Matrices commuting with a given normal tropical matrix

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Abstract

Consider the space M_n^{nor} of square normal matrices $X = (x_{ij})$ over $\mathbb{R} \cup \{-\infty\}$, i.e., $-\infty \leq x_{ij} \leq 0$ and $x_{ii} = 0$. Endow M_n^{nor} with the tropical sum $\oplus$ and multiplication $\odot$. Fix a real matrix $A \in M_n^{nor}$ and consider the set $\Omega(A)$ of matrices in M_n^{nor} which commute with A. We prove that $\Omega(A)$ is a finite union of alcoved polytopes; in particular, $\Omega(A)$ is a finite union of convex sets. The set $\Omega^A(A)$ of X such that $A \odot X = X \odot A = A$ is also a finite union of alcoved polytopes. The same is true for the set $\Omega'(A)$ of X such that $A \odot X = X \odot A = X$.

A topology is given to M_n^{nor}. Then, the set $\Omega^A(A)$ is a neighborhood of the identity matrix I. If A is strictly normal, then $\Omega'(A)$ is a neighborhood of the zero matrix. In one case, $\Omega(A)$ is a neighborhood of A. We give an upper bound for the dimension of $\Omega'(A)$. We explore the relationship between the polyhedral complexes $\operatorname{span} A$, $\operatorname{span} X$ and $\operatorname{span}(AX)$, when A and X commute. Two matrices, denoted $\underline{A}$ and $\overline{A}$, arise from A, in connection with $\Omega(A)$. The geometric meaning of them is given in detail, for one example. We produce examples of matrices which commute, in any dimension.

1 Introduction

Let $n \in \mathbb{N}$ and K be a field. Fix a matrix $A \in M_n(K)$ and consider $K[A]$, the algebra of polynomial expressions in A. In classical mathematics, the set $\Omega(A)$ of

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matrices commuting with A is well–known: $\Omega(A)$ equals $K[A]$ if and only if the characteristic and minimal polynomials of A coincide. Otherwise, $K[A]$ is a proper linear subspace of $\Omega(A)$; see [27], chap. VII.

In this paper we study the analogous of $\Omega(A)$ in the tropical setting. Moreover, we restrict ourselves to square *normal* matrices over $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$, i.e., matrices $A = (a_{ij})$ with $a_{ii} = 0$ and $-\infty \leq a_{ij} \leq 0$, for all i, j. The set of all such matrices, endowed with the tropical operations $\oplus = \max$ and $\odot = +$, is denoted M_n^{nor}.

For any $r \in \mathbb{R}_{\leq 0}$, the half–line $[-\infty, r) := \{x : -\infty \leq x < r\}$ is open in $\overline{\mathbb{R}}_{\leq 0}$ with the usual interval topology. A Cartesian product of such half–lines is open in $\overline{\mathbb{R}}_{\leq 0}^{n^2-n}$ with the usual product topology. The half–line $(r, 0] := \{x : r < x \leq 0\}$ is open in $\overline{\mathbb{R}}_{\leq 0}$. A Cartesian product of such half–lines is open in $\overline{\mathbb{R}}_{\leq 0}^{n^2-n}$.

The set M_n^{nor} can be identified with $\overline{\mathbb{R}}_{\leq 0}^{n^2-n}$ and, via this identification, M_n^{nor} gets a *topology.* Consider a matrix $X \in M_n^{nor}$ and a subset $V \subseteq M_n^{nor}$. We say that V is a *neighborhood* of X if there exists an open subset $U \subseteq M_n^{nor}$ such that $X \in U \subseteq V$ (we do not require V to be open).

Let $\Omega(A)$ be the subset of matrices commuting with a given real matrix A, i.e., $X \in M_n^{nor}$ such that $A \odot X = X \odot A$. The tropical analog of $K[A]$ inside M_n^{nor} is the set $\mathcal{P}(A)$ of tropical powers of A. In general, $\Omega(A)$ is larger than $\mathcal{P}(A)$ (see proposition 1).

Our new results are gathered in sections 3, 4 and 5. In section 3 we prove that

$$\Omega(A) = \bigcup_w \Omega_w(A)$$

is a finite union of alcoved polytopes, (see corollary 5). In particular, $\Omega(A)$ is a finite union of convex sets.

Two important subsets of $\Omega(A)$ are

$$\Omega^A(A) = \{X \in \Omega(A) : X \odot A = A \odot X = A\}$$

and

$$\Omega'(A) = \{X \in \Omega(A) : X \odot A = A \odot X = X\}.$$

Both are finite unions of alcoved polytopes (see theorems 9 and 12). Moreover, $\Omega^A(A)$ is a neighborhood (not necessarily open) of the identity matrix I. If A is strictly normal, then $\Omega'(A)$ is a neighborhood of the zero matrix 0 (see propositions 7 and 8).

The study of $\Omega^A(A)$ and $\Omega'(A)$ lead us to two matrices arising from A, denoted $\underline{A}$ and $\overline{A}$, and we prove

$$\underline{A} \leq A \leq \overline{A},$$

(see proposition 17). Moreover, $X \leq \underline{A}$ is a necessary condition for $A \odot X = X \odot A = A$, and $\overline{A} \leq X$ is a necessary condition for $A \odot X = X \odot A = X$ (see corollary 15). This provides an upper bound for the dimension of $\Omega'(A)$ (see corollary 16). The matrix $\underline{A}$ is explicitly given in expression (19), while the definition and computation of $\overline{A}$ is more involved (see definition 14).

In section 4 we study some instances of commutativity of matrices under perturbations. Theorem 20 is an easy way to produce two real matrices in M_n^{nor} which commute. Another way to obtain two such matrices is given in theorem 22. The geometry is different in both instances: in the first case, the polyhedral complexes (i.e., tropical column spans) associated to the matrices are convex, but not so in the second. Under certain hypothesis we prove that $\Omega(A)$ is a neighborhood of A (see corollary 21).

Section 5 has an exploratory nature. We examine the relationship among the complexes $\operatorname{span} A$, $\operatorname{span} B$, $\operatorname{span}(AB)$ and $\operatorname{span}(BA)$ when commutativity is present or absent. In addition, the geometric meaning of the matrices $\underline{A}$, A and $\overline{A}$ is given in full detail, for one example in the paper. We believe that classical convexity of $\operatorname{span} A$ depends on the matrices $\underline{A}$ and $\overline{A}$. We suspect that this is related to the question of commutativity. We leave two open questions in pages p. 15 and 20.

Alcoved polytopes play a crucial role in this paper. By definition, a polytope $\mathcal{P}$ in $\mathbb{R}^{n-1}$ is *alcoved* if it can be described by inequalities $c_i \leq x_i \leq b_i$ and $c_{ik} \leq x_i - x_k \leq b_{ik}$, for some $i, k \in [n-1]$, $i \neq k$, and $c_i, b_i, c_{ik}, b_{ik} \in \mathbb{R} \cup \{\pm\infty\}$. They are classically convex sets. Alcoved polytopes have been studied in [22, 23]. In connection with tropical mathematics, they appeared in [17, 18, 19, 29, 36]. *Kleene stars* are matrices A such that $A = A^*$, where $*$ is the so–called Kleene operator. Alcoved polytopes and Kleene stars are closely related notions; see [29, 32, 33]. See also [10] for tropical convexity issues.

