $\mathcal{P}$ in example 4.2. It is a dodecahedron. The generators, i.e., the columns of matrix A_0 are marked with big blue dots and labeled 1,2,3,4.

cant tuple is $c = (-1,-2,-3,-4,-5,-6)$, whence $m = (1,2,3,4,5,1)$, $M = (2,3,4,5,6,6)$, $\ell_j = 8$, so that

$$\sum_{j=1}^{6} \frac{m_j^2 M_j}{2} = \frac{428}{3}, \qquad \sum_{j=1}^{6} \frac{m_j^3}{6} = \frac{113}{3}, \qquad \sum_{j=1}^{6} \frac{c_j^2 \ell_j}{2} = 364$$

giving a total value $\operatorname{vol} \mathcal{P} = \operatorname{vol} \mathcal{P}(V) = 512 + \frac{428}{3} - \frac{113}{3} - 364 = \frac{760}{3}$.

Note that the former polyhedron $\mathcal{P}$ is maximal, i.e., its f–vector is $(f_0, f_1, f_2) = (20, 30, 12)$ (see [5, 6, 13, 15]) and $\mathcal{P}$ is a dodecahedron. For general n, the number f_0 of vertices is bounded above by $\binom{2n-2}{n-1}$, as proved in [6].

5 THE 2–MINORS OF A MATRIX

How do we compute the edge–lengths of the alcoved $\mathcal{P}(A)$ from the entries a_{ij} of A? The answer is easy if the matrix A is VNI: take the $b_{ij} = a_{i4}$ and the e_{ij}, for $A = B - E$, $B = [b_{ij}]$ box matrix, $E = [e_{ij}]$ perturbation matrix (see figure 3). In general, if A is only NI, the first thing we must do is computing the conjugate matrix $V = {}^D A$.

In this section we introduce the 2–minors of A and show that the entries of the conjugate matrix V are certain 2–minors of A. The same is true for the perturbation matrix E of V.

Definition 5.1. (1) The **difference** of matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $a + d - b - c$.

(2) The **2–minors of a matrix** $A = [a_{ij}]$ are the differences of the order 2 submatrices of A. If $i, j, k, l \in [n]$, $i < j$ and $k < l$, we write

$$a_{ij;kl} := a_{ik} + a_{jl} - a_{il} - a_{jk}, \quad \text{and} \tag{9}$$

$$a_{ij;kl} = -a_{ji;kl} = -a_{ij;lk} = a_{ji;lk}. \tag{10}$$

If $i = j$ or $k = l$, we write $a_{ij;kl} = 0$.

Straightforward computations yield the following.

Lemma 5.2 (Entries of conjugate matrix). *Let $V = {}^D A = [v_{ij}] \in M_4^{VNI}$ be the conjugate of $A = [a_{ij}] \in M_4^{NI}$ by $D = \operatorname{diag}(\operatorname{row}(A, 4))$ and write $V = B - E$, with perturbation matrix $E = [e_{ij}]$. Then*

$$v_{ij} = a_{4i;ij}, \qquad e_{ij} = a_{4i;j4}. \tag{11}$$

6 SYMMETRY

In this section we show that symmetry is transferred from a matrix S to the perturbation matrix E of its conjugate ${}^D S$, and conversely.

Lemma 6.1 (Conjugation and symmetry).

(1) *If $S \in M_4^{SNI}$ and $V = {}^D S = B - E$ is the conjugate of S by $D = \operatorname{diag}(\operatorname{row}(S, 4))$, then E is symmetric.*

(2) *If $V = B - E \in M_4^{VNI}$ with E symmetric, then ${}^D V \in M_4^{SNI}$, with $D = \operatorname{diag}(-v_{14}/2, -v_{24}/2, -v_{34}/2, 0)$.*

(3) *In either case if, in addition, B is a visualized cube matrix, then $s_{ij} = v_{ij}$, for $i, j \in [3]$.*

Proof. From lemma 5.2, $S = S^T$ and equalities (10), we get $e_{ij} = s_{4i;j4} = s_{j4;4i} = s_{4j;i4} = e_{ji}$, whence $E = E^T$, proving item 1.

For the proof of item 2 write $S = {}^D V$. We have $s_{ij} = v'_{ij;j4}$ if $j \in [3]$, $s_{i4} = v'_{i4}$, where $V' = [v'_{ij}]$ is an auxiliary matrix such that $v'_{i4} = v_{i4}/2$ and $v'_{ij} = v_{ij}$, otherwise. Then $S = S^T$ follows from $E = E^T$, S idempotent follows from V idempotent and S normal follows from $S = S^T$ and V normal.

