Isocanted alcoved polytopes

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December 27, 2019

Abstract

Through tropical normal idempotent matrices, we introduce isocanted alcoved polytopes, computing their f–vectors and checking the validity of the following five conjectures: Bárány, unimodality, 3^d, flag and cubical lower bound (CLBC). Isocanted alcoved polytopes are centrally symmetric, almost simple cubical polytopes. They are zonotopes. We show that, for each dimension, there is a unique combinatorial type. In dimension d, an isocanted alcoved polytope has $2^{d+1}-2$ vertices, its face lattice is the lattice of proper subsets of $[d+1]$ and its diameter is $d+1$. They are realizations of d–elementary cubical polytopes. **Keywords**: cubical polytope; isocanted; alcoved; centrally symmetric; almost simple; f–vector; cubical g–vector; unimodal; flag; face lattice; log–concave sequence; tropical normal idempotent matrix; symmetric matrix. **AMS classification**: 52B12, 15A80

1 Introduction

This paper deals with f–vectors of isocanted alcoved polytopes. A *polytope* is the convex hull of a finite set of points in $\mathbb{R}^d$. A polytope is a *box* if its facets are only of one sort: $x_i = cnst$, $i \in [d]$. A polytope is *alcoved* if its facets are only of two sorts: $x_i = cnst$ and $x_i - x_j = cnst$, $i, j \in [d]$, $i \neq j$. Every alcoved polytope can be viewed as the perturbation of a box. In a box we distinguish two opposite vertices and the perturbation consists on canting (i.e., beveling, meaning producing a flat face upon something) some (perhaps all) of the $(d-2)$–faces of the box *not meeting* the distinguished vertices. When the perturbation happens for all such $(d-2)$–faces and with the same positive magnitude, we obtain as a result an *isocanted alcoved polytope*. The notion makes sense only for $d \geq 2$.

The f–vector of a d–polytope $\mathcal{P}$ is the tuple $(f_0, f_1, \ldots, f_{d-1}, f_d)$, where f_j is the number of j– dimensional faces in $\mathcal{P}$, for $j = 0, 1, 2, \ldots, d-1$. Obviously, $f_d = 1$. It is well known that the f–vector of a d–box is

$$B_{d,j} = 2^{d-j}\binom{d}{j}, \quad j = 0, 1, \ldots, d. \tag{1}$$

The main result in the paper is that the f–vector of an isocanted d–alcoved polytope is given by

$$I_{d,j} = (2^{d+1-j} - 2)\binom{d+1}{j}, \; j = 0, 1, \ldots, d-1, \quad I_{d,d} = 1. \tag{2}$$

The numbers $I_{d,j}$ are even, for $j \leq d-1$, because isocanted alcoved d–polytopes are centrally symmetric. We verify several conjectures for f–vectors, namely, unimodality, Bárány, Kalai 3^d and flag conjectures and CLCB. Further properties are proved, showing that isocanted alcoved polytopes are cubical, almost simple zonotopes.

The paper is organized as follows. In section 3 we give the definition and, in Theorem 3.3 prove a crucial characterization: isocanted alcoved polytopes are those alcoved polytopes having a unique vertex for each proper subset of $[d+1]$. It follows that the face lattice of an isocanted alcoved d–polytope is the lattice of proper subsets of $[d+1]$. Cubicality and almost simplicity are easy consequences. In section 5 we prove that the five mentioned conjectures hold true for isocanted alcoved polytopes. Log–concavity provides a short proof of the unimodality of $I_{d,k}$, for fixed $d \geq 2$. However, a direct proof gives additional information: the maximum of $I_{d,j}$ is attained at the integer j closest to $\frac{d}{3}$. This proof is gathered in the Appendix.

This paper encompasses tropical matrices and classical polytopes, in the sense that tropical matrices are the means to describe certain polytopes. We use several sorts of *special matrices*, operated with tropical addition

$\oplus = \max$ and tropical multiplication $\odot = +$, such as: normal idempotent (wrt $\odot$), visualized normal idempotent matrices, symmetric normal idempotent matrices and, among these, box matrices, cube matrices and isocanted matrices.

Tropical linear algebra and tropical algebraic geometry are fascinating, new, fast growing areas of mathematics with new and important results. For our purposes we recommend [8, 9, 17, 18, 21, 26] among many others. Alcoved polytopes have been first studied in [28], then in [19, 20]. Cubical polytopes have been addressed in [1, 2, 5, 6, 14]. General references for polytopes are [3, 4, 12, 16, 22, 25, 30, 31]. Normal idempotent matrices have been used in [20, 29].

2 Background and notations

Well–known definitions and facts are presented here (see [3, 12, 30] among others). $[d+1]$ denotes the set $\{1, 2, \ldots, d+1\}$ and $\binom{[d+1]}{j}$ denotes the family of subsets of $[d+1]$ of cardinality j. A *polyhedron* in $\mathbb{R}^d$ is the intersection of a finite number of halfspaces. It may be unbounded. A d–polyhedron is a polyhedron of dimension d. A d–polyhedron $\mathcal{P}$ is *alcoved* if its facets are only of two types: $x_i = cnst$ and $x_i - x_j = cnst$, $i, j \in [d]$, $i \neq j$. A double index notation is useful here because, in this way, we can gather the coefficients in a matrix over $\mathbb{R} \cup \{\pm\infty\}$: indeed, write

$$a_{ij} \leq x_i - x_j \leq -a_{ji}$$

and, similarly,

$$a_{i,d+1} \leq x_i \leq -a_{d+1,i}.$$

Then, setting $a_{ij} = -\infty$ if the facet $x_i - x_j = cnst$ is not specified, and letting (by convention) $a_{ii} = 0$, for all $i \in [d+1]$, we get a square matrix $A = [a_{ij}] \in M_{d+1}(\mathbb{R} \cup \{\pm\infty\})$ from $\mathcal{P}$. We write $\mathcal{P} = \mathcal{P}(A)$ to express the former relationship between the alcoved polyhedron $\mathcal{P}$ and the matrix A. In addition to having null diagonal, the matrix A satisfies $-\infty \leq a_{ij} \leq -a_{ji} \leq +\infty$, for all $i, j \in [d+1]$. Different matrices A may give rise to the same polyhedron.

Let 0 denote the origin in $\mathbb{R}^d$. Important particular cases provide special matrices as follows:

1. $0 \in \mathcal{P}(A)$ if and only if A is *normal* (N) (meaning $a_{ii} = 0, a_{ij} \leq 0, \forall i, j$), (see [8, 29])
2. $0 \in \mathcal{P}(A)$ and A describes $\mathcal{P}(A)$ *optimally* if and only if A is *normal idempotent* (NI) (meaning that, in addition to normal, we have $a_{ij} + a_{jk} \leq a_{ik}, \forall i, j, k$) (see [19, 23]) and this matrix is unique; furthermore
   (a) $0 = \max \mathcal{P}(A)$ (maximum taken componentwise in $\mathbb{R}^d$, which is identified with hyperplane $x_{d+1} = 0$ in $\mathbb{R}^{d+1}$, throughout the paper) if and only if A is *visualized normal idempotent* (VNI) (in addition to NI, we have $a_{d+1,i} = 0, \forall i$), (see [8, 19, 20])
   (b) $\mathcal{P}(A) = -\mathcal{P}(A)$ if and only if A is *symmetric normal idempotent* (SNI) (in addition to NI, we have $a_{ij} = a_{ji}, \forall i, j$). (see [13, 20]).