By definition, a matrix $A = (a_{ij})$ over $\overline{\mathbb{R}}$ is *normal* if $a_{ii} = 0$ and $-\infty \leq a_{ij} \leq 0$, for all i, j. It is *strictly normal* if, in addition, $-\infty \leq a_{ij} < 0$, for all $i \neq j$. There are FOUR REASONS for us to restrict to normal matrices. First, it is not all too restrictive. Indeed, by the Hungarian Method (see [5, 6, 21, 26]), for every matrix A there exist a (not unique) similar matrix N which is normal. In practice, this means that by a relabeling of the columns of A and a translation, any A can be assumed to be normal. Second, normality of A has a clear geometric meaning in $\mathbb{R}^{n-1}$. Consider the alcoved polytope

$$C_A := \left\{ x \in \mathbb{R}^{n-1} : \begin{matrix} a_{in} \leq x_i \leq -a_{ni} \\ a_{ik} \leq x_i - x_k \leq -a_{ki} \end{matrix} ;\ 1 \leq i \neq k \leq n-1 \right\}. \qquad (1)$$

Then, A is normal if and only if the zero vector belongs to C_A and the columns of the matrix A_0 (see definition in p. 5), viewed as points in $\mathbb{R}^{n-1}$, lie around the zero vector and are listed in a predetermined order (and this order is a kind of orientation in $\mathbb{R}^{n-1}$); see [29] and also [14, 15, 16]. Third, when computing examples, normal matrices are easy to handle, due to inequalities (2). Fourth and last, normal matrices satisfy many max–plus properties (e.g., they are strongly definite; see [6, 7]).

Some aspects of commutativity in tropical algebra (also called max–plus algebra or max–algebra) have been addressed earlier. It is known that two commuting matrices have a common eigenvector; see [7], sections 4.7, 5.3.5 and 9.2.2. In [20] it is proved that the critical digraphs of two commuting irreducible matrices have a common node.

2 Background and notations

For $n \in \mathbb{N}$, set $[n] := \{1, 2, \ldots, n\}$. Let $\mathbb{R}_{\leq 0}$, $\mathbb{R}_{\geq 0}$, $\overline{\mathbb{R}}_{\leq 0}$, etc. have the obvious meaning. On $\overline{\mathbb{R}}_{\leq 0}$, i.e., on the closed unbounded half–line $[-\infty, 0]$, we consider the *interval topology*: an open set in $[-\infty, 0]$ is either a finite intersection or an arbitrary union of sets of the form $[-\infty, a)$ or $(b, 0]$, with $-\infty < a, b < 0$.

$\oplus = \max$ is the tropical sum and $\odot = +$ is the tropical product. For instance, $3 \oplus (-7) = 3$ and $3 \odot (-7) = -4$. Define tropical sum and product of matrices following the same rules of classical linear algebra, but replacing addition (multiplication) by tropical addition (multiplication). Consider order n square matrices. The *tropical multiplicative identity* is $I = (\alpha_{ij})$, with $\alpha_{ii} = 0$ and $\alpha_{ij} = -\infty$, for $i \neq j$. The *zero matrix* is denoted 0 (every entry of it is null). We will never use classical multiplication of matrices; thus $A \odot X$ will be written AX, for matrices A, X, for simplicity.

If $A = (a_{ij})$ and $B = (b_{ij})$ are matrices of the same order, then $A \leq B$ means $a_{ij} \leq b_{ij}$, for all i, j.

By definition, a square matrix $A = (a_{ij})$ over $\overline{\mathbb{R}}$ is *normal* if $a_{ii} = 0$ and $-\infty \leq a_{ij} \leq 0$, for all i, j. Thus, A is normal if and only if $I \leq A \leq 0$. Let us define A^0 to be the identity matrix I. So we have

$$I = A^0 \leq A \leq A^2 \leq A^3 \leq \cdots \leq 0 \tag{2}$$

since tropical multiplication by any matrix is monotonic (because it amounts to computing certain sums and maxima). By a theorem of Yoeli's (see [37]), we have $A^{n-1} = A^n = A^{n+1} = \cdots$ and we denote this matrix by A^* and call it the *Kleene star* of A. A matrix A is a Kleene star if $A = A^*$.

A normal matrix A is *strictly normal* if $a_{ij} < 0$, whenever $i \neq j$.

Let M_n^{nor} denote the family of order n normal matrices over $\overline{\mathbb{R}}$. It is in bijective correspondence with $\overline{\mathbb{R}}_{\leq 0}^{n^2-n}$. We consider the *product interval topology* on $\overline{\mathbb{R}}_{\leq 0}^{n^2-n}$.

The bijection carries this topology onto M_n^{nor}. The *border* of M_n^{nor} is the set of matrices A such that $a_{ij} = 0$ or $-\infty$, for some $i \neq j$.

We will write the coordinates of points in $\mathbb{R}^n$ in columns. Let $A \in \mathbb{R}^{n \times m}$ and denote by $a_1, \ldots, a_m \in \mathbb{R}^n$ the columns of A. The *(tropical column) span* of A is, by definition,

$$\begin{aligned} \operatorname{span} A: \quad &= \{(\mu_1 \odot a_1) \oplus \cdots \oplus (\mu_m \odot a_m) \in \mathbb{R}^n : \mu_1, \ldots, \mu_m \in \mathbb{R}\} \\ &= \max\{\mu_1 u + a_1, \ldots, \mu_m u + a_m : \mu_1, \ldots, \mu_m \in \mathbb{R}\} \end{aligned} \tag{3}$$

where $u = (1, \ldots, 1)^t$ and maxima are computed coordinatewise. We will never use classical linear spans in this paper. Clearly, the set $\operatorname{span} A$ is closed under classical addition of the vector μu, for $\mu \in \mathbb{R}$, since $\odot = +$. Therefore, the hyperplane section $\{x_n = 0\} \cap \operatorname{span} A$ determines $\operatorname{span} A$ completely. The set $\{x_n = 0\} \cap \operatorname{span} A$ is a connected polyhedral complex of impure dimension $\leq n-1$ and it is not convex, in general. Let A be normal. Then $\operatorname{span} A = C_A$ in (1) (and so it is convex) if and only if A is a Kleene–star; see [29, 32]. Throughout the paper, we will identify the hyperplane $\{x_n = 0\}$ inside $\mathbb{R}^n$ with $\mathbb{R}^{n-1}$. In particular, columns of order n matrices having zero last row are considered as points in $\mathbb{R}^{n-1}$.

For any $d \in \mathbb{R}^n$, $\operatorname{diag} d$ denotes the square matrix whose diagonal is d and is $-\infty$ elsewhere.

For any real matrix A, the matrix A_0 is defined as the tropical product

$$A \operatorname{diag}(-\operatorname{row}(A, n)). \tag{4}$$

Thus, the j–th column of A_0 is a tropical multiple of the corresponding column of A (i.e., the j–th column of A_0 is the sum of the vector $-a_{nj}u$ and the j–th column of A). The last row of A_0 is zero. Therefore, the matrix A_0 is used to draw the complex $\{x_n = 0\} \cap \operatorname{span} A$ inside $\mathbb{R}^{n-1}$. The sets $\operatorname{span} A$ and $\{x_n = 0\} \cap \operatorname{span} A$ determine each other.

The simplest objects in the tropical plane $\overline{\mathbb{R}}^2$ are lines. Given a tropical linear form

$$p_1 \odot X \oplus p_2 \odot Y \oplus p_3 = \max\{p_1 + X, p_2 + Y, p_3\}$$

a *tropical line* consists of the points $(x, y)^t$ where this *maximum is attained, at least, twice*. Such twice–attained–maximum condition is the tropical analog of the classical vanishing point set. Denote this line by L_p, where $p = (p_1, p_2, p_3) \in \mathbb{R}^3$. Lines in the tropical plane are *tripods*. Indeed, L_p is the union of three rays meeting at point $(p_3 - p_1, p_3 - p_2)^t$, in the directions west, south and north–east. The point is called the *vertex* of L_p.

