Workshop Operators and Banach lattices

Universidad Complutense de Madrid,
Facultad de Ciencias Matemáticas,
25-26 October 2012

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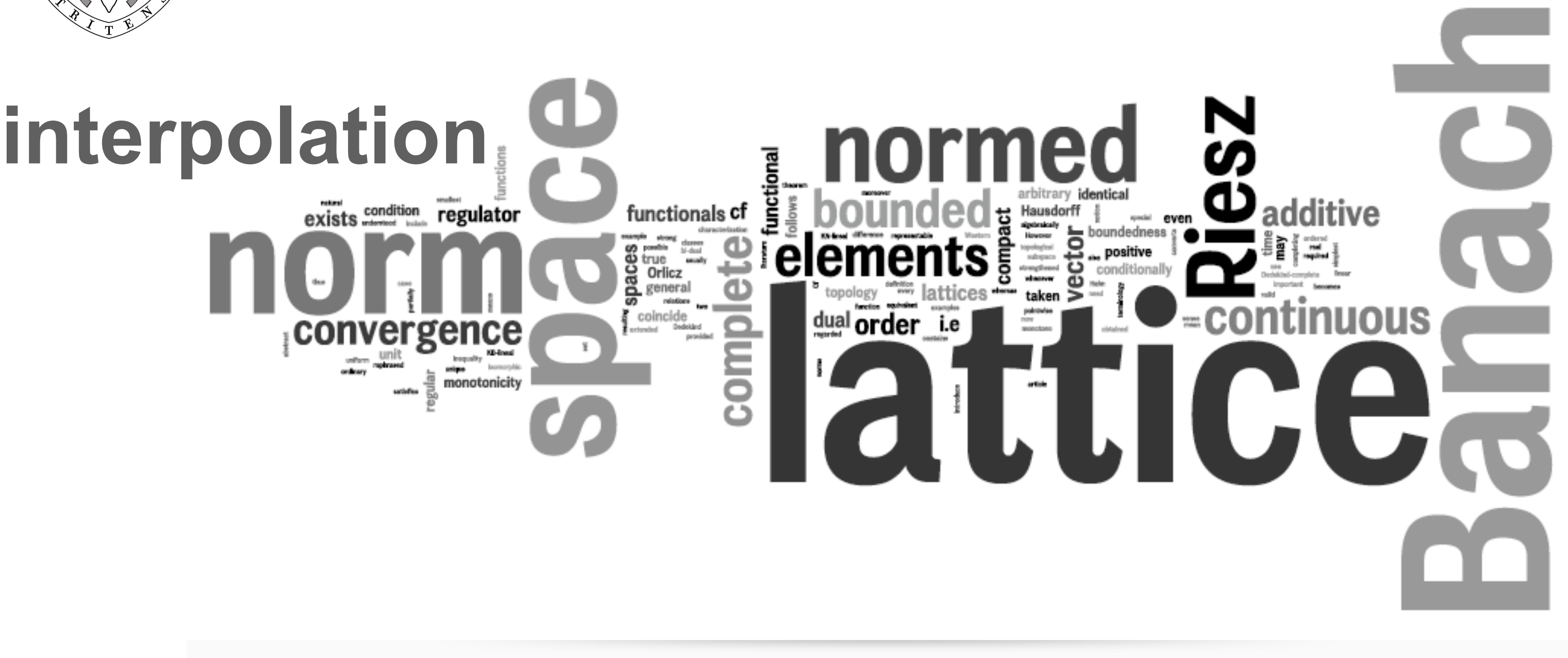

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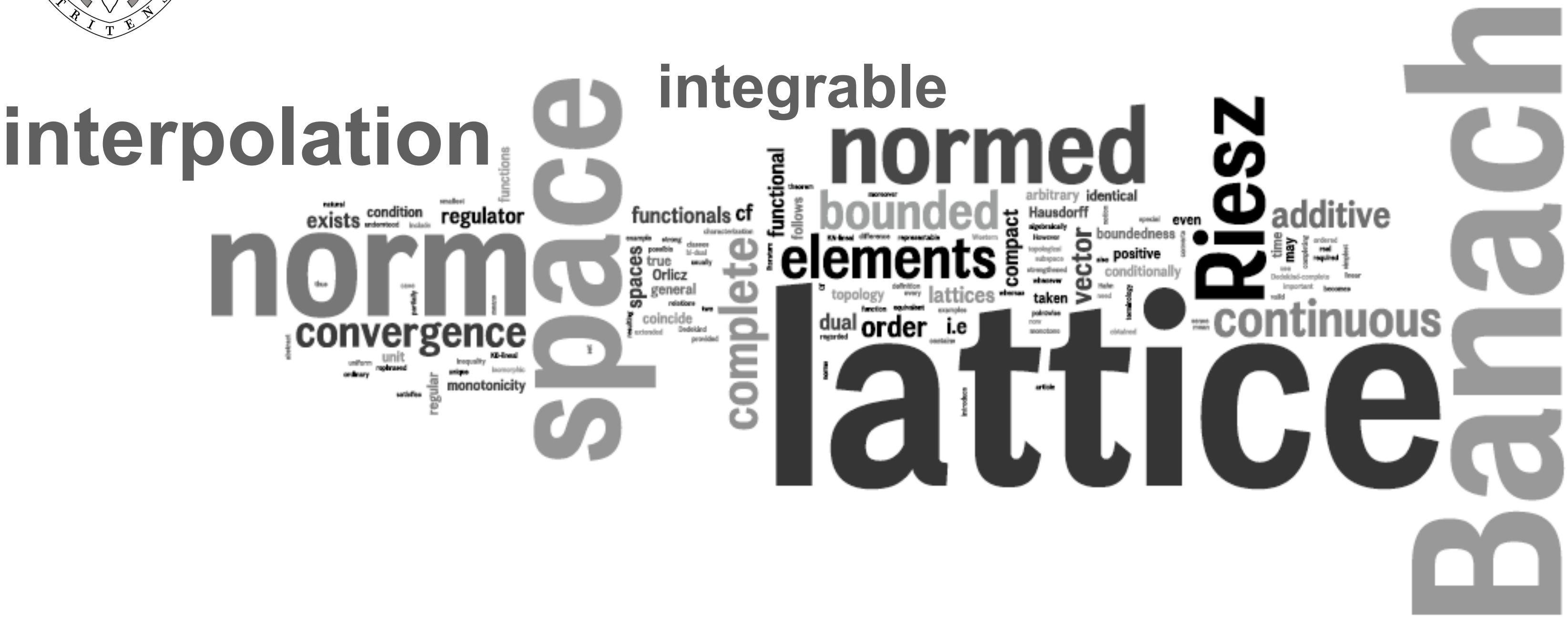