Last, if B is a visualized cube matrix, then $v_{i4} = v_{j4}$, for $i, j \in [3]$, whence $s_{ij} = v'_{ij;j4} = v_{ij} + v_{j4}/2 - v_{i4}/2 - v_{jj} = v_{ij}$, proving item 3. □

Corollary 6.2 (Volume formula for V such that $E = E^T$). *If $V = B - E \in M_4^{VNI}$ with symmetric E, then*

$$\operatorname{vol} \mathcal{P}(V) = \ell_1 \ell_2 \ell_3 + \sum_{j=1}^{3} m_j^2 M_j - \frac{m_j^3}{3} - c_j^2 \ell_j. \tag{12}$$

Proof. We have $c_j = c_{j+3}$, $m_j = m_{j+3}$ and $M_j = M_{j+3}$. □

7 POLARS AND MAHLER CONJECTURE

Let $r \in \mathbb{N}$ and $p_1, p_2, \ldots, p_r$ be vectors in $\mathbb{R}^n$. Let $\langle , \rangle$ denote the standard inner product. If $\mathcal{P} = \{x \in \mathbb{R}^n : \langle x, p_k \rangle \leq 1, \forall k \in [r]\}$, then the *polar* $\mathcal{P}^\circ$ is defined as $\operatorname{conv}(p_1, p_2, \ldots, p_r)$, the *convex hull* of vectors $p_1, p_2, \ldots, p_r$; see figure 7. Moreover, if O belongs to the interior of $\mathcal{P}$, then $(\mathcal{P}^\circ)^\circ = \mathcal{P}$.

Let (v_1, v_2, v_3) be the canonical basis in $\mathbb{R}^3$ and let x_1, x_2, x_3 be coordinates in $\mathbb{R}^3$.

We know that a SNI matrix S yields a centrally symmetric alcoved polyhedron $\mathcal{P}(S)$, by lemma 3.3.

Lemma 7.1 (Polar of a centrally symmetric alcoved polyhedron). *If $S = [s_{ij}] \in M_4^{SNI}$ with $s_{ij} < 0$, all $i \neq j$, then*

$$(\mathcal{P}(S))^\circ = \operatorname{conv}\left(\pm \frac{v_i}{s_{i4}}, \pm \frac{v_i - v_j}{s_{ij}} : i, j \in [3], i \neq j\right). \tag{13}$$

Proof. We know that $s_{ij} = s_{ji} \leq 0$. The alcoved polyhedron $\mathcal{P}(S)$ is defined by $s_{i4} \leq x_i \leq -s_{i4}$ and $s_{ij} \leq x_i - x_j \leq -s_{ij}$, or equivalently,

$$-1 \leq \frac{x_i}{s_{i4}} = \left\langle x, \frac{v_i}{s_{i4}} \right\rangle \leq 1 \text{ and } -1 \leq \frac{x_i - x_j}{s_{ij}} = \left\langle x, \frac{v_i - v_j}{s_{ij}} \right\rangle \leq 1,$$

$i, j \in [3],\ i \neq j$, whence the result follows. □

A normal matrix without zeros outside the diagonal (as in the former lemma) is called *strictly normal*.

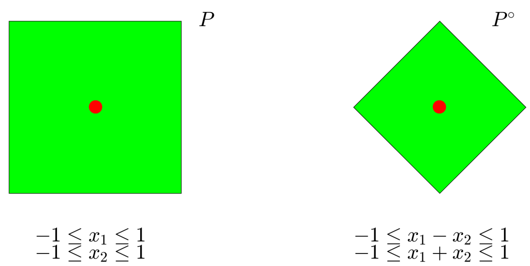

Figure 7: Unit square centered at O and its polar. The origin is marked in red

8 MAHLER CONJECTURE HOLDS FOR ALCOVED POLYHEDRA

The *Mahler volume product* of $\mathcal{P}$ is the product $\operatorname{vol}(\mathcal{P}) \operatorname{vol}(\mathcal{P}^\circ)$, by definition. It is well known that Mahler volume product is invariant with respect to affine–linear transformations and homotheties. For a centrally symmetric convex body, the *Mahler conjecture* is

$$\operatorname{vol}(\mathcal{P}) \operatorname{vol}(\mathcal{P}^\circ) \geq \frac{4^3}{3!} \tag{14}$$

with equality if and only if $\mathcal{P}$ is a box. It dates back to 1938. A recent survey on the conjecture is [16]; see also [11, 24] and the bibliography therein. A proof of the conjecture in 3–dimensional space is announced in [12].