Take $p = 0$. The line L_0 splits the plane $\overline{\mathbb{R}}^2$ into three closed sectors $S_1 := \{x \geq 0,\ x \geq y\}$, $S_2 := \{x \leq y,\ y \geq 0\}$ and $S_3 := \{x \leq 0,\ y \leq 0\}$. An order 3 real matrix A is normal if and only if (omitting the last row in A_0, which is zero) each column of A_0 lies in the corresponding sector i.e., $\mathrm{col}(A_0, j) \in S_j$, for $j = 1, 2, 3$. For instance, consider the normal matrix B and take B_0 in example 11, figure 3 top centre, p. 20. Notice that $(5, 1)^t \in S_1$, $(-3, 0)^t \in S_2$ and $(-1, -6)^t \in S_3$. An analogous statement holds for $\overline{\mathbb{R}}^{n-1}$ and order n matrices. See [3, 4, 11, 12, 13, 24, 25, 30, 34] for an introduction to tropical geometry. See [1, 2, 5, 7, 8, 9, 35, 38] for an introduction to tropical (or max–plus) algebra.

3 Normal matrices which commute with A

The set M_2^{nor} is commutative, since $AB = BA = A \oplus B$, for any $A, B \in M_2^{nor}$. Thus, we will study the set

$$\Omega(A) := \{X \in M_n^{nor} : AX = XA\}, \tag{5}$$

for a real matrix $A \in M_n^{nor}$ and $n \geq 3$.

If $A \in M_n^{nor}$ is real and $\lambda \in \mathbb{R}$, then $\lambda \odot A = \lambda u + A$ is normal if and only if $\lambda = 0$, where u denotes the order n one matrix. Together with (2), this means that the tropical analog of $K[A]$ inside M_n^{nor} is the set of powers of A together with the zero matrix

$$\mathcal{P}(A) := \{I = A^0, A, A^2, \dots, A^{n-1} = A^*, 0\}. \tag{6}$$

For $A \in M_n^{nor}$ real, set

$$m(A) := \min_{i,j \in [n]} a_{ij} = \min_{i \neq j \in [n]} a_{ij} \in \mathbb{R}_{\leq 0}, \qquad M(A) := \max_{i \neq j \in [n]} a_{ij} \in \mathbb{R}_{\leq 0}. \tag{7}$$

For each $r \in \mathbb{R}$, and $i, j \in [n]$, $i \neq j$, let $E_{ij}(r) \in M_n^{nor}$ denote the matrix whose (i, j) entry equals r, being zero everywhere else. For a generic $A \in M_n^{nor}$ the matrix $E_{ij}(r)$ is not a power of A.

The following proposition shows that, in general, $\Omega(A)$ is larger than $\mathcal{P}(A)$.

Proposition 1. *For any real $A \in M_n^{nor}$ there exist $\epsilon > 0$ and $i, j \in [n]$ with $i \neq j$ such that $E_{ij}(-\epsilon) \in \Omega(A)$.*

Proof. Fix i, j and ϵ. We have $AE_{ij}(-\epsilon) = E_{ij}(\alpha)$ and $E_{ij}(-\epsilon)A = E_{ij}(\beta)$, where

$\alpha = \max\{a_{i1}, \dots, a_{i,i-1}, -\epsilon, a_{i,i+1}, \dots, a_{in}\}$ and
$\beta = \max\{a_{1j}, \dots, a_{j-1,j}, -\epsilon, a_{j+1,j}, \dots, a_{nj}\}$.

If $a_{ij} = 0$, then $\alpha = \beta = a_{ij} = 0$, whence $AE_{ij}(-\epsilon) = E_{ij}(-\epsilon)A = 0$.

Assume now that A is strictly normal. Then $M(A) < 0$. For any ϵ with $M(A) < -\epsilon < 0$ and any $i \neq j$, we have $\alpha = \beta = -\epsilon$, whence $AE_{ij}(-\epsilon) = E_{ij}(-\epsilon)A = E_{ij}(-\epsilon)$. □

Let W_n be the set of empty–diagonal order n matrices with entries in $[n]^2$ (the diagonal is irrelevant in these matrices). Each $w \in W_n$ is called a *winning position* or a *winner*. Set

$$\begin{aligned}\Omega_w(A) := \{X \in \Omega(A) : &(AX)_{ij} = a_{i,w(i,j)_1} + x_{w(i,j)_1,j} = \\ &(XA)_{ij} = x_{i,w(i,j)_2} + a_{w(i,j)_2,j}, \text{ for } i,j \in [n], i \neq j\}.\end{aligned} \tag{8}$$

Example 2. *Consider*

$$A = \begin{bmatrix} 0 & -4 & -6 & -3 \\ -6 & 0 & -4 & -3 \\ -3 & -6 & 0 & -3 \\ -6 & -3 & -3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -4 & -4 & -6 \\ -2 & 0 & -3 & -4 \\ -5 & -6 & 0 & -5 \\ -6 & -5 & -2 & 0 \end{bmatrix}.$$

Then

$$AB = BA = \begin{bmatrix} 0 & -4 & -4 & -3 \\ -2 & 0 & -3 & -3 \\ -3 & -6 & 0 & -3 \\ -5 & -3 & -2 & 0 \end{bmatrix}$$

so that $B \in \Omega_w(A)$ *with*

$$w = \begin{bmatrix} & (1,1) & (1,3) & (4,1) \\ (2,1) & & (2,3) & (4,2) \\ (1,3) & (2,2) & & (4,3) \\ (2,3) & (2,4) & (4,3) & \end{bmatrix}. \quad \square$$

Example 3. *For any real* $A \in M_n^{nor}$*,*

- *if* tr *denotes the transposition operator, then* $I \in \Omega_{\mathrm{tr}}(A)$*,*
- *if* id *denotes the identity operator, then* $0, A^* \in \Omega_{\mathrm{id}}(A)$*.*

Proposition 4. *For any real* $A \in M_n^{nor}$*,* $\Omega_w(A)$ *is an alcoved polytope.*

Proof. Fix $i,j \in [n], i \neq j$. Then (8) means that

$$a_{i,w(i,j)_1} + x_{w(i,j)_1,j} = x_{i,w(i,j)_2} + a_{w(i,j)_2,j} \tag{9}$$

and the following $2n-2$ inequalities hold

$$a_{is} + x_{sj} \leq a_{i,w(i,j)_1} + x_{w(i,j)_1,j}, \text{ for } s \neq w(i,j)_1, \tag{10}$$

$$x_{it} + a_{tj} \leq x_{i,w(i,j)_2} + a_{w(i,j)_2,j}, \text{ for } t \neq w(i,j)_2. \tag{11}$$

Equalities and inequalities (9), (10) and (11) show that $X \in \Omega_w(A)$ if and only if $X = (x_{ij})$ belongs to certain alcoved polytope in $\overline{\mathbb{R}}_{\leq 0}^{n^2-n} \simeq M_n^{nor}$. $\square$

Remark 1: Given a winner w, if there exist $i, j, s, t \in [n]$ with $i \neq j$ and $s \neq t$ such that

$$(i,j) \neq (s,t) \neq (j,i), \quad w(i,j) = (s,t), \quad w(s,t) = (i,j), \quad a_{is}+a_{si} \neq a_{jt}+a_{tj}, \tag{12}$$

then $\Omega_w(A)$ is empty. Indeed, by (9), the following two parallel hyperplanes

$$a_{is} + x_{sj} = x_{it} + a_{tj}, \quad a_{si} + x_{it} = x_{sj} + a_{jt},$$

take part in the description of $\Omega_w(A)$.

For instance, back to A in example 2, if $\tau \in W_n$ is such that $\tau(1,3) = (2,4)$ and $\tau(2,4) = (1,3)$, then $\Omega_\tau(A) = \emptyset$, because $a_{12}+a_{21} = -10 \neq a_{34}+a_{43} = -6$.