A bounded polyhedron is called a *polytope* and every polytope is the convex hull of a finite set of points. If a polytope $\mathcal{P}(A)$ is alcoved, then its unique defining NI matrix A is real.

Our aim is to compute the f–vector of an alcoved isocanted polytope.[1] But, what is already known about vertices of an alcoved polytope $\mathcal{P}(A)$ in $\mathbb{R}^d$? First, the number of vertices of $\mathcal{P}(A)$ is bounded above by $\binom{2d}{d}$ and this bound is sharp (see [9, 26]). Which points are vertices of $\mathcal{P}(A)$? Let the auxiliary matrix $A_0 = [\alpha_{ij}]$ be defined by $\alpha_{ij} := a_{ij} - a_{d+1,j}$ and notice that $\mathcal{P}(A) = \mathcal{P}(A_0)$ (the property holds because columns in A and A_0 are proportional, although A_0 is not normal). The columns of A_0, used to picture $\mathcal{P}(A) \subset \mathbb{R}^d$, are some of the vertices of $\mathcal{P}(A)$, called the *generators* of $\mathcal{P}(A)$. Further, each $W \in \binom{[d+1]}{j}$ with $1 \leq j \leq d$ determines a $(j-1)$–dimensional tropical linear space, denoted $L(W)$. The space $L(W)$ has a finite number of vertices (however, an upper bound on how many is not known in all cases; see [26]). The vertices of $\mathcal{P}(A)$ are all the vertices of all spaces $L(W)$, for $W \in \binom{[d+1]}{j}$. The case $j = 1$ gives points, namely, the $d+1$ generators of $\mathcal{P}(A)$.

Every translate of an alcoved polytope is alcoved and, from [20], we know that translation of an alcoved polyhedron $\mathcal{P}(A)$ corresponds to conjugation of its matrix A by a diagonal matrix (with null last diagonal entry).

[1] In the future, we might be able to compute the f–vector of a general alcoved polytope.

The easiest alcoved polytopes are boxes, determined by equations $x_i = cnst$. We fix a convenient matrix notation for boxes in special cases 2a and 2b above.

Notation 2.1 (Box matrices)**.** Given real numbers $\ell_i > 0$, $i \in [d]$, consider

1. $B^{VNI}(d+1; \ell_1, \ell_2, \ldots, \ell_d) = [b_{ij}] \in M_{d+1}(\mathbb{R})$ with $b_{ij} = \begin{cases} -\ell_i, & d+1 \neq i \neq j, \\ 0, & \text{otherwise}, \end{cases}$. This matrix is VNI (easily checked) and called the *VNI box matrix with edge–lengths* ℓ_j. In particular, we have the *VNI cube matrix* $Q^{VNI}(d+1; \ell) := B^{VNI}(d+1; \ell, \ldots, \ell)$.
2. The conjugate matrix $D \odot B^{VNI}(d+1; \ell_1, \ell_2, \ldots, \ell_d) \odot D^{-1}$ is SNI (easily checked), where $D = \operatorname{diag}(\ell_1/2, \ell_2/2, \ldots, \ell_d/2, 0)$. It is denoted $B^{SNI}(d+1; \ell_1, \ell_2, \ldots, \ell_d) = [c_{ij}]$ and we have $c_{ij} = \begin{cases} -\ell_i/2, & j = d+1, \\ -\ell_j/2, & i = d+1, \\ 0, & i = j, \\ (-\ell_i - \ell_j)/2, & \text{otherwise}. \end{cases}$ Similarly we have the cube matrix $Q^{SNI}(d+1; \ell)$.
3. A *box matrix* is any conjugate of the above, i.e., $D' \odot B \odot D'^{-1}$, where $D' = \operatorname{diag}(d'_1, d'_2, \ldots, d'_d, 0)$ with $d'_j \in \mathbb{R}$ and $B = B^{VNI}(d+1; \ell_1, \ell_2, \ldots, \ell_d)$ or $B = B^{SNI}(d+1; \ell_1, \ell_2, \ldots, \ell_d)$. It is NI (easily checked).

Definition 2.2 (from Puente [20])**.** Any non–positive real matrix $E \in M_{d+1}(\mathbb{R})$ with null diagonal, last row and column is called *perturbation matrix*. In symbols, $E = [e_{ij}]$ with $e_{ii} = e_{d+1,i} = e_{i,d+1} = 0$ and $e_{ij} \leq 0, \forall i, j$.

3 Definition, characterization and f–vector of IAPs

In [20] it is proved that for any NI matrix $A \in M_{d+1}(\mathbb{R})$ (not necessarily VNI or SNI), there exists a unique decomposition $A = B - E$, where B a NI box matrix and E is a perturbation matrix. The polytope $\mathcal{P}(B)$ is called the *bounding box* of the alcoved polytope $\mathcal{P}(A)$.

Definition 3.1 (Isocanted alcoved polytope (IAP))**.** Let $A \in M_{d+1}(\mathbb{R})$ be a NI matrix with decomposition $A = B - E$. The alcoved polytope $\mathcal{P}(A)$ is *isocanted* if E is a *constant perturbation matrix*, i.e., there exists $a > 0$ such that $e_{ij} = -a$, for all $i, j \in [d+1]$, $i \neq j$. The number a is called *cant parameter* of $\mathcal{P}(A)$. We write $E = [-a]$, by abuse of notation.

Notation 3.2 (Special matrices for visualized IAPs and symmetric IAPs, with cubic bounding boxes)**.** Given real numbers a, ℓ, consider the constant perturbation matrix $E = [-a] \in M_{d+1}(\mathbb{R})$ as above and the matrices (as in Notation 2.1)

1. $I^{VNI}(d+1; \ell, a) := Q^{VNI}(d+1; \ell) - E$,
2. $I^{SNI}(d+1; \ell, a) := Q^{SNI}(d+1; \ell) - E$.

It is an easy computation to check that, for these matrices to be NI, it is necessary and sufficient that $0 < a < \ell$.[2]

The following is the crucial step of the paper. Its proof contains the only tropical computations in what follows.

Theorem 3.3 (Characterization of IAPs)**.** *An alcoved d–polytope $\mathcal{P}$ is* isocanted *if and only if, for each $1 \leq j \leq d$ and each $W \in \binom{[d+1]}{j}$, the tropical linear space $L(W)$ has a unique vertex.*

Proof. Without loss of generality, we can assume that the bounding box of $\mathcal{P}$ is a cube (of edge–length $\ell > 0$) since an affine bijection does not affect the result. We can also assume that $\mathcal{P}$ is located in d–space so that $\max \mathcal{P}$ is the origin, because a translation does not affect the result. Then $\mathcal{P} = \mathcal{P}(C)$, with $C = Q^{VNI}(d+1; \ell) - E$, for some positive ℓ, as in in Notation 2.1 and Definition 2.2.