In this section we show that Mahler conjecture holds true, for centrally symmetric alcoved polyhedra.

First, we compute the volume of the polar of a centrally symmetric alcoved polyhedron $\mathcal{P}$. We write a defining matrix $S \in M_4^{SNI}$ for $\mathcal{P}$ and, if $V = DS$ is a conjugate of S, then $\operatorname{vol} \mathcal{P} = \operatorname{vol} \mathcal{P}(V)$. Now corollary 6.2 applies to V, because the symmetry is transferred from S to $E = B - V$ (by item 1 in lemma 6.1).

It is no restriction to assume that $\mathcal{P}$ is maximal with respect to f–vector. Then the matrix $S \in M_4^{SNI}$ satisfies $s_{ij} \neq 0$, unless $i = j$. By affine invariance of the Mahler volume product, we may assume that the bounding box of $\mathcal{P}$ is the unit cube (of edge–length 2), centered at the origin. Further, we may assume that $-1 \leq e_{12} \leq e_{13} \leq e_{23} \leq 0$, without loss of generality. Thus, our matrices are

$$S = \begin{bmatrix} 0 & -2 - e_{12} & -2 - e_{13} & -1 \\ -2 - e_{21} & 0 & -2 - e_{23} & -1 \\ -2 - e_{31} & -2 - e_{31} & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix} \tag{15}$$

$$V = \begin{bmatrix} 0 & -2 - e_{12} & -2 - e_{13} & -2 \\ -2 - e_{21} & 0 & -2 - e_{23} & -2 \\ -2 - e_{31} & -2 - e_{32} & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

with $E = E^T$. For simplicity of notation, we write $x = e_{23}$, $y = e_{13}$ and $z = e_{12}$, with

$$-1 \leq z \leq y \leq x \leq 0. \tag{17}$$

The cant sequence is $c = (x, y, z, x, y, z)$ and $m = (|x|, |y|, |x|, |x|, |y|, |x|)$, $M = (|y|, |z|, |z|, |y|, |z|, |z|)$. Using formula (12), we get

$$\operatorname{vol}(\mathcal{P}) = 8 - x^2(y + z) - y^2 z + \frac{1}{3}\left(2x^3 + y^3\right) - 2\left(x^2 + y^2 + z^2\right). \tag{18}$$

Assume $y = x = 0$. We get

$$\operatorname{vol}(\mathcal{P}|_{y=x=0}) = 8 - 2z^2.$$

Indeed, $\mathcal{P}|_{y=x=0}$ is the unit cube Q (of volume 8) canted by the two planes of equations

$$\frac{x_2 - x_1}{2 + z} = \pm 1.$$

Since the polar of Q is an octahedron, then the polar of $\mathcal{P}|_{y=x=0}$ is Q° with two additional vertices:

$$\pm \frac{v_2 - v_1}{2 + z}.$$

Since both $\mathcal{P}$ and $\mathcal{P}^\circ$ are symmetric, we look at the upper half of these bodies. There, the plane $\frac{x_2 - x_1}{2+z} = 1$ yields a 4–gonal facet in $\mathcal{P}$, whence the vertex $\frac{v_2 - v_1}{2+z}$ yields 4 concurrent facets in $\mathcal{P}^\circ$. The difference between $\mathcal{P}^\circ$ and Q° is that tetrahedra with vertices 0, $\frac{v_2 - v_1}{2+z}$, v_2, $\pm v_3$ and tetrahedra with vertices 0, $\frac{v_2 - v_1}{2+z}$, $-v_1$, $\pm v_3$ have been added (to Q°) and tetrahedra with vertices 0, v_2, $-v_1$, $\pm v_3$ have been removed (from Q°). The volume of any of the added tetrahedra is $\frac{1}{6(2+z)}$ and the volume of any of the removed tetrahedra is $\frac{1}{6}$. So, when passing from Q° to $\mathcal{P}^\circ$, we have a total volume gain of

$$2\left(\frac{4}{6(2+z)} - \frac{2}{6}\right) = \frac{-2z}{3(2+z)}$$

and we get

$$\operatorname{vol}((Q)^\circ) = \frac{4}{3}, \qquad \operatorname{vol}((\mathcal{P}|_{y=x=0})^\circ) = \frac{4}{3} + \frac{-2z}{3(2+z)}.$$