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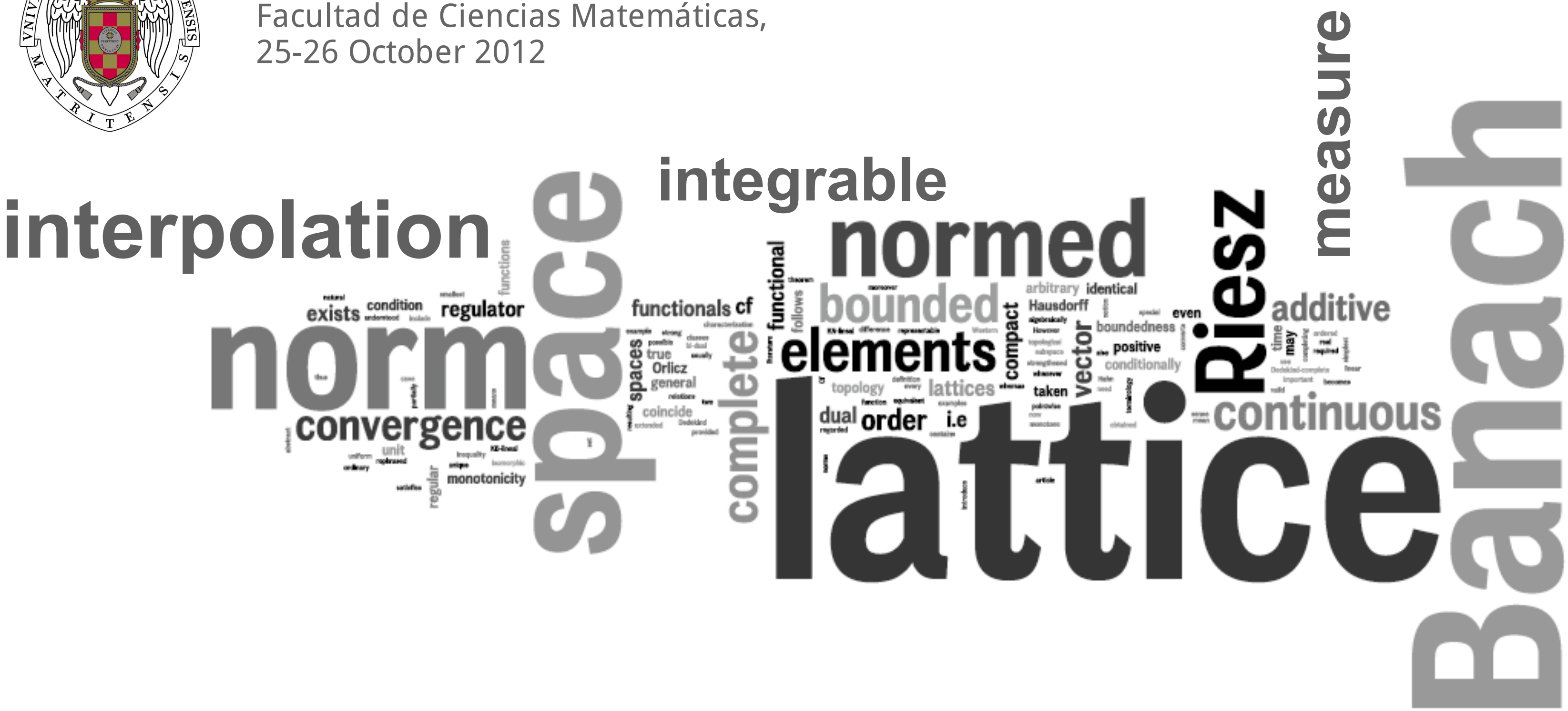