Remark 2: Given a winner w and $i, j \in [n], i \neq j$, if

$$w(i,j) = (i,j) \text{ or } w(i,j) = (j,i), \tag{13}$$

then equality (9) is tautological. In particular,

$$\dim \Omega_w(A) \leq n^2 - n - \operatorname{card} P_w^c, \tag{14}$$

where $P_w := \{(i,j) : 1 \leq i < j \leq n \text{ with } w(i,j) = (i,j) \text{ or } w(i,j) = (j,i)\}$ and ^{c} denotes complementary.

Example 2. *(Continued) For w, the pairs which do not satisfy (13) are $w(1,2) = (1,1)$, $w(3,2) = (2,2)$ and $w(4,1) = (2,3)$, so that $P_w^c = \{(1,2),(3,2),(4,1)\}$. It follows that $x_{12} = -4$, $x_{32} = -6$ and $x_{21} = x_{43}$ are some of the equations describing $\Omega_w(A)$. Besides, condition (12) is satisfied for no pairs, whence*

$$0 < \dim \Omega_w(A) \leq 16 - 4 - 3 = 9. \quad \square$$

Clearly,

$$\Omega(A) = \bigcup_{w \in W_n} \Omega_w(A) \tag{15}$$

and the set W_n is finite, whence the following corollary is a straightforward consequence of proposition 4.

Corollary 5. *For any real $A \in M_n^{nor}$, $\Omega(A)$ is a finite union of alcoved polytopes.* $\square$

The sets $\Omega_w(A)$ are not too natural. On the contrary, the sets $\Omega^S(A)$ described below are more natural but harder to study. For any $S \in M_n^{nor}$, let

$$\Omega^S(A) := \{X \in \Omega(A) : XA = AX = S\}, \tag{16}$$

so that

$$\Omega(A) = \bigcup_{S \in M_n^{nor}} \Omega^S(A) \tag{17}$$

is a disjoint union. For instance, $B \in \Omega^S(A)$, for $S := BA$ in example 2. We also consider the set

$$\Omega'(A) := \{X \in \Omega(A) : XA = AX = X\}. \tag{18}$$

It is immediate to see that

1. $A^{j-1} \in \Omega^{A^j}(A)$, for $j \in [n]$. In particular, $I = A^0 \in \Omega^A(A)$, i.e., $AI = IA = A$.

2. $A^* \in \Omega'(A)$, i.e., $AA^* = A^*A = A^*$.

3. $0 \in \Omega'(A)$, i.e., $A0 = 0A = 0$.

Proposition 6. *For any real $A, B \in M_n^{nor}$, if that $A^{n-2} \leq B \leq A^*$, then $B \in \Omega^{A^*}(A)$.*

Proof. $A^{n-1} = A^n = A^{n+1} = \cdots = A^*$, by Yoeli's theorem, and left or right multiplication by A is monotonic, so that $A^{n-2} \leq B \leq A^*$ implies $A^* \leq AB \leq A^*$ and $A^* \leq BA \leq A^*$. $\square$

Recall $m(A)$ and $M(A)$ defined in (7). Recall the topology in M_n^{nor}, defined in p. 2.

For $r \in \overline{\mathbb{R}}$, denote by $K(r) = (\alpha_{ij})$ the *constant matrix* such that $\alpha_{ii} = 0$ and $\alpha_{ij} = r$, for all $i \neq j$. For instance, $I = K(-\infty)$ and $0 = K(0)$.

Proposition 7. *For any real $A \in M_n^{nor}$, if $I \leq B \leq K(m(A))$, then $B \in \Omega^A(A)$. In particular, $\Omega^A(A)$ is a neighborhood of the identity matrix I.*

Proof. The hypothesis $I \leq B \leq K(m(A))$ means that B is normal and $b_{ij} \leq m(A)$, for all $i \neq j$.

If $i \neq j$, we have $(AB)_{ij} = \max_{k \in [n]} a_{ik} + b_{kj} = a_{ij}$, since $a_{ik} + b_{kj} \leq a_{ik} + m(A) \leq m(A) \leq a_{ij}$, when $k \neq j$, and $a_{ij} + b_{jj} = a_{ij}$. Similarly, $(BA)_{ij} = a_{ij}$. This shows $AB = BA = A$, so that $B \in \Omega^A(A)$.

The value $m(A)$ defined in (7) is real. The set $U = \{B : I \leq B < K(m(A))\}$ is in bijective correspondence with the Cartesian product of half–lines $[-\infty, m(A))^{n^2-n}$, which is open. Moreover, $I \in U \subseteq \Omega^A(A)$, proving the neighborhood condition. $\square$

Notice that $m(A)$ equals $-|||A|||$, as defined in [29]. There, it is proved that $|||A|||$ is the *(tropical) radius* of the section $\{x_n = 0\} \cap \operatorname{span} A$, i.e., the maximal tropical distance to the zero vector, from any point on $\{x_n = 0\} \cap \operatorname{span} A$. This conveys a geometrical meaning to proposition 7.

Proposition 8. *Suppose that $A \in M_n^{nor}$ is real and strictly normal. If B is such that $K(M(A)) \leq B \leq 0$, then $B \in \Omega'(A)$. In particular, $\Omega'(A)$ is a neighborhood of the zero matrix* 0.

Proof. We have $M(A) < 0$, by strict normality. The hypothesis on $B = (b_{ij})$ means that $M(A) \leq b_{ij}$, for every $i, j \in [n]$ with $i \neq j$.

For $i \neq j$, we get $(AB)_{ij} = \max_{k \in [n]} a_{ik} + b_{kj} = b_{ij}$, since $a_{ik} + b_{kj} \leq M(A) + b_{kj} \leq M(A) \leq b_{ij}$, when $k \neq i$, and $a_{ii} + b_{ij} = b_{ij}$. Similarly, $(BA)_{ij} = b_{ij}$. This shows $AB = BA = B$, so that $B \in \Omega'(A)$.

The set $U = \{B : K(M(A)) < B \leq 0\}$ is in bijective correspondence with the Cartesian product of half–lines $(M(A), 0]^{n^2-n}$, which is open. Moreover, $0 \in U \subseteq \Omega'(A)$, proving the neighborhood condition. □

Note that the former proposition is analogous to proposition 7, with the zero matrix playing the role of the identity matrix.

Below we describe the sets $\Omega^A(A)$ and $\Omega'(A)$ as finite union of alcoved polytopes. In order to do so, for $i \in [n]$, consider the matrices

- $R_A^i = (r_{kj}^i)$, with $r_{kj}^i = a_{ij} - a_{ik}$ (difference in i–th row; subscripts k, j get inverted),
- $C_A^i = (c_{kj}^i)$, with $c_{kj}^i = a_{ki} - a_{ji}$ (difference in i–th column; subscripts k, j don't get inverted).

Let $\oplus'$ denote $\min$. Write $R := \bigoplus'_{i \in [n]} R_A^i$ and $C := \bigoplus'_{i \in [n]} C_A^i$ and consider

$$\underline{A} := R \oplus' C = A \oplus' R \oplus' C, \tag{19}$$

the last equality being true since $r_{ij}^i = a_{ij}$ and $c_{kj}^j = a_{kj}$, by normality of A. Clearly, $\underline{A} \leq A$ and $\underline{A}$ is real and normal, if A is.

Notation: $(\leftarrow, \underline{A}] := \{X \in M_n^{nor} : X \leq \underline{A}\}$. This is an alcoved polytope of dimension $n^2 - n$.