[2] The limit case $a = \ell$ provides a polytope of dimension less than d. The limit case $a = 0$ provides the d–cube. Matrices $I^{VNI}(d+1; \ell_1, \ell_2, \ldots, \ell_d, a)$ and $I^{SNI}(d+1; \ell_1, \ell_2, \ldots, \ell_d, a)$ may be similarly defined, for $0 < a < \min_j \ell_j$, but we will not use them.

($\Rightarrow$) Assume $\mathcal{P}$ is IAP. Then $E = [-a]$ is constant and then $C = I^{VNI}(d+1;\ell,a) = [c_{ij}]$, as in Item 1 of Notation 3.2. In symbols, $c_{ij} = \begin{cases} -\ell, & i \neq j = d+1, \\ 0, & i = j \text{ or } i = d+1, \\ -\ell + a, & \text{otherwise,} \end{cases}$ with $0 < a < \ell$. Note that the tropical rank of C is $d+1$ (meaning that the maximum in the tropical permanent of C is attained only once.[3]) In particular, $\operatorname{rk}_{\text{tr}} C(W) = j$, for each proper subset $W \in \binom{[d+1]}{j}$.

For $j = 1$, $L(W)$ reduces to a point (i.e., a generator) and uniqueness is trivial. For $j \geq 2$, $L(W)$ is the tropical line determined by two generators in the set W. Consider a point $x \in \mathbb{R}^{d+1}_{\leq 0}$ with $x_{d+1} = 0$, and let $C(W)$ denote the $(d+1) \times j$ sized matrix whose columns are indexed by W and taken from C, and let $C(W,x)$ be $C(W)$ extended with column x. It is well–known (see [21, 26, 27]) that $x \in L(W)$ if and only if $\operatorname{rk}_{\text{tr}} C(W,x) \leq j$, (meaning that the maximum in *each* order $(j+1)$ tropical minor is attained at least twice). Besides, x is a vertex in $L(W)$ if and only if the maximum in *each* order $(j+1)$ tropical minor is attained $(j+1)$ times.

For better readability, we do the case $d = 3$ (the case $d > 3$ is similar, due to the structure of the matrix $C(W,x)$).

1. Case $j = 2$. The order 3 minors in $C(W,x)$ are

$$m_{123} = \max\{x_1 + m_{23},\ x_2 + m_{13},\ x_3 + m_{12}\}$$

$$m_{124} = \max\{x_1 + m_{24},\ x_2 + m_{14},\ m_{12}\}$$

$$m_{134} = \max\{x_1 + m_{34},\ x_3 + m_{14},\ m_{13}\}$$

$$m_{234} = \max\{x_2 + m_{34},\ x_3 + m_{24},\ m_{23}\}$$

where m_{ij} denotes the order 2 minor of $C(W)$ involving rows i–th and j-th, with $i < j$, and we have done Laplace expansions by the last column. Two cases arise.

(a) *If* $d + 1 \notin W$. For simplicity in writing and without loss of generality, assume $W = [2]$. Then, $x_3 = a - \ell$ follows from $c_{13} = c_{23} = -\ell + a$ and $\operatorname{rk}_{tr} C(W,x) \leq 2$. In addition, the values of the order 2 minors in $C(W)$ are $m_{12} = m_{14} = m_{24} = 0$, $m_{13} = m_{23} = m_{34} = -\ell + a < 0$. Hence

$$m_{123} = \max\{x_1 - \ell + a,\ x_2 - \ell + a,\ x_3\}$$

$$m_{124} = \max\{x_1,\ x_2,\ 0\}$$

$$m_{134} = \max\{x_1 - \ell + a,\ x_3,\ -\ell + a\}$$

$$m_{234} = \max\{x_2 - \ell + a,\ x_3,\ -\ell + a\}$$

Then $x_3 = -\ell + a$ and $x_1 = x_2 = 0$ provide triple maxima in all m_{ijk}, and no other values of x_j do. Thus $x^* = [0, 0, -\ell + a, 0]^T$ is a vertex of $L(W)$, the unique one.

(b) *If* $d + 1 \in W$. For simplicity in writing and without loss of generality, assume $W = \{1, 4\}$. Then, $x_2 = x_3$ follows from $c_{21} = c_{31}$, $c_{24} = c_{34}$ and $\operatorname{rk}_{tr} C(W,x) \leq 2$. Now, the values of the order 2 minors in $C(W)$ are $m_{14} = 0$, $m_{12} = m_{13} = -\ell$, $m_{23} = -2\ell + a < 0$, $m_{24} = m_{34} = -\ell + a < 0$ which yield

$$m_{123} = \max\{x_1 + -2\ell + a,\ x_2 - \ell,\ x_3 - \ell\}$$

$$m_{124} = \max\{x_1 - \ell + a,\ x_2,\ -\ell\}$$

$$m_{134} = \max\{x_1 - \ell + a,\ x_3,\ -\ell\}$$

$$m_{234} = \max\{x_2 - \ell + a,\ x_3 - \ell + a,\ -2\ell + a\}$$

Then $x_1 = -a$, $x_2 = x_3 = -\ell$ provide triple maxima in all m_{ijk}, and no other values do. Then, $x^* = [-a, -\ell, -\ell, 0]^T$ is a vertex of $L(W)$, the unique one.

[3] We have $\operatorname{per}_{tr} C = 0$, attained only at the identity permutation. For tropical permanent and tropical rank, see [8, 10, 11].

2. Case $j = 3$. A Laplace expansion yields $\mathrm{per}_{tr}\, C(W, x) = \max\{x_1 + m_{234},\ x_2 + m_{134},\ x_3 + m_{124},\ m_{123}\}$, where m_{ijk} denote order 3 minors of $C(W)$. Two cases arise.

   (a) *If* $d + 1 \notin W$. We have $W = [3]$ and $m_{123} = m_{124} = m_{134} = m_{234} = 0$. Thus, $\mathrm{per}_{tr}\, C(W, x) = \max\{x_1,\ x_2,\ x_3,\ 0\}$ is attained 4 times if and only if $x_1 = x_2 = x_3 = 0$. The unique solution is $x^* = [0, 0, 0, 0]^T$, the origin.

   (b) *If* $d + 1 \in W$. For simplicity in writing and without loss of generality, assume $W = \{1, 2, 4\}$. Then $m_{123} = -\ell$, $m_{124} = 0$, $m_{134} = m_{234} = -\ell + a$, whence $\mathrm{per}_{tr}\, C(W, x) = \max\{x_1 - \ell + a,\ x_2 - \ell + a,\ x_3,\ -\ell\}$ is attained 4 times if and only if $x_1 = x_2 = -a$ and $x_3 = -\ell$. The unique solution is $x^* = [-a, -a, -\ell, 0]^T$.