Assume now $x = 0$. From (18), we have

$$\operatorname{vol}(\mathcal{P}|_{x=0}) = 8 - 2z^2 - (2 + z)y^2 + \frac{y^3}{3}.$$

Indeed, $\mathcal{P}|_{x=0}$ is the unit cube Q canted by the 4 planes of equations

$$\frac{x_2 - x_1}{2 + z} = \pm 1, \qquad \frac{x_3 - x_1}{2 + y} = \pm 1. \tag{19}$$

We look at the upper half of $\mathcal{P}$ and $\mathcal{P}^\circ$ and note that two vertices appear in the polar $\mathcal{P}^\circ$

$$\frac{v_2 - v_1}{2 + z}, \qquad \frac{v_3 - v_1}{2 + y}. \tag{20}$$

The planes (19) yield a 4–gon and an adjacent 5–gon as facets in $\mathcal{P}$. Thus, the vertices (20) yield a 4–pyramid and an adjacent 5–pyramid in $\mathcal{P}^\circ$. Some computations show that

$$\operatorname{vol}((\mathcal{P}|_{x=0})^\circ) = \frac{4}{3} + g(y, z), \qquad g(y, z) := \frac{-3y - 4z - 3yz}{3(2 + y)(2 + z)}.$$

Note that $g(0, z) = \frac{-2z}{3(2+z)}$ agrees with the case $y = x = 0$ above.

General case: similar computations show that $\mathrm{vol}(\mathcal{P}^\circ) = \frac{4}{3} + h(x,y,z)$ with $h(x,y,z) := \frac{2/3}{(2+x)(2+y)} + \frac{2/3}{(2+y)(2+z)} + \frac{2/3}{(2+z)(2+x)} + \frac{1/3}{2+y} + \frac{2/3}{2+z} - 1$ and $h(0,y,z) = g(y,z)$. We have proved

Theorem 8.1. *If $\mathcal{P} \subset \mathbb{R}^3$ is the centrally symmetric alcoved polyhedron given by the inequalities $-1 \le x_j \le 1$, $j \in [3]$, $-2-z \le x_1 - x_2 \le 2+z$, $-2-x \le x_2 - x_3 \le 2+x$, $-2-y \le x_3 - x_1 \le 2+y$, then*

$$\mathrm{vol}(\mathcal{P}^\circ) = \frac{1}{3} + \frac{2/3}{(2+x)(2+y)} + \frac{2/3}{(2+y)(2+z)} + \frac{2/3}{(2+z)(2+x)} + \frac{1/3}{2+y} + \frac{2/3}{2+z}. \tag{21}$$

Proof. $\mathcal{P} = \mathcal{P}(S)$ with $S = \begin{bmatrix} 0 & -2-z & -2-y & -1 \\ -2-z & 0 & -2-x & -1 \\ -2-y & -2-x & 0 & -1 \\ -1 & -1 & -1 & 0 \end{bmatrix}$. □

Using (18) and (21), and clearing denominators, we transform *Mahler conjecture* (14) into the question of whether the below defined polynomial MC is non–negative on the simplex $\mathcal{S}$ given by $-1 \le z \le y \le x \le 0$, where $MC = \sum_{j=1}^{6} MC_j$, with MC_j homogeneous in x,y,z of degree j:

$$\begin{aligned}
MC_6 &= 2x^4yz - 3x^3y^2z - 3x^3yz^2 + xy^4z - 3xy^3z^2,\\
MC_5 &= 8x^4y + 6x^4z - 12x^3y^2 - 23x^3yz - 9x^3z^2\\
&\quad -6x^2y^2z - 6x^2yz^2\\
&\quad +4xy^4 - 15xy^3z - 9xy^2z^2 - 6xyz^3 + 2y^4z - 6y^3z^2,\\
MC_4 &= 24x^4 - 40x^3y - 38x^3z - 30x^2y^2 - 66x^2yz\\
&\quad -24x^2z^2 - 12xy^3 - 54xy^2z\\
&\quad -24xyz^2 - 18xz^3 + 10y^4 - 34y^3z - 24y^2z^2 - 12yz^3,\\
MC_3 &= -8x^3 - 156x^2y - 144x^2z - 72xy^2 - 72xyz\\
&\quad -72xz^2 - 28y^3 - 144y^2z - 60yz^2 - 48z^3,\\
MC_2 &= -192x^2 - 96xy - 120xz - 192y^2 - 144yz - 192z^2,\\
MC_1 &= -96x - 144y - 192z.
\end{aligned}$$