Theorem 9. *For any real $A \in M_n^{nor}$, $\Omega^A(A)$ is a finite union of alcoved polytopes. Moreover,*

$$\Omega_{\mathrm{tr}}(A) \subseteq \Omega^A(A) \subseteq (\leftarrow, \underline{A}].$$

Proof. $AX = XA = A$ if and only if

$$\max_{k \in [n]} a_{ik} + x_{kj} = a_{ij}, \quad \max_{k \in [n]} x_{ik} + a_{kj} = a_{ij}, \text{ for } i, j \in [n], i \neq j. \tag{20}$$

Now, for each $X = (x_{ij}) \in \Omega^A(A)$ there exists some winner w_X such that, for each pair (i, j) with $i \neq j$, the maxima in (20) are attained at $w_X(i, j)$. Since W_n is finite, then (20) describe a finite union of alcoved polytopes in the variables x_{ij}. Moreover, $X \leq \underline{A}$ follows from (19) and (20). In addition, the maxima in (20) are attained, at least, for the transposition operator. Therefore, $\Omega_{\mathrm{tr}}(A) \subseteq \Omega^A(A)$. □

Algorithm 10. *To compute $\underline{A}$, we proceed as follows: for $1 \le i < j \le n$,*

- *compute the minimum and maximum of* $\operatorname{row}(A,i) - \operatorname{row}(A,j)$*, denoted* mr_{ij} *and* MR_{ij}*, respectively,*
- *compute the minimum and maximum of* $\operatorname{col}(A,i) - \operatorname{col}(A,j)$*, denoted* mc_{ij} *and* MC_{ij}*, respectively,*
- $\underline{A}_{ij} = \min\{a_{ij}, \mathrm{mr}_{ij}, -\mathrm{MC}_{ij}\}$,
- $\underline{A}_{ji} = \min\{a_{ji}, -\mathrm{MR}_{ij}, \mathrm{mc}_{ij}\}$.

A sorting algorithm *is needed to compute* $\mathrm{mr}_{ij}, \mathrm{mc}_{ij}, \mathrm{MR}_{ij}, \mathrm{MC}_{ij}$*. For instance,* Mergesort *has $O(n \log n)$ complexity, whence the complexity of the computation of $\underline{A}$ is $O(n^3 \log n)$.*

Example 11. *For*

$$B = \begin{bmatrix} 0 & -3 & -1 \\ -4 & 0 & -6 \\ -5 & 0 & 0 \end{bmatrix} \text{ we get } \underline{B} = \begin{bmatrix} 0 & -3 & -3 \\ -5 & 0 & -6 \\ -5 & -2 & 0 \end{bmatrix}. \tag{21}$$

On the other hand, for A in example 2, we get $A = \underline{A}$. $\square$

Notation: $\underline{[A, \rightarrow)} := \{X \in M_n^{nor} : A \le \underline{X}\}$. It is an alcoved polytope, since the definition of $\underline{X}$ involves differences $x_{ij} - x_{kl}$ of two entries.

The proof of the theorem below is similar to the proof of theorem 9. Alternatively, theorem 12 is a corollary of theorem 9, using that $X \in \Omega^A(A)$ if and only if $A \in \Omega'(X)$.

Theorem 12. *For any real $A \in M_n^{nor}$, $\Omega'(A)$ is a finite union of alcoved polytopes. Moreover,*

$$\Omega_{\mathrm{id}}(A) \subseteq \Omega'(A) \subseteq \underline{[A, \rightarrow)}. \quad \square$$

The sets $(\leftarrow, \underline{A}]$ and $\underline{[A, \rightarrow)}$ are alcoved polytopes, but $\underline{[A, \rightarrow)}$ is trickier than $(\leftarrow, \underline{A}]$. We can compute a *tight description* of any of them, as explained in [29]. It goes as follows. For any $m \in \mathbb{N}$, any real matrix $H \in M_m^{nor}$ yields the alcoved polytope C_H (see (1)), and it turns out that $C_H = C_{H^*}$. Moreover, the description of this convex set given by H^* is *tight*.

Example 11. *(Continued) Let us compute a tight description of $\underline{[B, \rightarrow)}$, for B in (21). The matrix $\underline{X}$ is defined in (19) and we have $B \le \underline{X}$ if and only if*

$$\begin{array}{ll}
-3 \leq x_{12} & -6 \leq x_{23} \\
-3 \leq x_{32} - x_{31} & -6 \leq x_{13} - x_{12} \\
-3 \leq x_{13} - x_{23} & -6 \leq x_{21} - x_{31} \\
-1 \leq x_{13} & -5 \leq x_{31} \\
-1 \leq x_{23} - x_{21} & -5 \leq x_{21} - x_{23} \\
-1 \leq x_{12} - x_{32} & -5 \leq x_{32} - x_{12} \\
-4 \leq x_{21} & \ \ 0 \leq x_{32} \\
-4 \leq x_{31} - x_{32} & \ \ 0 \leq x_{12} - x_{13} \\
-4 \leq x_{23} - x_{13} & \ \ 0 \leq x_{31} - x_{21}.
\end{array}$$

Now, in order to write down the matrix H, we perform a relabeling *of the unknowns; for instance:*

$$y_1 = x_{12},\ y_2 = x_{13},\ y_3 = x_{21},\ y_4 = x_{23},\ y_5 = x_{31},\ y_6 = x_{32},$$

so that,

$$\begin{array}{rr}
-3 \leq y_1 & 0 \leq y_1 - y_2 \leq 6 \\
-1 \leq y_2 & -1 \leq y_1 - y_6 \leq 5 \\
-4 \leq y_3 & -3 \leq y_2 - y_4 \leq 4 \\
-6 \leq y_4 & -5 \leq y_3 - y_4 \leq 1 \\
-5 \leq y_5 & -6 \leq y_3 - y_5 \leq 0 \\
0 \leq y_6 & -4 \leq y_5 - y_6 \leq 3
\end{array}$$

and we get $\underline{[B, \to)} = C_H$*, with*

$$H = \begin{bmatrix}
0 & 0 & -\infty & -\infty & -\infty & -1 & -3 \\
-6 & 0 & -\infty & -3 & -\infty & -\infty & -1 \\
-\infty & -\infty & 0 & -5 & -6 & -\infty & -4 \\
-\infty & -4 & -1 & 0 & -\infty & -\infty & -6 \\
-\infty & -\infty & 0 & -\infty & 0 & -4 & -5 \\
-5 & -\infty & -\infty & -\infty & -3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.$$

Then $H^3 = H^4 = H^*$*, with*

$$H^* = \begin{bmatrix} 0 & 0 & -1 & -1 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 & -1 & -1 & -1 \\ -4 & -4 & 0 & -4 & -4 & -4 & -4 \\ -5 & -4 & -1 & 0 & -5 & -5 & -5 \\ -4 & -4 & 0 & -4 & 0 & -4 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

so that $\underline{[B, \to)} = C_H = C_{H^*}$*, by [29], and this set is described* tightly *as follows:*

$$\begin{array}{ll} -1 \le y_1 \le 0 & -1 \le y_1 - y_4 \le 5 \\ -1 \le y_2 \le 0 & -1 \le y_1 - y_5 \le 4 \\ -4 \le y_3 \le 0 & -1 \le y_2 - y_3 \le 4 \\ -5 \le y_4 \le 0 & -1 \le y_2 - y_4 \le 4 \\ -4 \le y_5 \le 0 & -1 \le y_2 - y_5 \le 4 \\ 0 = y_6 & -4 \le y_3 - y_4 \le 1 \\ 0 \le y_1 - y_2 \le 1 & -4 \le y_3 - y_5 \le 0 \\ -1 \le y_1 - y_3 \le 4 & -5 \le y_4 - y_5 \le 4. \end{array}$$

In particular, $\dim \underline{[B, \to)} = \dim C_{H^*} = 9 - 3 - 1 = 5$*. Undoing the relabeling, we get*

$$\begin{array}{ll} -1 \le x_{12} \le 0 & -1 \le x_{12} - x_{23} \le 5 \\ -1 \le x_{13} \le 0 & -1 \le x_{12} - x_{31} \le 4 \\ -4 \le x_{21} \le 0 & -1 \le x_{13} - x_{21} \le 4 \\ -5 \le x_{23} \le 0 & -1 \le x_{13} - x_{23} \le 4 \\ -4 \le x_{31} \le 0 & -1 \le x_{13} - x_{31} \le 4 \\ 0 = x_{32} & -4 \le x_{21} - x_{23} \le 1 \\ 0 \le x_{12} - x_{13} \le 1 & -4 \le x_{21} - x_{31} \le 0 \\ -1 \le x_{12} - x_{21} \le 4 & -5 \le x_{23} - x_{31} \le 4. \end{array}$$

Write

$$\overline{B} = \begin{bmatrix} 0 & -1 & -1 \\ -4 & 0 & -5 \\ -4 & 0 & 0 \end{bmatrix} \tag{22}$$

and notice that $\overline{B} \le X$ *follows from the first six inequalities above.*

Computations as in the former example can be done for any real matrix $A \in M_n^{nor}$, as follows.