($\Leftarrow$) We have $\mathcal{P} = \mathcal{P}(C)$, where $C = Q^{VNI}(d+1; \ell) - E$ is an NI matrix. Assume that, for each $1 \le j \le d$ and each $W \in \binom{[d+1]}{j}$, the tropical linear space $L(W)$ has a unique vertex. We want to prove that perturbation matrix E is constant.

The matrix E is symmetric, because $\mathcal{P}$ is centrally symmetric, using [20]. Then $e_{ij} = e_{ji}, \forall i, j$. Take $W = [2]$ and compute the order 2 minors of the matrix $C([2])$ (using the inequalities guaranteed by NI, in Items 1 and 2 in p. 11). They are $m_{12} = m_{14} = m_{24} = 0$, $m_{13} = -\ell - e_{32}$, $m_{23} = -\ell - e_{31}$, $m_{34} = \max\{-\ell - e_{31},\ -\ell - e_{32}\}$. Let $x^* \in \mathbb{R}^{d+1}_{\le 0}$ be the unique vertex of $L([2])$, with $x^*_{d+1} = 0$. Then, for each order 3 minor m_{ijk} of the matrix $C([2], x^*)$, the maximum is attained three times. We have $m_{124} = \max\{x^*_1,\ x^*_2,\ 0\}$, whence $x^*_1 = x^*_2 = 0$ and substitution in the remaining m_{ijk} yield

$$m_{123} = \max\{x^*_1 - \ell - e_{31},\ x^*_2 - \ell - e_{32},\ x^*_3\} = \max\{-\ell - e_{31},\ -\ell - e_{32},\ x^*_3\}$$

$$m_{134} = \max\{x^*_1 + m_{34},\ x^*_3,\ -\ell - e_{32}\} = \max\{m_{34},\ x^*_3,\ -\ell - e_{32}\}$$

$$m_{234} = \max\{x^*_2 + m_{34},\ x^*_3,\ -\ell - e_{31}\} = \max\{m_{34},\ x^*_3,\ -\ell - e_{31}\}$$

and we conclude $e_{31} = e_{32}$. Other instances of W provide $e_{ij} = e_{st}$ whenever $i \ne j$ and $s \ne t$, for $i, j, s, t \in [d]$ and we are done.

□

Remark 3.4. *In the proof above we have obtained* $x^* = \bigoplus_{j \in W} \underline{j}$*, whenever* $d + 1 \notin W$.

Notation 3.5 (Labeling of vertices of IAP)**.** Given any isocanted alcoved d–polytope $\mathcal{P}$, it follows from Theorem 3.3 that the vertices of $\mathcal{P}$ are in bijection with the *proper* subsets $W \subset [d + 1]$. The label of the vertex corresponding to $W \subset [d + 1]$ is $\underline{W}$ (underlined). The cardinality $|W|$ is called *length* of W.

Notation 3.6 (Parent and child)**.** Assume $\mathcal{P}$ is an isocanted alcoved d–polytope. Two vertices in $\mathcal{P}$ are joined by an edge in $\mathcal{P}$ if and only if they are labeled $\underline{W}$ and $\underline{W'} \subset [d+1]$ with $\emptyset \ne W \subset W'$ and $|W| + 1 = |W'|$. We say that $\underline{W}$ is a *parent* of $\underline{W'}$ and $\underline{W'}$ is a *child* of $\underline{W}$. A 2–face of $\mathcal{P}$ is determined by four vertices with labels $\underline{jW}$, $\underline{jkW}$, $\underline{jrW}$, $\underline{jkrW}$, with $\emptyset \ne W \subset [d + 1] \setminus \{j, k, r\}$, for j, k, r pairwise different in $[d + 1]$. (jW is shorthand for $\{j\} \cup W$.)

Theorem 3.7 (f–vector for IAP)**.** $I_{d,j} = (2^{d+1-j} - 2)\binom{d+1}{j}, \quad 0 \le j \le d - 1.$

Proof. First, $I_{d,0} = \left|\bigcup_{j=1}^{d} \binom{[d+1]}{j}\right| = 2^{d+1} - 2$ is the number of proper subsets of $[d + 1]$.

Second, let us count facets. We mentioned in p. 1 that an alcoved polytope is obtained from a box, where we can cant only the $(d - 2)$–faces not meeting two distinguished opposite vertices; thus, we can cant half of the $(d-2)$–faces of the box. In an IAP we do cant every cantable $(d-2)$–face, so that $I_{d,d-1} = B_{d,d-1} + B_{d,d-2}/2 = (d + 1)d$, using (1).

For $1 \le j \le d$, the number of vertices of length j is $\binom{d+1}{j}$, by Theorem 3.3.

Assume $2 \le j \le d$. A vertex of length j has j parents, by Notation 3.6. The total number of edges is $\sum_{j=2}^{d} \binom{d+1}{j} j = (d + 1) \sum_{j=2}^{d} \binom{d}{j-1} = (d + 1) \sum_{k=1}^{d-1} \binom{d}{k} = (d + 1)(2^d - 2) = I_{d,1}$, (where we have used the equalities $\binom{d+1}{j} j = (d + 1)\binom{d}{j-1}$ and $2^d = \sum_{j=0}^{d} \binom{d}{j}$).

Assume $3 \le j \le d$. A vertex of length j has $\binom{j}{2}$ grandparents (i.e., parent of parent). The total number of 2–faces is $\sum_{j=3}^{d}\binom{d+1}{j}\binom{j}{2} = \binom{d+1}{2}\sum_{j=3}^{d}\binom{d-1}{j-2} = \binom{d+1}{2}\sum_{k=1}^{d-2}\binom{d-1}{k} = \binom{d+1}{2}(2^{d-1}-2) = I_{d,2}$ (where we have used the equality $\binom{d+1}{j}\binom{j}{2} = \binom{d+1}{2}\binom{d-1}{j-2}$).

Similarly, the total number of r–faces is $\sum_{j=r+1}^{d}\binom{d+1}{j}\binom{j}{r} = \binom{d+1}{r}\sum_{j=r+1}^{d}\binom{d+1-r}{j-r} = \binom{d+1}{r}\sum_{k=1}^{d-r}\binom{d+1-r}{k} = \binom{d+1}{r}(2^{d+1-r}-2) = I_{d,r}$ (where we have used the equality $\binom{d+1}{j}\binom{j}{r} = \binom{d+1}{r}\binom{d+1-r}{j-r}$). □

Remark 3.8. *Notice the coincidence with the triangular sequence OEIS A259569 (triangle of numbers where $T(d,k)$ is the number of k-dimensional faces on the polytope that is the convex hull of all permutations of the list $(0,1,...,1,2)$, where there are $d-1$ ones) and absolute values of OEIS A138106 (triangle of numbers based on a Morse potential type function).*

Remark 3.9. *IAPs are maximal in facets, among alcoved polytopes, because in an IAP we cant every possible cantable face. Notice that IAPs are neither simplicial nor simple and far from being neighborly.*