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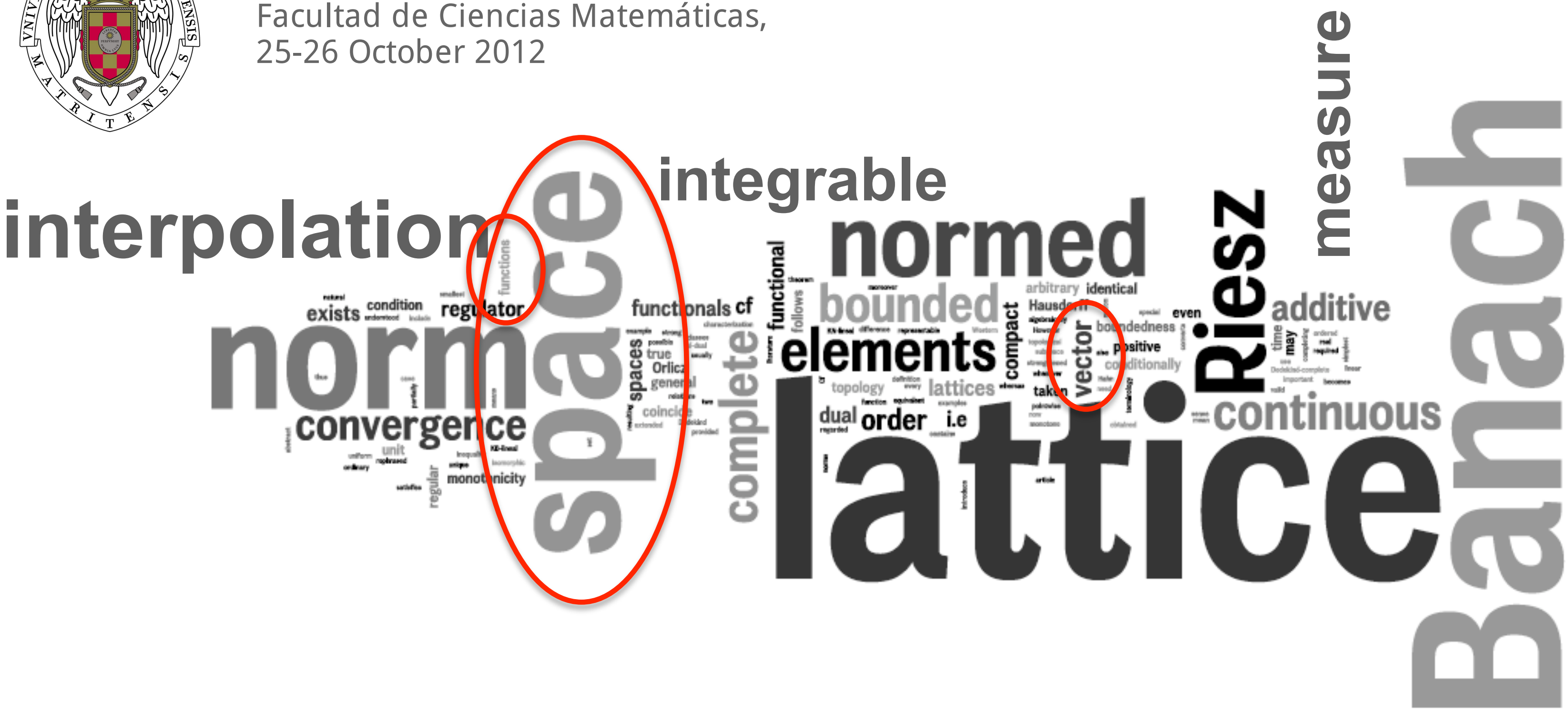