Definition 13. *For $n \in \mathbb{N}$, a* relabeling *is a bijection between two sets of variables: $\{x_{ij} : (i,j) \in [n]^2,\ i \neq j\}$ and $\{y_k : k \in [n^2 - n]\}$. By abuse of notation, we write $y_k = x_{ij}$, for corresponding y_k and x_{ij}.*

Definition 14. *Given $A \in M_n^{nor}$ real, suppose that $\underline{[A, \rightarrow)}$ equals C_{H^*}, for some idempotent matrix $H^* = (h^*_{ij}) \in M^{nor}_{n^2-n+1}$ and some relabeling $y_k = x_{ij}$. Then $\overline{A} = (\alpha_{ij}) \in M_n^{nor}$, with $\alpha_{ij} = h^*_{k,n^2-n+1}$, i.e., the entries of $\overline{A}$ are obtained form the last column of H^*.*

The matrix $\overline{A}$ does not depend on the relabeling. The arithmetical complexity of computing $\overline{A}$ is that of H^*, which is $O((n^2-n)^3) = O(n^6)$, by the Floyd–Warshall algorithm.

Corollary 15. *For any $A, X \in M_n^{nor}$ with A real, $A \leq \underline{X}$ implies $\overline{A} \leq X$. In particular, $\Omega'(A) \subseteq [\overline{A}, \rightarrow)$.*

Proof. We proceed as in example above and we use theorem 12. □

Corollary 16. *Given $A \in M_n^{nor}$ real, suppose that $\underline{[A, \rightarrow)}$ equals C_{H^*}, for some idempotent matrix $H^* = (h^*_{ij}) \in M^{nor}_{n^2-n+1}$. Then*

$$\dim \Omega'(A) \leq n^2 - n - \operatorname{card} Q,$$

where $Q = \{(i, n^2 - n + 1) : h^*_{i,n^2-n+1} = h^*_{n^2-n+1,i} = 0,$ with $1 \leq i < n^2 - n + 1\} \cup \{(i,k) : h^*_{ik} = h^*_{ki} = 0,$ with $1 \leq i < k \leq n^2 - n + 1\}$.

Proof. The description of $\underline{[A, \rightarrow)}$ via H^* is tight, by proposition 2.6 in [29]. Thus, the dimension of $\underline{[A, \rightarrow)}$ drops by one unit each time that a chain of two inequalities in expression (1) (for H^* instead of A), turns into two equalities, which occurs whenever $h^*_{ik} = h^*_{ki} = 0$, by normality of H^*. Thus, $\dim \underline{[A, \rightarrow)} = n^2 - n - \operatorname{card} Q$ and this is an upper bound for $\dim \Omega'(A)$. □

Proposition 17. *For any $A \in M_n^{nor}$ real, we have $\underline{A} \leq A \leq \overline{A}$.*

Proof. The inequality $\underline{A} \leq A$ was explained in p. 10. Now consider X such that $A \leq \underline{X}$. Then,

$$A \leq \underline{X} \leq X,$$

by the same reason, so that $A \leq X$. By definition 14, the matrix $\overline{A}$ is obtained from the last column of H^* and, by [29], the description of the alcoved polytope $\underline{[A, \rightarrow)}$ as C_{H^*} is tight. Part of this description is $\overline{A} \leq X$. Therefore, $A \leq \overline{A} \leq X$, by tightness. □

Some questions arise, such as:

1. We know that $\underline{A} \le A \le \overline{A}$. Does every X with $\underline{A} \le X \le \overline{A}$ commute with A? The answer is NO. Example: take B in (21) and

$$X = \begin{bmatrix} 0 & -2 & -2 \\ -4 & 0 & -5 \\ -4 & 0 & 0 \end{bmatrix}, BX = \overline{B} \neq XB = \begin{bmatrix} 0 & -2 & -1 \\ -4 & 0 & -5 \\ -4 & 0 & 0 \end{bmatrix}.$$

2. We know that A^* and 0 belong to $\Omega'(A)$. Does every X with $A^* \le X \le 0$ commute with A? The answer is NO. Example: for B in (21), we have $B^* = \overline{B}$ in (22) and

$$X = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} = XB \neq BX = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}.$$

4 Perturbations

Definition 18. *Assume $a, b \in \mathbb{R}_{\ge 0}$ with $a \le b$. Then a, b are of the* same size *if $b \le 2a$. Otherwise, $2a < b$ and we say that a is* small with respect to *b.*

In the topological space $M_n^{nor} \simeq \overline{\mathbb{R}}_{\le 0}^{n^2-n}$ the following is expected to hold true, for any real matrix $A \in M_n^{nor}$:

1. for $j \in [n]$ and each *sufficiently small perturbation* X of A^{j-1}, we have $AX = XA$, and this is a perturbation of A^j, (including the case that X is a perturbation of $I = A^0$ or of $A^* = A^{n-1}$)

2. for each *sufficiently small perturbation* X of 0, we have $AX = XA$, and this is a perturbation of 0.

The point here is, of course, to give a precise meaning of *sufficiently small perturbation*. Although we are not able to do it yet, we believe that the statement will be about linear inequalities in terms of the non–zero entries a_{ij} of A and some perturbing constants $\pm\epsilon_1, \dots, \pm\epsilon_s$, with $\epsilon_k \ge 0$ for $k = 1, \dots, s$, and some $s \ge 0$. We further believe that the perturbing constants must be *small* with respect to every non–zero absolute value $|a_{ij}|$, according to definition 18. Recall that $\Omega(A)$ is larger than $\mathcal{P}(A)$ (see p. 2). An intriguing related QUESTION is the following: is every $X \in \Omega(A)$ a *small perturbation* of some member of $\mathcal{P}(A)$?

Below we present some partial results.

For brevity, write $A \oplus B := M = (m_{ij})$.