In order to answer the question $MC|_{\mathcal{S}} \ge 0$, we consider the polynomials $w_1 = 1+z$, $w_2 = y-z$, $w_3 = x-y$, $w_4 = -x$, which determine the simplex $\mathcal{S}$ and are non–negative on $\mathcal{S}$. We have $x = -w_4$, $y = -w_3 - w_4$, $z = -w_2 - w_3 - w_4$ and the relation

$$1 = w_1 + w_2 + w_3 + w_4. \tag{22}$$

For each $j \in [6]$, we compute $MC_j = 1^{6-j} MC_j = (w_1 + w_2 + w_3 + w_4)^{6-j} MC_j$, a degree 6 polynomial, homogeneous in the w_j's, and we add from 1 to 6, getting the following expression

$$\begin{aligned}
MC = {} & 192w_1^5w_2 + 336w_1^5w_3 + 432w_1^5w_4 + 768w_1^4w_2^2 + 2112w_1^4w_2w_3\\
&+2472w_1^4w_2w_4 + 1152w_1^4w_3^2 + 2568w_1^4w_3w_4 + 1224w_1^4w_4^2\\
&+1200w_1^3w_2^3 + 4524w_1^3w_2^2w_3 + 5076w_1^3w_2^2w_4\\
&+4824w_1^3w_2w_3^2 + 10440w_1^3w_2w_3w_4 + 4992w_1^3w_2w_4^2\\
&+1528w_1^3w_3^3 + 4896w_1^3w_3^2w_4 + 4740w_1^3w_3w_4^2 + 1380w_1^3w_4^3\\
&+912w_1^2w_2^4 + 4392w_1^2w_2^3w_3 + 4830w_1^2w_2^3w_4 + 6960w_1^2w_2^2w_3^2\\
&+14850w_1^2w_2^2w_3w_4 + 7146w_1^2w_2^2w_4^2 + 4442w_1^2w_2w_3^3\\
&+14034w_1^2w_2w_3^2w_4 + 13656w_1^2w_2w_3w_4^2 + 4050w_1^2w_2w_4^3\\
&+972w_1^2w_3^4 + 4092w_1^2w_3^3w_4 + 6072w_1^2w_3^2w_4^2 + 3702w_1^2w_3w_4^3\\
&+774w_1^2w_4^4 + 336w_1w_2^5 + 1980w_1w_2^4w_3 + 2160w_1w_2^4w_4\\
&+4176w_1w_2^3w_3^2 + 8862w_1w_2^3w_3w_4 + 4302w_1w_2^3w_4^2\\
&+4030w_1w_2^2w_3^3 + 12657w_1w_2^2w_3^2w_4 + 12414w_1w_2^2w_3w_4^2\\
&+3744w_1w_2^2w_4^3 + 1790w_1w_2w_3^4 + 7475w_1w_2w_3^3w_4 + 11163w_1w_2w_3^2w_4^2\\
&+6918w_1w_2w_3w_4^3 + 1482w_1w_2w_4^4 + 292w_1w_3^5 + 1534w_1w_3^4w_4\\
&+3120w_1w_3^3w_4^2 + 2988w_1w_3^2w_4^3 + 1326w_1w_3w_4^4 + 216w_1w_4^5\\
&+48w_2^6 + 336w_2^5w_3 + 366w_2^5w_4 + 888w_2^4w_3^2 + 1884w_2^4w_3w_4\\
&+924w_2^4w_4^2 + 1152w_2^3w_3^3 + 3615w_2^3w_3^2w_4 + 3582w_2^3w_3w_4^2\\
&+1098w_2^3w_4^3 + 776w_2^2w_3^4 + 3233w_2^2w_3^3w_4 + 4875w_2^2w_3^2w_4^2\\
&+3072w_2^2w_3w_4^3 + 672w_2^2w_4^4 + 256w_2w_3^5 + 1340w_2w_3^4w_4\\
&+2752w_2w_3^3w_4^2 + 2682w_2w_3^2w_4^3 + 1218w_2w_3w_4^4 + 204w_2w_4^5\\
&+32w_3^6 + 204w_3^5w_4 + 540w_3^4w_4^2 + 728w_3^3w_4^3 + 516w_3^2w_4^4\\
&+180w_3w_4^5 + 24w_4^6.
\end{aligned}$$