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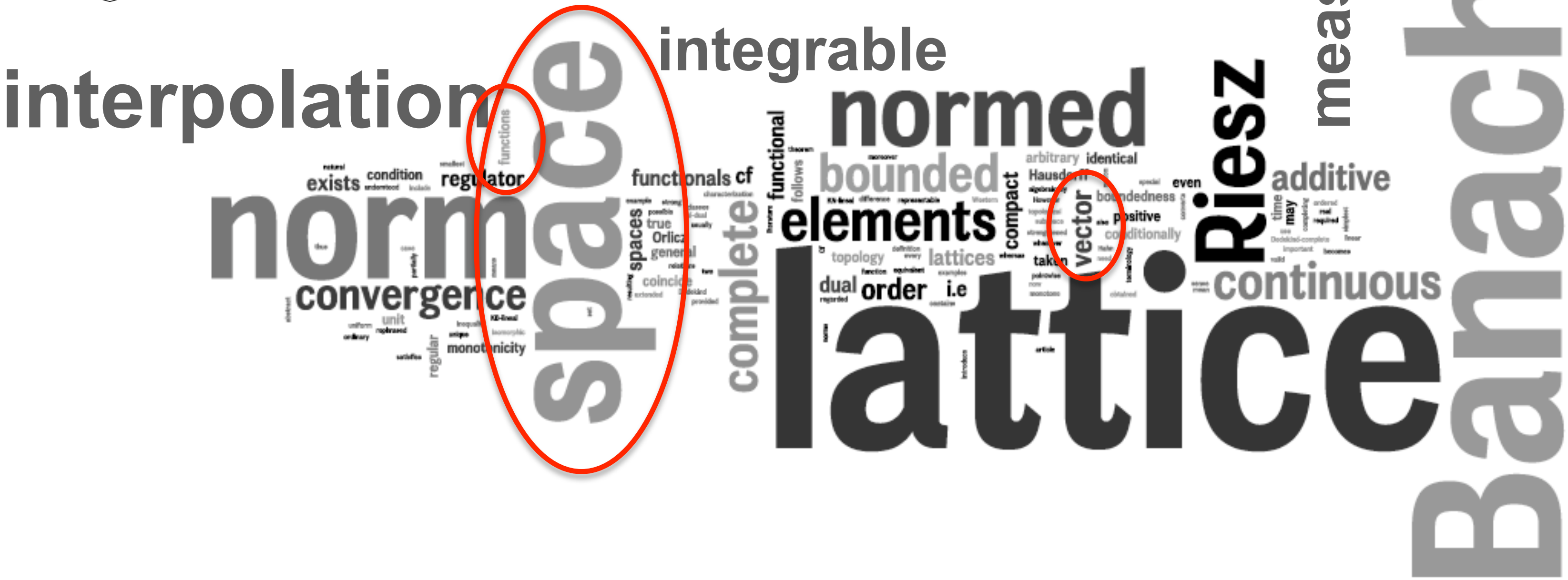

Interpolation of spaces of integrable functions with respect to a vector measure

Antonio Fernández (Universidad de Sevilla)

The team (FQM–133).

The Bartle–Dunford–Schwartz integral.

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The ingredients: Ω set, Σ sigma-algebra ($\mathcal{R}$ delta-ring), X Banach space and $m : \Sigma \to X$ ($\nu : \mathcal{R} \to X$) vector measure.

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$$\mathcal{R} \xrightarrow{\nu} X \xrightarrow{x'} \mathbb{K}, \quad x' \in X'$$

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A set $A \in \mathcal{R}^{\text{loc}}$ is said to be null if $\|\nu\|(A) = 0$.

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The definition. A measurable function $f:\Omega\longrightarrow\mathbb{K}$ is said to be integrable if:

I) $f\in L^1(|\langle\nu,x'\rangle|)$ for each $x'\in X'$, and

II) For every $A\in\mathcal{R}^{\mathrm{loc}}$ there exists $\int_A f\,d\nu\in X$ such that
$$\left\langle\int_A f\,d\nu, x'\right\rangle=\int_A f\,d\langle\nu,x'\rangle,\quad x'\in X'.$$

The function spaces.

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$$
\begin{aligned}
L^1_w(\nu) &:= \{f : \Omega \longrightarrow \mathbb{K} : \mathbf{I)}\}\,, \\
L^1(\nu) &:= \{f : \Omega \longrightarrow \mathbb{K} : \mathbf{I)} + \mathbf{II)}\}\,.
\end{aligned}
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In general, $L^p(\nu) \subsetneq L^p_w(\nu)$.

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7) $L^p(m) \rightsquigarrow L^p[0,1]$ and $L^p(\nu) \rightsquigarrow L^p(\mathbb{R})$

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If $1 \leq p_0 \neq p_1 < \infty$ and $\nu : \mathcal{R} \to X$ sigma-finite

$$\Omega = \bigcup_{n \geq 1} \Omega_n \cup N, \quad (\Omega_n)_n \subseteq \mathcal{R}, \; \|\nu\|(N) = 0,$$

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we can consider the following (compatible) couples of Banach spaces:

$$(L^{p_0}(\nu), L^{p_1}(\nu)) \quad (L^{p_0}_w(\nu), L^{p_1}_w(\nu))$$

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The ambient space: $L^0(\nu)$.

Complex methods. [Calderón (1964)]

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4) $[L^{p_0}(\mu), L^{p_1}(\mu)]_{[\theta]} = (L^{p_0}(\mu))^{1-\theta} (L^{p_1}(\mu))^{\theta}$

$$\frac{1}{p} := \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \qquad = [L^{p_0}(\mu), L^{p_1}(\mu)]^{[\theta]} = L^p(\mu).$$

$1 \le p_0, p_1 \le \infty$

Complex methods: the problem.

For $\nu : \mathcal{R} \to X$ sigma-finite, and

$$0 < \theta < 1 \leq p_0 \neq p_1 \leq \infty$$

describe

$$[L^{p_0}(\nu), L^{p_1}(\nu)]_{[\theta]}, \qquad [L^{p_0}(\nu), L^{p_1}(\nu)]^{[\theta]},$$

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$L^{p_1}_w(m)$	$L^p(m)_{(\Sigma)}$	$L^p(m)_{(\mathcal{R})}$

$[\cdot,\cdot]^\theta$	$L^{p_0}(m)$	$L^{p_0}_w(m)$
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It is not difficult to show that

$$L^p(\nu) = c_0 \subseteq L^p_w(\nu) = \ell^\infty, \ 1 \leq p < \infty.$$

Then, for $1 \leq p_0, p_1 < \infty$,

$$[L^{p_0}_w(\nu), L^{p_1}_w(\nu)]_{[\theta]} \; = \; [\ell^\infty, \ell^\infty]_{[\theta]} = \ell^\infty = L^p_w(\nu),$$

$$[L^{p_0}(\nu), L^{p_1}(\nu)]^{[\theta]} \; = \; [c_0, c_0]^{[\theta]} = c_0 = L^p(\nu).$$

Intermediate spaces (sigma-algebra).