Proposition 19. *Assume $A, B \in M_n^{nor}$ are such that $a_{ik} + b_{kj} \le m_{ij}$, for all $i, j, k \in [n]$. Then $AB = BA = M$. In particular, $B \in \Omega^M(A)$.*

Proof. By normality, $I \leq A \leq 0$ and $I \leq B \leq 0$, whence $A \leq AB \leq 0$ and $B \leq AB \leq 0$, since (tropical) left or right multiplication by any matrix is monotonic. Thus, $M \leq AB$ and, similarly, $M \leq BA$ and, by hypothesis, $AB \leq M$ and $BA \leq M$. Therefore $AB = BA = M$. □

Theorem 20. *For each $n \in \mathbb{N}$ and each non positive real number r, any two order n matrices A, B having zero diagonal and all off–diagonal entries in the closed interval $[2r, r]$ satisfy $AB = BA = M$. In particular, $B \in \Omega^M(A)$.*

Proof. Let $a_{ii} = b_{ii} = 0$ and $2r \leq a_{ij}, b_{ij} \leq r \leq 0$, for $i, j \in [n]$. Fix $i, j \in [n]$ with $i \neq j$. For each $k \in [n]$, we have $a_{ik} + b_{kj} \leq 2r \leq a_{ij}, b_{ij}$, and we can apply the previous proposition to conclude. □

That is an easy way to produce two real matrices which commute! Moreover, the matrices A, B and M are idempotent. Indeed, $A \leq A^2$ by normality and, since $a_{ij} + a_{jk} \leq 2r \leq a_{ik}$, we get $A^2 \leq A$, whence $A = A^2$; similarly $B = B^2$ and $M = M^2$. Here $B \in \Omega(A)$ is a perturbation of A and $AB = BA = M$ is a perturbation of $A^2 = A$, so this is an example of item 1 in p. 15, for $j = 2$.

In the former theorem, notice that the absolute value of the entries $|a_{ij}|$ and $|b_{ij}|$ of A and B are of the *same size*, taken by pairs, as in definition 18. The reader should compare theorem 20 with example 2, where $M^2 = AB = BA \neq M$, these matrices being different only at entry $(4, 1)$. There A, B and AB are idempotent, but M is not.

Corollary 21. *For each $n \in \mathbb{N}$ and each negative real number r, take a_{ij} in the open interval $(2r, r)$, whenever $i \neq j$ and $a_{ii} = 0$, all $i, j \in [n]$. Then $A = (a_{ij})$ is strictly normal and $\Omega(A)$ is a neighborhood of A.*

Proof. The Cartesian product of intervals $U = (2r, r)^{n^2-n}$ is open in $\overline{\mathbb{R}}_{\leq 0}^{n^2-n}$. The image U' of U in M_n^{nor} satisfies $A \in U' \subseteq \Omega(A)$, by theorem 20, proving the neighborhood condition. □

Corollary 21 is an instance of item 1 in p. 15. Below we present another one.

For $n \geq 3$, consider $p = (p_1, \ldots, p_n) \in \mathbb{R}^n{}_{\geq 0}$ and $\epsilon \geq 0$ and set

$$P(-p, -\epsilon) := \begin{bmatrix} 0 & -\epsilon & \cdots & -\epsilon & -p_n \\ -p_1 & 0 & -\epsilon & \cdots & -\epsilon \\ -\epsilon & -p_2 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -\epsilon \\ -\epsilon & \cdots & -\epsilon & -p_{n-1} & 0 \end{bmatrix} \in M_n^{nor}, \tag{23}$$

and for $n \geq 4$, set

$$Q(-p,-\epsilon) := \begin{bmatrix} 0 & 0 & \cdots & 0 & -\epsilon & -p_n \\ -p_1 & 0 & \cdots & \cdots & 0 & -\epsilon \\ -\epsilon & -p_2 & 0 & \cdots & \cdots & 0 \\ 0 & -\epsilon & -p_3 & 0 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\epsilon & -p_{n-1} & 0 \end{bmatrix} \in M_n^{nor}. \tag{24}$$

The matrices $P(-p,-\epsilon)$ and $Q(-p,-\epsilon)$ are perturbations of $P(-p,0) = Q(-p,0)$.

Theorem 22. *Let $p \in \mathbb{R}^n{}_{\geq 0}$ and let $\delta, \epsilon \geq 0$ be such that $\delta + \epsilon \leq \min_{i\in[n]} p_i$. Write $m = \min\{\delta, \epsilon\}$. Then*

1. $P(-p,-\delta)P(-p,-\epsilon) = P(-p,-\epsilon)P(-p,-\delta) = P(-(\delta+\epsilon,\ldots,\delta+\epsilon),-m)$.
2. $Q(-p,-\delta)Q(-p,-\epsilon) = Q(-p,-\epsilon)Q(-p,-\delta) = Q(-(m,\ldots,m),0)$.

Proof. Straightforward computations. □

Example 23. *Take $p = (4,3,5)$, $\epsilon = 1$ and $\delta = 2$,*

$$P(-p,-2) = \begin{bmatrix} 0 & -2 & -5 \\ -4 & 0 & -2 \\ -2 & -3 & 0 \end{bmatrix}, \; P(-p,-1) = \begin{bmatrix} 0 & -1 & -5 \\ -4 & 0 & -1 \\ -1 & -3 & 0 \end{bmatrix}. \tag{25}$$

By theorem 22, we have

$$P(-p,-2)P(-p,-1) = P(-p,-1)P(-p,-2) = P(-(3,3,3),-1) = \begin{bmatrix} 0 & -1 & -3 \\ -3 & 0 & -1 \\ -1 & -3 & 0 \end{bmatrix}. \tag{26}$$

Pictures for this example are shown in figure 1. Write $A = P(-p,-2)$, $B = P(-p,-1)$, $C = AB = BA$. In $\mathbb{R}^2$ we have sketched the intersection of the classical hyperplane $\{x_3 = 0\}$ with $\operatorname{span} A, \operatorname{span} P(-p,0)$ and $\operatorname{span} B$ on top, and with $\operatorname{span} C$ bottom. To do so, we have used the matrices A_0, $P(-p,0)_0$, B_0 and C_0 as defined in p. 5:

$$A_0 = \begin{bmatrix} 2 & 1 & -5 \\ -2 & 3 & -2 \\ 0 & 0 & 0 \end{bmatrix}, P(-p,0)_0 = \begin{bmatrix} 0 & 3 & -5 \\ -4 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B_0 = \begin{bmatrix} 1 & 2 & -5 \\ -3 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$C_0 = \begin{bmatrix} 1 & 2 & -3 \\ -2 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

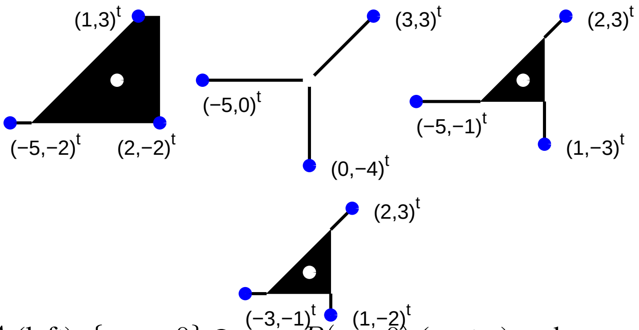

Figure 1: Top: $\{x_3 = 0\} \cap \operatorname{span} A$ (left), $\{x_3 = 0\} \cap \operatorname{span} P(-p, 0)$ (center) and $\{x_3 = 0\} \cap \operatorname{span} B$ (right), for $p = (4, 3, 5)$. Bottom: $\{x_3 = 0\} \cap \operatorname{span} C$, with $C = AB = BA$. In each case, the zero vector is marked in white and generators are represented in blue. The matrices A, B and C are perturbations of $P(-p, 0)$.

5 Geometry

Let $A, B \in M_n^{nor}$ be real. Here we study the role played by the geometry of the complexes $\operatorname{span} A$ and $\operatorname{span} B$ in order to have $AB = BA$. To do so, we bear in mind how the maps f_A and f_B act, where $f_A : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ transforms a column vector X into the product AX. For $n = 3$, f_A is described in detail in see [28]; see also [31].

Before, we have met two instances where the geometry explains why $AB = BA$. Namely, in remarks after propositions 7 and 8. In the first (resp. second) case we have $AB = BA = A$ (resp. $AB = BA = B$) because $\operatorname{span} B$ is much larger (resp. smaller) than $\operatorname{span} A$.