A d–*cuboid* is a d–polytope combinatorially equivalent to a cube. A d–cuboid is denoted $\mathcal{C}^d$. A polytope is *cubical* if every face in it is a cuboid. A d–polytope is *almost simple* if the valence of each vertex is d or $d+1$. A d–polytope $\mathcal{P}$ is *liftable* if its boundary complex $\partial\mathcal{P}$ is combinatorially equivalent to a subcomplex of the complex $\partial\mathcal{C}^{d+1}$. A d–polytope $\mathcal{P}$ is *d–elementary* if the complex $\partial\mathcal{P}$ is combinatorially equivalent to the subcomplex $\partial\mathcal{C}_V^{d+1}$ defined as follows: we take a vertex V in $\mathcal{C}^{d+1}$ and consider the subcomplex $\mathcal{F}_V^d$ of $\partial\mathcal{C}^{d+1}$ determined by the facets of $\mathcal{C}^{d+1}$ meeting V. Consider the subcomplex $\mathcal{C}_V^{d+1}$ of $\mathcal{F}_V^{d+1}$ determined by the outer faces of $\mathcal{F}_V^{d+1}$ (the underlying set of $\mathcal{C}_V^{d+1}$ is $\partial\mathcal{F}_V^d$), according to [5, 6].

The following are immediate consequences of Theorem 3.3 and Notation 3.5.

Corollary 3.10. *For each $d \ge 2$,*

1. *the face lattice of an d–IAP is the lattice of proper subsets of $[d+1]$,*
2. *there exists a unique combinatorial type of d–IAP,*
3. *every IAP is cubical and almost simple.* □

Remark 3.11. *Since the combinatorial type is unique, we can fix a notation for a d–IAP: it is denoted $\mathcal{I}^d$ in what follows. Notice a* duality *on vertices in $\mathcal{I}^d$, due to the lattice order–reversing isomorphism $W \mapsto [d+1] \setminus W$.*

Corollary 3.12. *$\mathcal{I}^d$ is d–elementary and liftable, for $d \ge 2$.*

Proof. For $d \ge 4$, d–elementarity follows from the main theorem in [5], using that number of vertices in $\mathcal{C}_V^{d+1}$ and $\mathcal{I}^d$ coincide (it is $I_{d,0} = 2^{d+1}-2$). Liftability follows from d–elementarity. For $d=2$ the two properties are easily checked: $\mathcal{I}^2$ is a hexagon, and it is combinatorially equivalent to $\mathcal{F}_V^2$, the border of a 3–cube cask at a vertex. Case $d=3$ is explained in section 4. □

Call $\mathcal{F}_V^d$ *cask at V*. The f–vector of a cask is

$$C_{d,j} = \left(2^{d-j}-1\right)\binom{d}{j}, \quad j = 0,1,\dots,d-2. \tag{3}$$

The relation with (2) is easily verified to be

$$I_{d,j} = 2C_{d,j} + I_{d-1,j-1}. \tag{4}$$

It is a consequence of d–elementarity of $\mathcal{I}^d$.

Recall that a *zonotope* is a (Minkowski) sum of segments.

Corollary 3.13. *Every IAP is a zonotope.*

Proof. A known characterization of zonotope is that it is a polytope all whose 2–faces are centrally symmetric, and this is satisfied by IAPs. Another proof is direct: $\mathcal{I}^d$ may be obtained from a d–box $\mathcal{B} = \mathcal{B}(\ell_1,\ell_2,\dots,\ell_d) \subset \mathbb{R}^d$ with $\max\mathcal{B}$ at the origin, edge–lengths $\ell_j > 0$ and cant parameter a with $0 < a < \min_j \ell_j$, satisfies $\mathcal{I}^d = \mathcal{B} + [0, av_{d+1}]$ where $(v_1, v_2, \dots, v_d)$ is the standard basis and $v_{d+1} = v_1 + v_2 + \cdots + v_d$. □

4 Cases $d = 3, 4$.

Fix $d \geq 2$. Two opposite vertices in $\mathcal{I}^d$ are distinguished: $\mathcal{N} := \max \mathcal{I}^d$, called the *North Pole*, and $\mathcal{S} := \min \mathcal{I}^d$ called the *South Pole* of $\mathcal{I}^d$ (maximum and minimum computed coordinatewise). The label of $\mathcal{N}$ is $\underline{12\cdots d}$, and the label of $\mathcal{S}$ is $\underline{d+1}$. The complex $\mathcal{F}^d_{\mathcal{N}} \subset \partial\mathcal{I}^d$ introduced in paragraph in p. 6 (resp. $\mathcal{F}^d_{\mathcal{S}}$) is called *North Polar Cask* (resp. *South Polar Cask*) of $\mathcal{I}^d$. Vertices included in the North (resp. South) Polar Cask are exactly those omitting (resp. including) digit $d+1$ in their label. The Equatorial Belt is, by definition, the subcomplex of $\partial\mathcal{I}^d$ determined by all faces of $\mathcal{I}^d$ not meeting the poles. The $(d-1)$–faces appearing in the Equatorial Belt are characterized by the property that they contain edges of $\mathcal{I}^d$ in the direction of vector $(1, 1, \ldots, 1)^T$. These are the edges joining vertices $\underline{W}$ and $\underline{Wd+1}$, for proper subsets $W \subset [d]$. The complex $\partial\mathcal{I}^d$ is the union of the Polar Casks and the Equatorial Belt. This idea, which goes back to Kepler, has been developed for isocanted in [20].

A Polar Cask is homeomorphic to a closed $(d-1)$–disk. The Equatorial Belt is homeomorphic to a closed $(d-1)$–cylinder, i.e., $S^{d-2} \times [-1, 1]$ (the Cartesian product of a $(d-2)$–sphere and a closed interval).

Case $d = 3$**:** we have $\mathcal{N} = \underline{123}$ and the North Cask is homeomorphic to a 2–disk with one interior point labeled $\underline{123}$, points in the circumference labeled $\underline{1}, \underline{12}, \underline{2}, \underline{23}, \underline{3}, \underline{13}$ and inner edges joining $\underline{12}, \underline{23}, \underline{13}$ to $\underline{123}$ (see figure 1). The South Pole is $\mathcal{S} = \underline{4}$ and the South Cask is homeomorphic to a 2–disk with one interior point labeled $\underline{4}$, points in the circumference labeled $\underline{14}, \underline{124}, \underline{24}, \underline{234}, \underline{34}, \underline{134}$ and inner edges joining $\underline{14}, \underline{24}, \underline{34}$ to $\underline{4}$ (see figure 2). The Equatorial Belt is homeomorphic to a cylindrical surface (see figure 3). Identification of borders of polar casks with border components of cylinder is easily done by using vertex labels. The f–vector of a 2–polar cask is the sum of the f–vector of the circumference $(6, 6)$ and of the internal subdivision $(1, 3)$, yielding $(7, 9)$, which agrees with $(C_{3,0}, C_{3,1})$ in (3).