This is a certificate of non–negativeness of MC on $\mathcal{S}$, since all the coefficients are non–negative. The only missing term in MC (as a homogeneous polynomial in w_1, w_2, w_3, w_4) is w_1^6. This means that the only real root of MC is given by $w_1 = 1$ and $w_2 = w_3 = w_4 = 0$, equivalently, by $x = y = z = 0$. This shows that *equality is only attained by boxes, among centrally–symmetric alcoved polyhedra.* We have proved *Mahler conjecture* for alcoved polyhedra. The conjecture also holds for limits of centrally symmetric alcoved polyhedra. □

Inspiration came from [9] and 2.24 in [10], for the former proof.

9 SUMMARY, FINAL REMARKS AND FUTURE DEVELOPMENTS

We have computed the volume of alcoved polyhedra and we have verified Mahler conjecture. Geometry tells us that our polyhedron is a perturbed box, more precisely, a canted box. In general, it is a dodecahedron with 20 vertices and 30 edges. Our method is to represent a d–dimensional polytope by a normal idempotent square matrix of order $n = d + 1$. Then, geometric questions about the polytope are answered through matrix computations. We have worked out the case $d = 3$. By a translation of the polyhedron, the vertex having larger coordinates can be moved to the origin. The corresponding matrix is then visualized, in which case the matrix splits as a (classical) sum of a box matrix and a perturbation matrix.

The entries of these two matrices are precisely the edge–lengths of the polyhedron. The volume formula follows from here.

What is specific of our method in dimension 3 is a natural question to ask. The particularities of dimensions $d = 2$ and 3 are various. First, we know the f–vectors of maximal alcoved polytopes, which are $(6, 6)$, for $d = 2$, and $(20, 30, 12)$ for $d = 3$. In other words, we have hexagons, for $d = 2$, and dodecahedra with 20 vertices and 30 facets, for $d = 3$. Second, we know how many cantable faces are there, and how are they organized. Indeed, for $d = 2$, we have two cantable vertices in a rectangle, and for $d = 3$ we have a 3–dimensional box with six (out of a total of 30) cantable edges, $\ell_1, \ell_2, \ldots, \ell_6$, arranged in a topological 1–dimensional sphere or cycle. At the matrix level, the counterpart are the entries e_{ij} of the perturbation matrix, organized in a cycle, renamed $(c_1, c_2, \ldots, c_6)$ and called cant tuple. The cyclic structure is essential for us to obtain formula 4.1.

What do we know for higher dimensions? For $n - 1 = d = 4$ we can prove that $(f_0, f_1, f_2, f_3) = (70, 140, 90, 20)$ is the f–vector of maximal alcoved polytopes (using simplicity and zero Euler characteristic). However, we only know two entries of this vector for $n - 1 = d > 4$, namely, the number of vertices is $f_0 = \binom{2n-2}{n-1}$ and the number of facets is $f_{n-1} = n^2 - n$, in the alcoved case. We also know that maximal alcoved polytopes are simple. Which faces are cantable and how are they arranged?

Another question is whether one can use a similar method to obtain volume formulas for polytopes arising from other root systems. Connections with Ehrhart theory could be sought. We do not have answers yet.

An application of our volume expression is the possibility of producing a formula for mixed volumes, in the alcoved case. We will report this in a subsequent paper.

Our second contribution on the volume product lower bound is a simple proof of Mahler conjecture in a particular case. We are expectant to learn whether the 67 pages long preprint by Iriyeh and Shibata provides a correct proof of the conjecture in 3 dimensions.

ACKNOWLEDGMENTS

The first author has been partially supported by Ministerio de Economía y Competitividad, Proyecto I+D MTM2016–76808–P, www.mat.ucm.es/~cruizb/Investigacion/Proyecto-Web-16.

REFERENCES

[1] M. Akian, R. Bapat, and S. Gaubert. 2007. Max–plus algebra. In *Handbook of linear algebra*, L. Hobgen (Ed.). Chapman and Hall, Chapter 25.