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If $1 \leq p_0 \neq p_1 \leq \infty$ and $r < p < s$, where

$$r := \min\{p_0, p_1\} < s := \max\{p_0, p_1\},$$

then

$$L^s(m) \subseteq L^s_w(m) \subseteq L^p(m) \subseteq L^p_w(m) \subseteq L^r(m) \subseteq L^r_w(m).$$

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$$L^s(m) \subseteq L^s_w(m) \subseteq L^p(m) \subseteq L^p_w(m) \subseteq L^r(m) \subseteq L^r_w(m).$$

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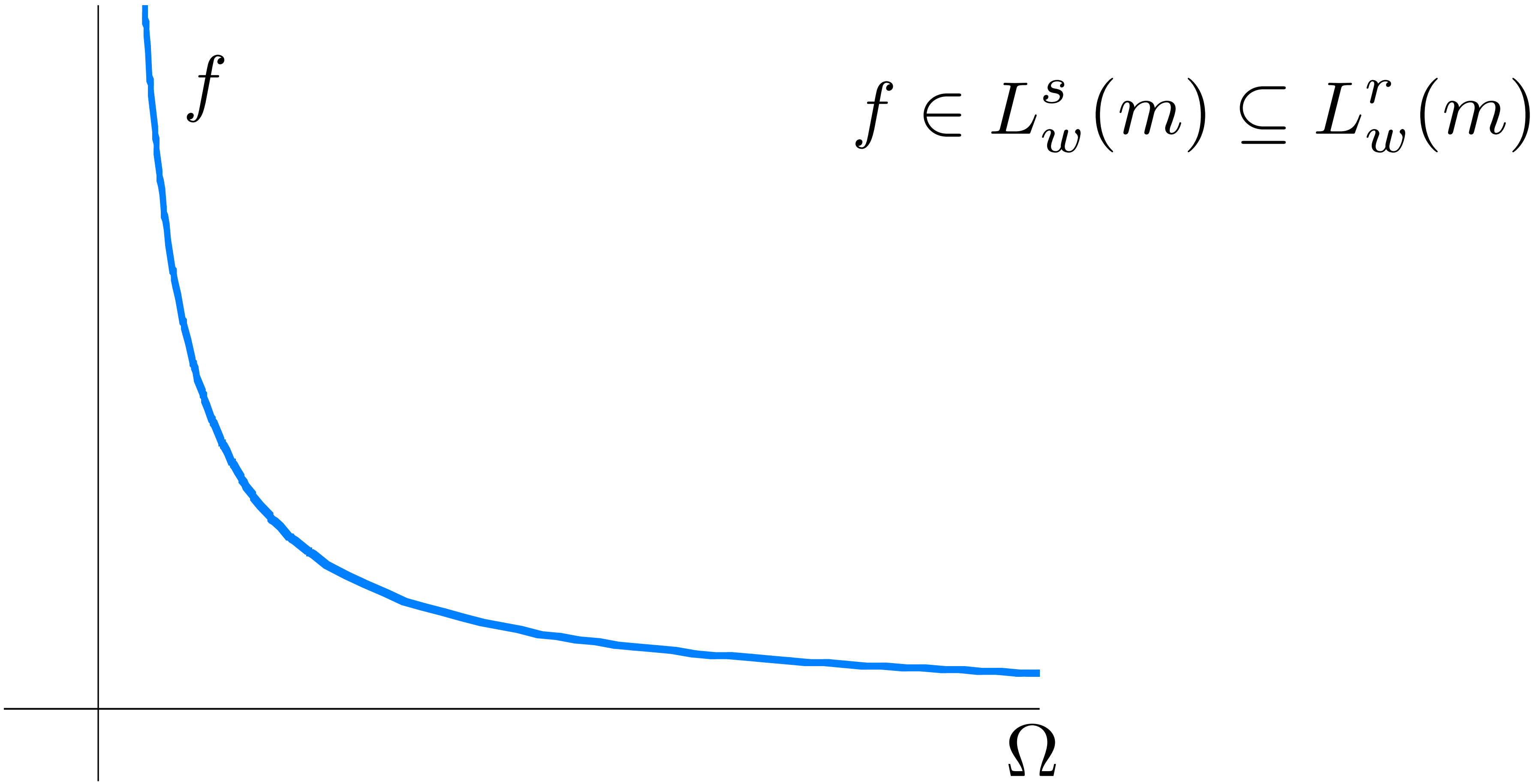