Case $d = 4$**:** the North Cask is homeomorphic to a solid 3–sphere with one interior point labeled $\mathcal{N} = \underline{1234}$, points on the surface labeled $\underline{i}$, $\underline{ij}$, and $\underline{ijk}$, with $i, j, k \in [4]$, pairwise different. Edges join parent and child (see Notation 3.6). Combinatorially, the cask is equivalent to a solid rhombic dodecahedron with an interior point labeled $\underline{1234}$ and six quadrangular inner 2–faces given by $\underline{ij}, \underline{ijk}, \underline{ijr}, \underline{1234}$, with $\{i, j, k, r\} = [4]$ (see figure 4).

The South Cask is homeomorphic to a solid 3–sphere with one interior point labeled $\mathcal{S} = \underline{5}$, points on the surface labeled $\underline{i5}$, $\underline{ij5}$, and $\underline{ijk5}$, with $i, j, k \in [4]$, pairwise different. Edges are determined by Notation 3.6. Combinatorially, the cask is equivalent to a solid rhombic dodecahedron with an interior point labeled $\underline{5}$ and six quadrangular inner 2–faces given by $\underline{i5}, \underline{ij5}, \underline{ik5}, \underline{ijk5}$, with $i, j, k \in [4]$ pairwise different (see figure 5).

The f–vector of a rhombic dodecahedron is $(14, 24, 12)$ and the internal subdivision adds $(1, 4, 6)$, so that the sum $(15, 28, 18)$ is the f–vector of a 3–polar cask, which agrees with $(C_{4,0}, C_{4,1}, C_{4,2})$ in (3).

The Equatorial Belt is homeomorphic to a 3–cylinder $S^2 \times [-1, 1]$. Identification of borders of polar casks with border components of cylinder is easily done by using vertex labels.

Remark 4.1. *We have* $I_4 = (30, 70, 60, 20)$ *and*

1. fatness *is* $\frac{f_1+f_2-20}{f_0+f_3-10} = \frac{11}{4}$ *(notice that fatness lays between the known bounds holding for simplicial and simple 4–polytopes:* $\frac{5}{2} < \frac{11}{4} < 3$*; see [31]),*

2. *The number of vertex–facet incidences in* $\mathcal{I}^4$ *is* $f_{03} = 160$*, since there are* $I_{4,3} = (d+1)d = 20$ *3–cubes (with 8 vertices each) and no other 3–faces.*

5 Five conjectures proved for IAPs

Consider the set $\mathcal{M}$ of lower triangular infinite matrices with both entries **and** indices in $\mathbb{Z}_{\geq 0}$. Examples of matrices in $\mathcal{M}$ are the *2–power matrix*, denoted T, defined by $T_{d,k} = \begin{cases} 2^{d-k}, & 0 \leq k \leq d, \\ 0, & \text{otherwise,} \end{cases}$ and the *Pascal matrix*, denoted P, defined by $P_{d,k} = \begin{cases} \binom{d}{k}, & 0 \leq k \leq d, \\ 0, & \text{otherwise.} \end{cases}$ With the Hadamard or entry–wise product, multiply the former matrices, obtaining $B := T \circ P = P \circ T \in \mathcal{M}$ and notice that the d–th row of B shows the f–vector

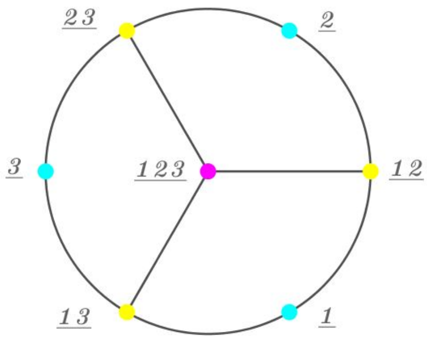

Figure 1: North Polar Cask for $d = 3$.

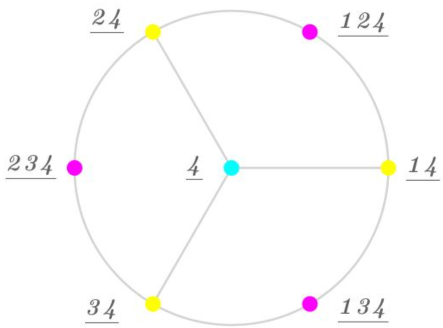

Figure 2: South Polar Cask for $d = 3$.

of a d–box, for $d \in \mathbb{Z}_{\geq 0}$; so we call B is the *box matrix*. Next, consider the matrix $H \in \mathcal{M}$ defined by

$$H_{d,k} = \begin{cases} (2^{d-k} - 1)\binom{d+1}{k}, & 0 \leq k \leq d-1, \\ 1/2, & k = d, \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

For fixed $d \geq 2$, we study the growth[4] of the sequence $H_{d,k}$, with $0 \leq k < k+1 \leq d-1$.

Proposition 5.1. *For each $d \geq 0$, we have $H_{d,d-1} \leq H_{d,0}$ with equality only for $d = 0, 1, 2$.*

Proof. The inequality $(d+1)d/2 \leq 2^d - 1$ is easily proved by induction on d (degree 2 polynomials grow slower than 2–powers.) □

Recall that a sequence a_k is *log–concave* if $a_{k+1}^2 \geq a_k a_{k+2}, \forall k$; see [7, 24].

Proposition 5.2. *For $d \geq 2$, the sequence $\{H_{d,k} : 0 \leq k \leq d-1\}$ is log–concave.*

Proof. For fixed d, the sequence $T_{d,k} - 1 = 2^{d-k} - 1$ is log–concave, because $(T_{d,k+1} - 1)^2 - (T_{d,k} - 1)(T_{d,k+2} - 1) = 2^{d-k-2} > 2 > 0$, for $0 \leq k < k+2 \leq d-1$. It is easy to check that any row of Pascal's triangle is a log–concave sequence. Since the termwise product of two log–concave sequences (with the same number of terms) is log–concave, then the result follows for $H_{d,k}$. □

Notice $I_{d,k} = 2H_{d,k}$, for $0 \leq k \leq d$.

[4] $H_{d,k}$ is an expression involving 2–powers and binomial coefficients. Precisely, $H_{d,k} = (T_{d,k} - 1)P_{d+1,k}$ is the product of two factors. For sufficiently small k, the first factor dominates (meaning, is larger than the other factor), as in the cases $H_{d,0} = 2^d - 1$, $H_{d,1} = (2^{d-1} - 1)(d+1)$ and $H_{d,2} = (2^{d-2} - 1)(d+1)d/2$. However, for sufficiently large k, the second factor dominates, as in the cases $H_{d,d-3} = 7(d+1)d(d-1)(d-2)/24$, $H_{d,d-2} = (d+1)d(d-1)/2$ and $H_{d,d-1} = (d+1)d/2$.