[2] F. Ardila, C. Benedetti, and J. Doker. 2010. Matroid polytopes and their volumes. *Discrete and Computational Geometry* 43, 4 (June 2010), 841–854. arXiv:0810.3947v3(2011)

[3] A. Ben-Israel. 1992. A volume associated with $m \times n$ matrices. *Linear Algebra Appl.* 167 (1992), 87–111.

[4] P. Butkovič. 2010. *Max–plus linear systems: theory and algorithms*. Springer.

[5] M.J. de la Puente. 2013. On tropical Kleene star matrices and alcoved polytopes. *Kybernetika* 49, 6 (2013), 897–010.

[6] M. Develin and B. Sturmfels. 2004. Tropical convexity. *Doc. Math.* 9 (2004), 1–27. Erratum in Doc. Math. **9** (electronic), (2004) 205–206.

[7] M. Fiedler. 2007. Some applications of matrices and graphs in Euclidean Geometry. In *Handbook of linear algebra*, L. Hobgen (Ed.). Chapman and Hall, Chapter 66.

[8] P. Gritzmann and V. Klee. 1994. On the complexity of some basic problems in computational convexity II: Volume and Mixed Volume. In *Polytopes: abstract, convex and computational*, T. Bisztriczky, P. McMullen, R. Schneider, and I. Weiss (Eds.). NATO ASI Series, SeiresC: Mathematical and Physical Sciences, Vol. 440. Kluwer Academic Publishers, 373–466.

[9] D.F. Handelman. 1988. Representing polynomials by positive linear functions on compact convex polyhedra. *Pacific J. Math* 132, 1 (1988), 35–62.

[10] G.H. Hardy, J.E. Littlewood, and G. Pólya. 2010. *Inequalities* (2nd ed.). Cambridge Univ. Press.

[11] M. Henze. 2008. *The Mahler conjecture*. Master's thesis. Fakultat für Mathematik, Institut für Algebra und Geometrie.

[12] H. Iriyeh and M. Shibata. 2017. Symmetric Mahler conjecture for the volume product in the three dimensional case. (Oct. 2017). https://arxiv.org/abs/1706.01749

[13] A. Jiménez and M.J. de la Puente. 2012. Six combinatorial clases of maximal convex tropical polyhedra. (Oct. 2012). https://arxiv.org/abs/1706.01749v2

[14] M. Johnson and M. Kambites. 2014. Idempotent tropical matrices and finite metric spaces. *Advances in Geometry* 14, 2 (2014). https://doi.org/10.1515/advgeom-2013-0034 arXiv: 1203.2480, 2012.

[15] M. Joswig and K. Kulas. 2010. Tropical and ordinary convexity combined. *Advances in Geometry* 10 (2010), 333-352.

[16] E. Makai Jr. 2015. The recent status of the volume product problem. (2015). https://arxiv.org/abs/1507.01473

[17] H.W. Kuhn. 1955. The Hungarian method for the assignment problem. *Naval Research Logistics* 2 (1955), 83–97.

[18] T. Lam and A. Postnikov. 2007. Alcoved polytopes I. *Discrete and Computational Geometry* 38, 3 (2007), 453–478.

[19] T. Lam and A. Postnikov. 2012. Alcoved polytopes II. (2012). arXiv:1202.4015v1

[20] G.L. Litvinov and S.N. Sergeev (Eds.). 2009. *Tropical and Idempotent Mathematics. International workshop tropical -07*. Contemporary Mathematics, Vol. 495. AMS. ISBN: 987-0-8218-4782-4.

[21] A. Miné. 2004. *Domaines numériques abstraits faiblement relationnels*. Ph.D. Dissertation. L'École Polytechnique, Université Paris IX Dauphine.

[22] M. Senechal (Ed.). 2012. *Shaping Space. Exploring Polyhedra in Nature, Art, and the Geometrical Imagination*. Springer.

[23] S. Sergeev, H. Scheneider, and P. Butkovič. 2009. On visualization, subeigenvectors and Kleene stars in max algebra. *Linear Algebra Appl.* 431 (2009), 2395–2406.

[24] T. Tao. 2007. Open question: the Mahler conjecture on convex bodies. (2007). https://terrytao.wordpress.com/2007/03/08/open-problem-the-mahler-conjecture-on-convex-bodies/

[25] A. Werner and J. Yu. 2014. Symmetric alcoved polytopes. *The electronic journal of combinatorics* 21, 1 (2014). arXiv:1201.4378v1(2012)