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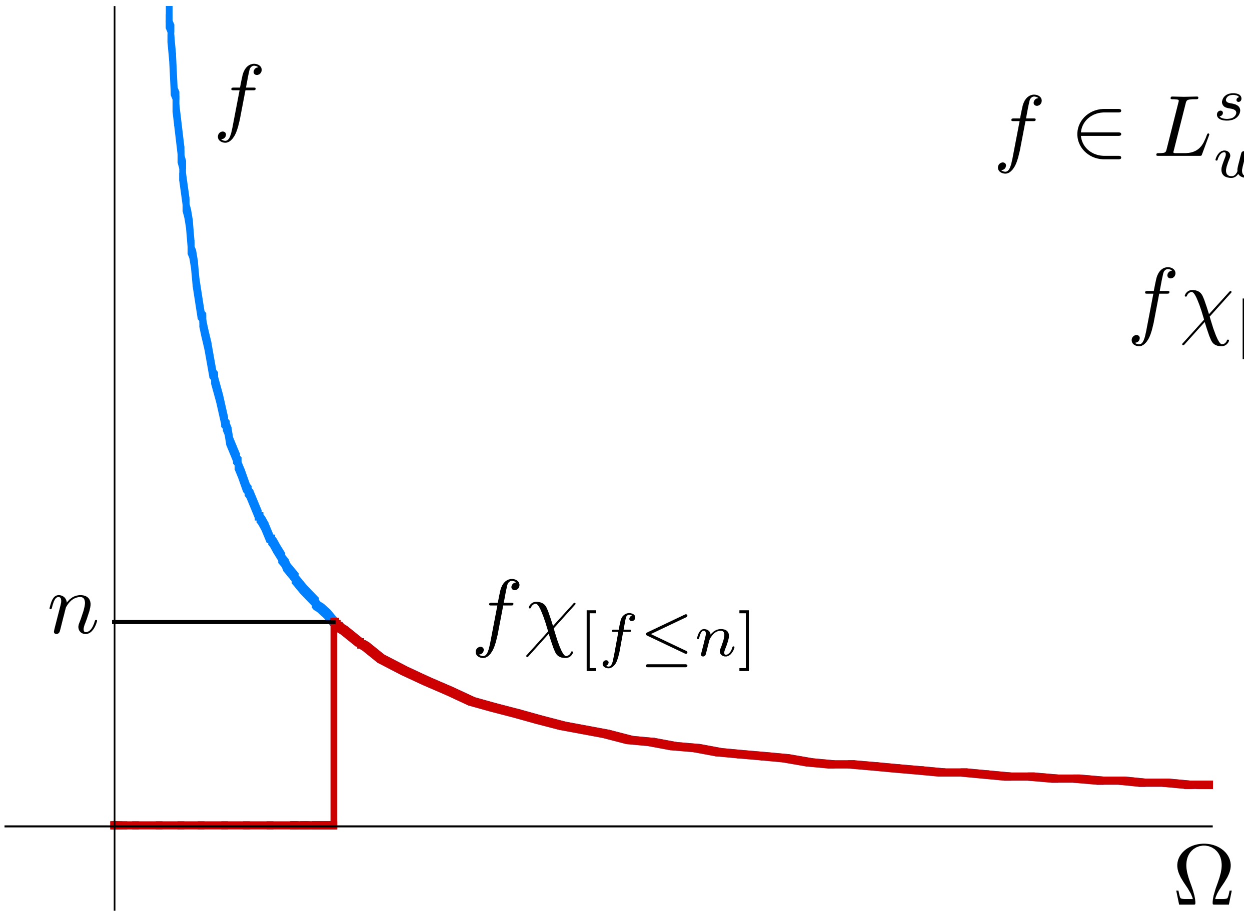