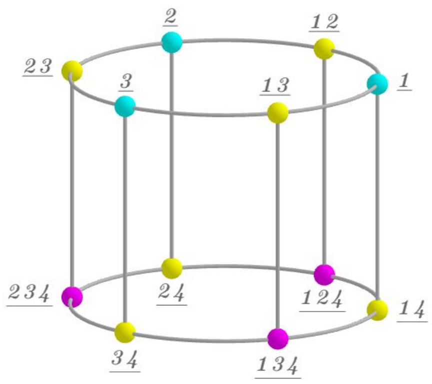

Figure 3: Ecuatorial Belt for $d = 3$.

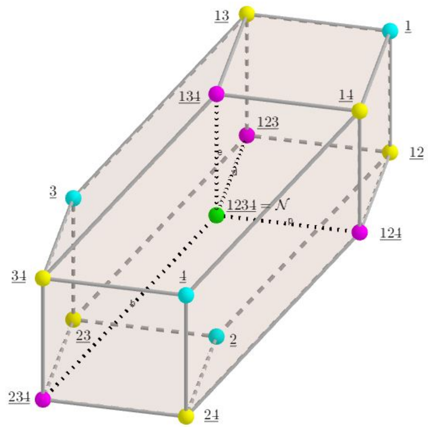

Figure 4: North Polar Cask for $d = 4$.

Corollary 5.3 (Unimodality holds for isocanted)**.** *For each $d \geq 2$, the sequence $\{I_{d,k} : 0 \leq k \leq d-1\}$ is unimodal.*

Proof. It is easy to show that every log–concave sequence is unimodal (but not conversely). The sequence $H_{d,k}$ is unimodal and so is its double. $\square$

Corollary 5.4 (Bárány conjecture holds for isocanted)**.** *If $d \geq 2$ and $0 \leq k < k+1 \leq d-1$, then $I_{d,k} \geq \min\{I_{d,0}, I_{d,d-1}\} = I_{d,d-1} = (d+1)d$.*

Proof. Use unimodality and Proposition 5.1. $\square$

Corollary 5.5 (3^d conjecture holds for isocanted)**.** *For $d \geq 2$, it holds $\sum_{k=0}^{d} I_{d,k} = 3^{d+1} - 2^{d+2} + 2$ and this is larger than 3^d.*

Proof. The binomial theorem $(x+y)^d = \sum_{j=0}^{d} x^j y^{d-j} \binom{d}{j}$ with $x = 1$ yields $2^d = \sum_{j=0}^{d} \binom{d}{j}$ and $3^d = \sum_{j=0}^{d} 2^{d-j} \binom{d}{j}$. Then

$$3^{d+1} - 2 \times 2^{d+1} = \sum_{j=0}^{d+1} 2^{d+1-j} \binom{d+1}{j} - 2 \sum_{j=0}^{d+1} \binom{d+1}{j} = \sum_{j=0}^{d+1} (2^{d+1-j} - 2) \binom{d+1}{j} = \sum_{j=0}^{d-1} I_{d,j} + \text{two summands.} \tag{6}$$

Summand for $j = d$ is zero and for $j = d+1$ is -1, whence, using $I_{d,d} = 1$, we get the claimed equality. Proof of the inequality: we have $2^3 = 8 = 3^2 - 1$ and $2^{d-2} < 3^{d-2}$. Multiply termwise and get $2^{d+1} \leq 3^{d-2}(3^2 - 1) = 3^d - 3^{d-2} < 3^d + 1$ whence $2(2^{d+1} - 1) < 2 \times 3^d = 3^{d+1} - 3^d$. $\square$

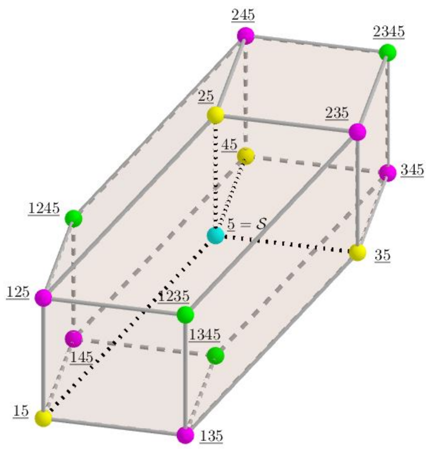

Figure 5: South Polar Cask for $d = 4$.

Remark 5.6. *Recall that* Stirling number of the second kind *is the number of ways to partition* $[d]$ *into* k *non–empty subsets, and it is denoted* $S(d,k)$. *We have* $3^{d+1} - 2^{d+2} + 2 = 2S(d+2,3) + 1$ *(see Wikipedia and OEIS A101052, OEIS A028243 and OEIS A000392).*

Recall that a *Hanner polytope* is obtained from closed intervals, by using two operations any finite number of times: Cartesian product and polar.

Remark 5.7. *Is* $\mathcal{I}^d$ *a Hanner polytope? Conversely, is some Hanner polytope an IAP? Since Hanner polytopes satisfy the* 3^d *conjecture and they attain the minimal conjectured value, then the answer is NO in both cases.*

Recall that a *complete flag* in a polytope $\mathcal{P}$ is a maximal chain of faces of $\mathcal{P}$ with increasing dimensions. Next, we count complete flags (and call them flags, for short). The number of flags in a d–box is $2^d d!$, because there are 2^d vertices and, at each one, there are $d!$ flags. The flag conjecture (stated by Kalai in 2008; see [15, 22]) yields that boxes minimize flags among centrally symmetric polytopes.

Corollary 5.8 (Flag conjecture holds for isocanted)**.** *The number of flags in* $\mathcal{I}^d$ *is* $(d+1)(d-1)!(2^{d+1}-4)$ *and it is larger than* $2^d d!$*, for* $d \geq 2$.

Proof. In $\mathcal{I}^d$ there are $2(d+1)$ vertices of valence d, and the remaining $2(2^d - d - 2)$ vertices have valence $d+1$. Indeed, the vertices of length 1 or d have valence d. A vertex of length $2 \leq t \leq d-1$ has valence $d+1$, because it has t parents and $d+1-t$ children. Now, using Corollary 3.10 and reasoning as in boxes, we find $d!$ flags beginning at a vertex of valence d, but $(d+1)(d-1)!$ flags beginning at a vertex of valence $d+1$. Thus, adding up, $2(d+1) \times d! + 2(2^d - d - 2) \times (d+1)(d-1)! = (d+1)(d-1)!(2^{d+1}-4)$ is the total number of flags. Further, we have $(2^{d-1}-1)(d+1) > 2^{d-2}d$, for $d \geq 2$, whence the claimed inequality. ☐

The *cubical lower bound conjecture (CLBC)* was posed by Jockusch in 1993 and rephrased, in terms of the cubical g–vector g^c, by Adin et al. in 2019 as follows: is $g^c_{d,2} \geq 0$?; see [2, 14].

Proposition 5.9 (CLBC holds for isocanted)**.** $g^c_{d,2} \geq 0$ *holds true for* $\mathcal{I}^d$*, for* $d \geq 2$.