More generally, we explore the relationship among the sets $\operatorname{span} A$, $\operatorname{span} B$, $\operatorname{span}(AB)$ and $\operatorname{span}(BA)$ when commutativity is present or absent. In general, we have $\operatorname{span}(AB) \subseteq \operatorname{span} A$ and $\operatorname{span}(BA) \subseteq \operatorname{span} B$. In particular, if $AB = BA$ then $\operatorname{span}(AB) \subseteq \operatorname{span} A \cap \operatorname{span} B$.

Proposition 24. *Let $A, B \in M_n^{nor}$. If $A \leq B = B^2$ and A is real, then $A \in \Omega^B(B)$ and $\operatorname{span} A \supseteq \operatorname{span} B$.*

Proof. By normality, we have $I \leq A \leq B \leq 0$ and left or right tropical multiplication by any matrix is monotonic. Therefore, $B \leq AB \leq B^2 = B$ and

$B \leq BA \leq B^2 = B$, whence $AB = BA = B$ and $A \in \Omega^B(B)$. Moreover, whatever the matrices A and B may be, we have $\operatorname{span} A \supseteq \operatorname{span}(AB)$ and, in our case, $\operatorname{span}(AB) = \operatorname{span} B$. $\square$

The hypothesis $B = B^2$ cannot be removed in the previous proposition, as the following example shows.

Example 25. *Consider*

$$A = \begin{bmatrix} 0 & -1 & -3 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix} \leq B = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix},$$

then

$$AB = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \neq BA = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix},$$

and $\operatorname{span} A \not\supseteq \operatorname{span} B$*; see figure 2.*

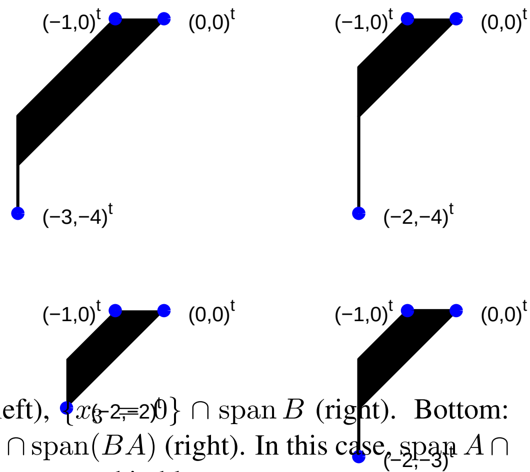

Figure 2: Top: $\{x_3 = 0\} \cap \operatorname{span} A$ (left), $\{x_3 = 0\} \cap \operatorname{span} B$ (right). Bottom: $\{x_3 = 0\} \cap \operatorname{span}(AB)$ (left), $\{x_3 = 0\} \cap \operatorname{span}(BA)$ (right). In this case, $\operatorname{span} A \cap \operatorname{span} B = \operatorname{span}(AB)$. Generators are represented in blue.

Below we explore the properties of the matrices $\underline{A}$, A and $\overline{A}$ and of the corresponding polyhedral complexes.

Example 11. *(Continued) By proposition 17, we have*

$$\underline{B} = \begin{bmatrix} 0 & -3 & -3 \\ -5 & 0 & -6 \\ -5 & -2 & 0 \end{bmatrix} \leq B = \begin{bmatrix} 0 & -3 & -1 \\ -4 & 0 & -6 \\ -5 & 0 & 0 \end{bmatrix} \leq \overline{B} = \begin{bmatrix} 0 & -1 & -1 \\ -4 & 0 & -5 \\ -4 & 0 & 0 \end{bmatrix}$$

and we can easily check, in this case, that

$$\operatorname{span} \underline{B} \supseteq \operatorname{span} B \supseteq \operatorname{span} \overline{B}.$$

See figure 3, where we are using the matrices

$$\underline{B}_0 = \begin{bmatrix} 5 & -1 & -3 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{bmatrix}, B_0 = \begin{bmatrix} 5 & -3 & -1 \\ 1 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix}, \overline{B}_0 = \begin{bmatrix} 4 & -1 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{bmatrix},$$

as defined in p. 5. Notice that $\{x_3 = 0\} \cap \operatorname{span} B$ *is the union of one closed 2–dimensional cell (called* soma*) and three closed 1–dimensional cells (called* antennas*); see [28] for the definition of soma, antennas and co–antennas (with a slightly different notation and language). In figure 3, bottom, we can see* $\{x_3 = 0\} \cap \operatorname{span} B$ *together with its co–antennas.*

In this example,

$$\overline{B} = B^*$$

and the matrix $\underline{B}$ *is idempotent. Therefore, the sets* $\operatorname{span} \underline{B}$ *and* $\operatorname{span} \overline{B}$ *are classically convex, and so are the sections* $\{x_3 = 0\} \cap \operatorname{span} \underline{B}$ *and* $\{x_3 = 0\} \cap \operatorname{span} \overline{B}$.

Consider $\mathcal{H}$*, the classical convex hull of* $\{x_3 = 0\} \cap \operatorname{span} B$*: its the vertices are* $(5,0)^t, (5,1)^t, (-2,1)^t, (-3,0)^t, (-3,-6)^t$ *and* $(-1,-6)^t$*, going counterclockwise. Notice that* $\{x_3 = 0\} \cap \operatorname{span} \underline{B}$ *is* strictly larger *than* $\mathcal{H}$*. Actually,* $\{x_3 = 0\} \cap \operatorname{span} \underline{B}$ *is the convex hull of the* union *of* $\{x_3 = 0\} \cap \operatorname{span} B$ *and the* co–antennas *of it. On the other hand,* $\{x_3 = 0\} \cap \operatorname{span} \overline{B}$ *is the* soma *of* $\{x_3 = 0\} \cap \operatorname{span} B$*, i.e., it is the maximal convex set contained there.* □

We wonder whether the statements in the former example are true for any real $B \in M_n^{nor}$. This is an open QUESTION.

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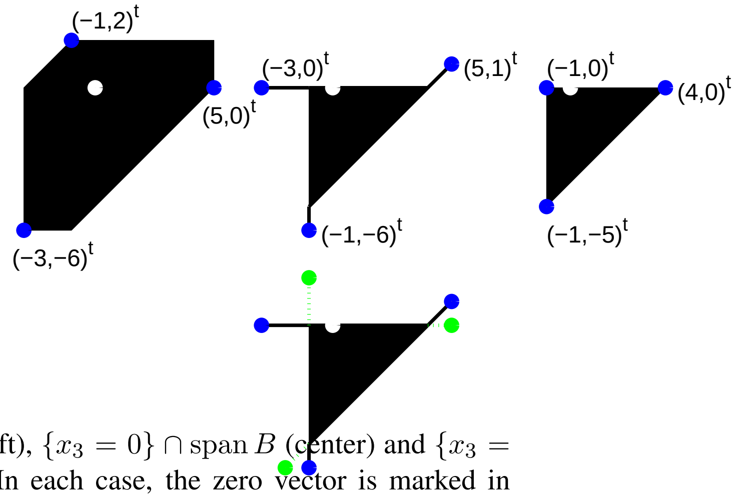

Figure 3: Top: $\{x_3 = 0\} \cap \operatorname{span} \underline{B}$ (left), $\{x_3 = 0\} \cap \operatorname{span} B$ (center) and $\{x_3 = 0\} \cap \operatorname{span} \overline{B}$ (right), for B in (21). In each case, the zero vector is marked in white, and generators (i.e., columns of the corresponding matrix $\underline{B}_0$, B_0 and $\overline{B}_0$) are represented in blue. The hyperplane section $\{x_3 = 0\} \cap \operatorname{span} B$ has three antennas. Bottom: $\{x_3 = 0\} \cap \operatorname{span} B$ is represented together with its co–antennas, which appear dotted in green. The convex hull of the bottom figure is the top left one.

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