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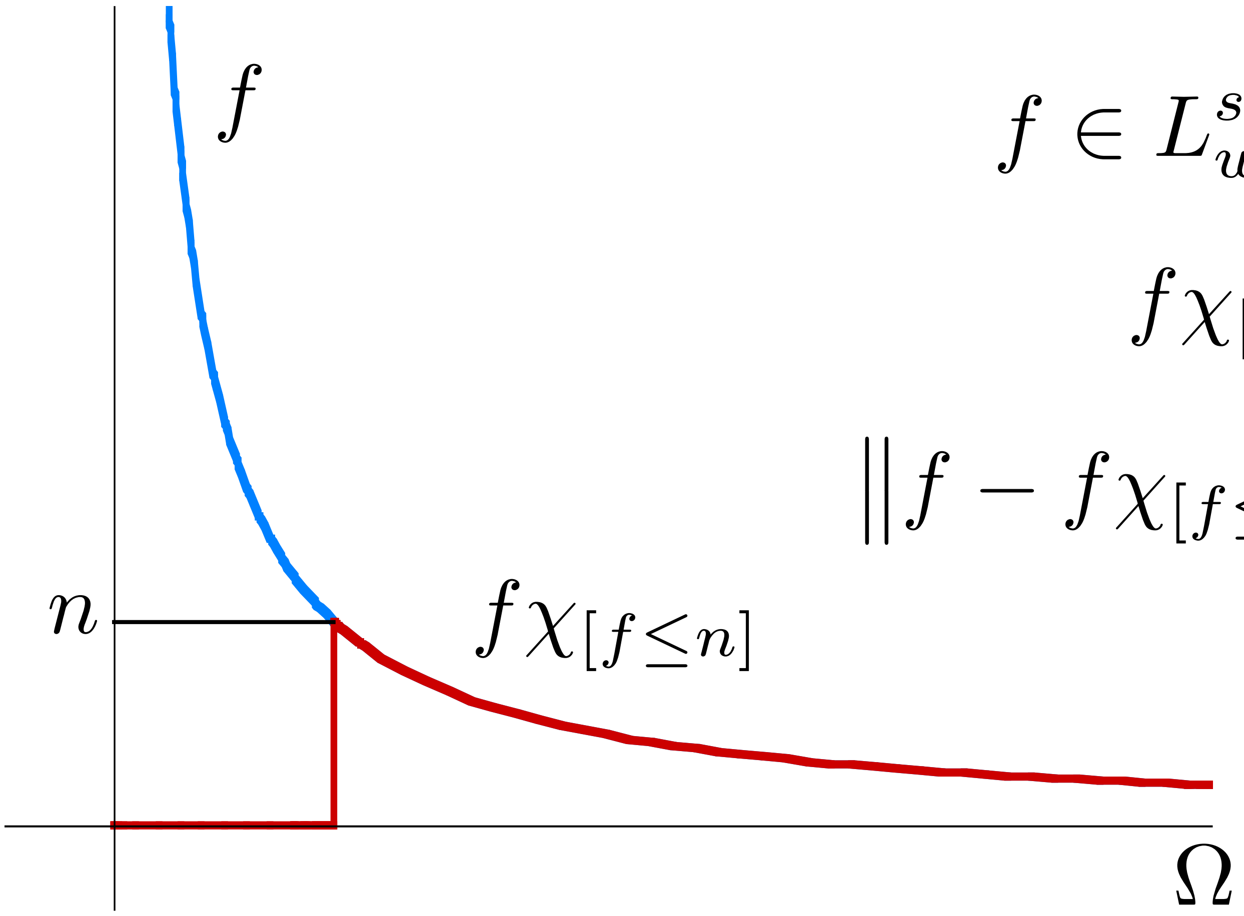

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Nevertheless,

$$\begin{aligned} L^{p_0}_w(\nu) \cap L^{p_1}_w(\nu) \quad &\not\subseteq \quad L^{p}(\nu), \\ L^{p}_w(\nu) \quad &\not\subseteq \quad L^{p_0}(\nu) + L^{p_1}(\nu). \end{aligned}$$

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$\nu : \mathcal{R} \to X$ is said to be locally strongly additive if

$\lim_{n\to\infty} \|\nu(A_n)\|_X = 0$ for each disjoint sequence $(A_n)_n \subseteq \mathcal{R}$ such that $\|\nu\| \left(\bigcup_{n\geq 1} A_n \right) < \infty$.

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2) $\nu : A \in \mathcal{P}_f(\mathbb{N}) \longrightarrow \nu(A) := \chi_A \in c_0$ is not locally strongly additive.

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3) Every $m : \Sigma \to X$ is (locally) strongly additive.

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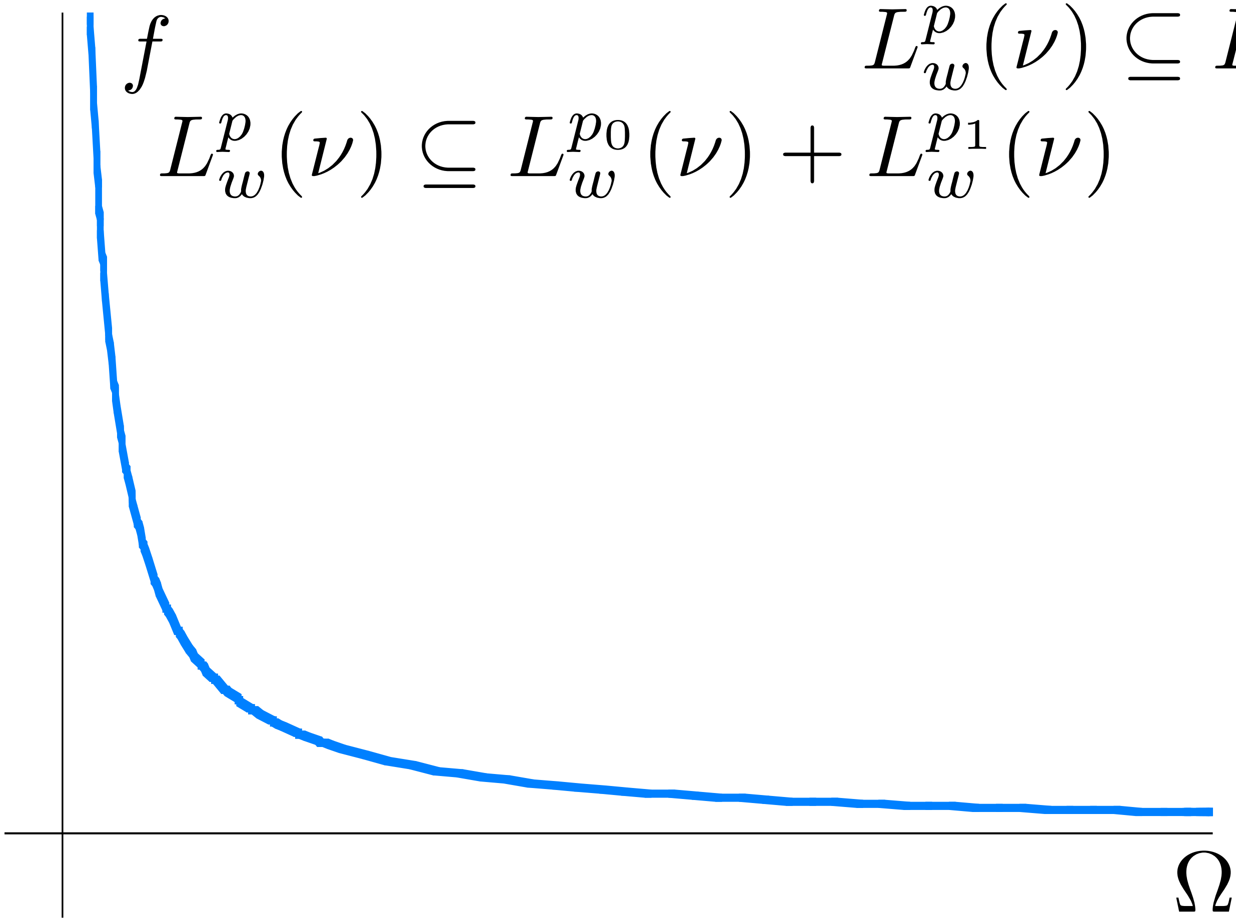