Proof. We have computed the sequence $g^c_{d,2}$ for IAPs, obtaining $6, 20, 50, 112, 238, \ldots$; see OEIS A052515. ☐

Recall that the *distance* between two vertices of a polytope is the minimum number of edges in a path joining them. The *diameter* of a polytope is the greatest distance between two vertices of the polytope.

Corollary 5.10 (Diameter of isocanted)**.** *The diameter of* $\mathcal{I}^d$ *is* $d+1$.

Proof. Consider different proper subsets $W, W' \subset [d+1]$ and assume $|W \cap W'| = i$, $|W| = i+w$, $|W'| = i+w'$, with $i, w, w' \geq 0$ and $i + w + w' \leq d+1$. To go from vertex $\underline{W}$ to vertex $\underline{W'}$ one must drop (one at a time) the w digits in $W \setminus W'$ and one must gain (one at a time) the w' digits in $W' \setminus W$, whence $\mathrm{d}(\underline{W}, \underline{W'}) = w + w'$. In the particular case that W' is complementary to W, we get the greatest distance $\mathrm{d}(\underline{W}, \underline{W'}) = d+1$. ☐

6 Appendix

For fixed $d \geq 2$, the following is a direct proof of unimodality of the sequence $H_{d,k}$ defined in (5). Consider the quotient

$$Q_{d,k+1} := \frac{H_{d,k+1}}{H_{d,k}} = \frac{(2^{d-k-1}-1)(d-k+1)}{(2^{d-k}-1)(k+1)}, \quad 0 \leq k < k+1 \leq d-1 \tag{7}$$

where the equality is due to factor simplifications. Furthermore, clearing the positive denominator $(2^{d-k}-1)(k+1)$ and grouping terms, we get that $H_{d,k+1} \geq H_{d,k}$ if and only if $Q_{d,k+1} \geq 1$ if and only if $L_{d,k+1} \geq R_{d,k+1}$, where

$$L_{d,k+1} := 2^{d-k-1}(d-3k-1), \quad R_{d,k+1} := d-2k. \tag{8}$$

Note that the exponent $d-k-1$ appearing in $L_{d,k+1}$ is at least 1.

Remark 6.1 (Two easy cases)**.** *For $0 \leq k < k+1 \leq d-1$, it holds*

1. *if $d-3k-1=0$, then $L_{d,k+1}=0<R_{d,k+1}=k+1$ and $Q_{d,k+1}=\frac{2^{2k+1}-2}{2^{2k+1}-1}<1$ (and $Q_{d,k+1}$ is nearly 1),*
2. *if $d-2k=0$, then $L_{d,k+1}=2^{k-1}(-k-1)<0=R_{d,k+1}$ and $Q_{d,k+1}=\frac{2^{k-1}-1}{2^k-1}<1$.*

In both cases, the sequence $H_{d,k}$ is decreasing.

Proposition 6.2. *If $d \geq 2$ and $0 \leq k < k+1 \leq d-1$, then*

1. *if $4k \leq d-2$, then $Q_{d,k+1} \geq 1$,*
2. *if $2d-5 \leq 3k$, then $Q_{d,k+1} \leq 1$,*
3. *if $\frac{d-2}{4} \leq k \leq \frac{2d-5}{3}$ and*
   (a) *if $d-3k-1 \geq 1$, then $Q_{d,k+1} \geq 1$,*
   (b) *if $d-3k-1 \leq -1$, then $Q_{d,k+1} \leq 1$.*

Proof. 1. We have $L_{d,k+1} \geq 2(d-3k-1) \geq d-2k = R_{d,k+1} \geq 2k+2 > 0$ whence $Q_{d,k+1} \geq 1$ follows. In this case, the factor $(d-3k-1)$ appearing in $L_{d,k+1}$ is, at least, 1.

2. We have $2d-5 \leq 3k \leq 3d-6 \Rightarrow -2d+5 \geq -3k \geq -3d+6 \Rightarrow 0 > -2 \geq -d+4 \geq d-3k-1 \geq -2d+5$ so one factor of $L_{d,k+1}$ is negative and $\frac{-4d+10}{3} \geq -2k \geq -2d+4 \Rightarrow 0 \geq \frac{-d+10}{3} \geq d-2k \geq -d+4$ so we have $R_{d,k+1} \geq -d+4 \geq d-3k-1 \geq 2(d-3k-1) \geq L_{d,k+1}$; then $Q_{d,k+1} \leq 1$ follows.

3. We have $\frac{d-2}{4} \leq k \leq \frac{2d-5}{3} \Rightarrow \frac{d+2}{2} \geq d-2k = R_{d,k+1} \geq \frac{-d+10}{3}$ and the exponent in $L_{d,k+1}$ is bounded above and below as follows: $\frac{3d-2}{4} \geq d-k-1 \geq \frac{d+2}{3}$.
   (a) If $d-3k-1 \geq 1$, then $L_{d,k+1} \geq 2^{\frac{d+2}{3}}(d-3k-1) \geq 2^{\frac{d+2}{3}} \geq \frac{d+2}{2} \geq R_{d,k+1}$ and so $Q_{d,k+1} \geq 1$.
   (b) If $d-3k-1 \leq -1$, then $R_{d,k+1} \geq \frac{-d+10}{3} \geq -2^{\frac{d+2}{3}} \geq 2^{\frac{d+2}{3}}(d-3k-1) \geq L_{d,k+1}$ and so $Q_{d,k+1} \leq 1$.

□

Recall $I_{d,k} = 2H_{d,k}$, for $k < d$, $I_{d,d} = 1$.

Corollary 6.3 (Unimodality holds for isocanted)**.** *If $d \geq 2$ and $0 \leq k < k+1 \leq d-1$, then*

1. *if $k < \frac{d-1}{3}$ then $I_{d,k+1} \geq I_{d,k}$,*
2. *if $\frac{d-1}{3} \leq k$ then $I_{d,k+1} \leq I_{d,k}$.*

Proof. The cases $2 \leq d \leq 4$ are checked directly. Besides, $d \geq 5 \Rightarrow \frac{d-2}{4} \leq \frac{d-1}{3} \leq \frac{2d-6}{3} < \frac{2d-5}{3}$ and then

1. follows from Items 1 and 3a from Proposition 6.2.

2. follows from Items 2 and 3b from Proposition 6.2, and Item 1 from Remark 6.1.

□

Corollary 6.4. *For fixed $d \geq 2$, the maximum in the sequence $I_{d,k}$ is attained at the integer k closest to $\frac{d}{3}$.*

Proof. For fixed d we have found the maximum in $H_{d,k}$ attained at $k = \begin{cases} \frac{d}{3}, & d \equiv 0 \mod 3, \\ \frac{d-1}{3}, & d \equiv 1 \mod 3, \\ \frac{d+1}{3}, & d \equiv 2 \mod 3. \end{cases}$ □

Key to colors: blue dots are generators, yellow dots are vertices of length 2, magenta dots are vertices of length 3.

Acknowledgments

The first author is partially supported by Ministerio de Economía y Competitividad, Proyecto I+D MTM2016-76808-P and by UCM research group 910444.

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