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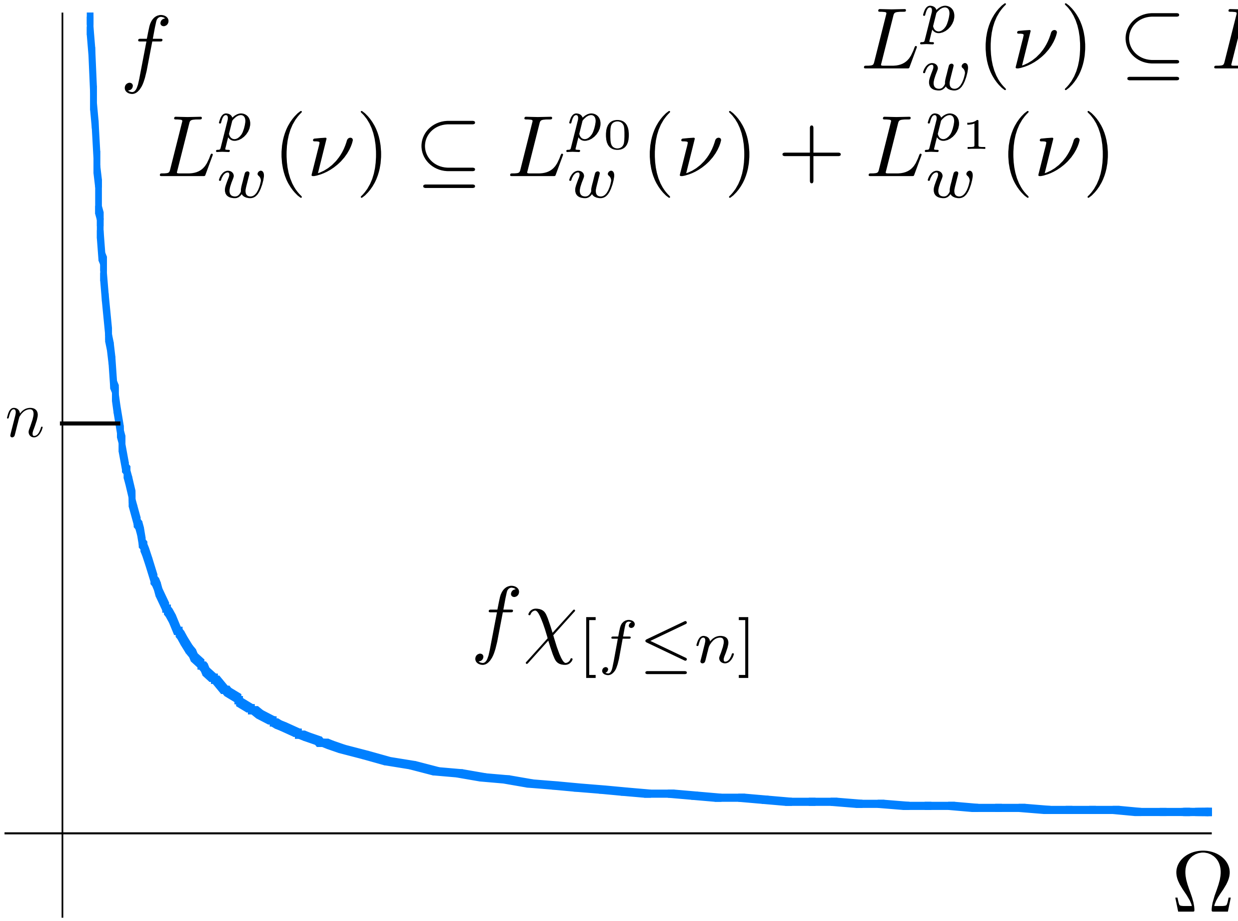

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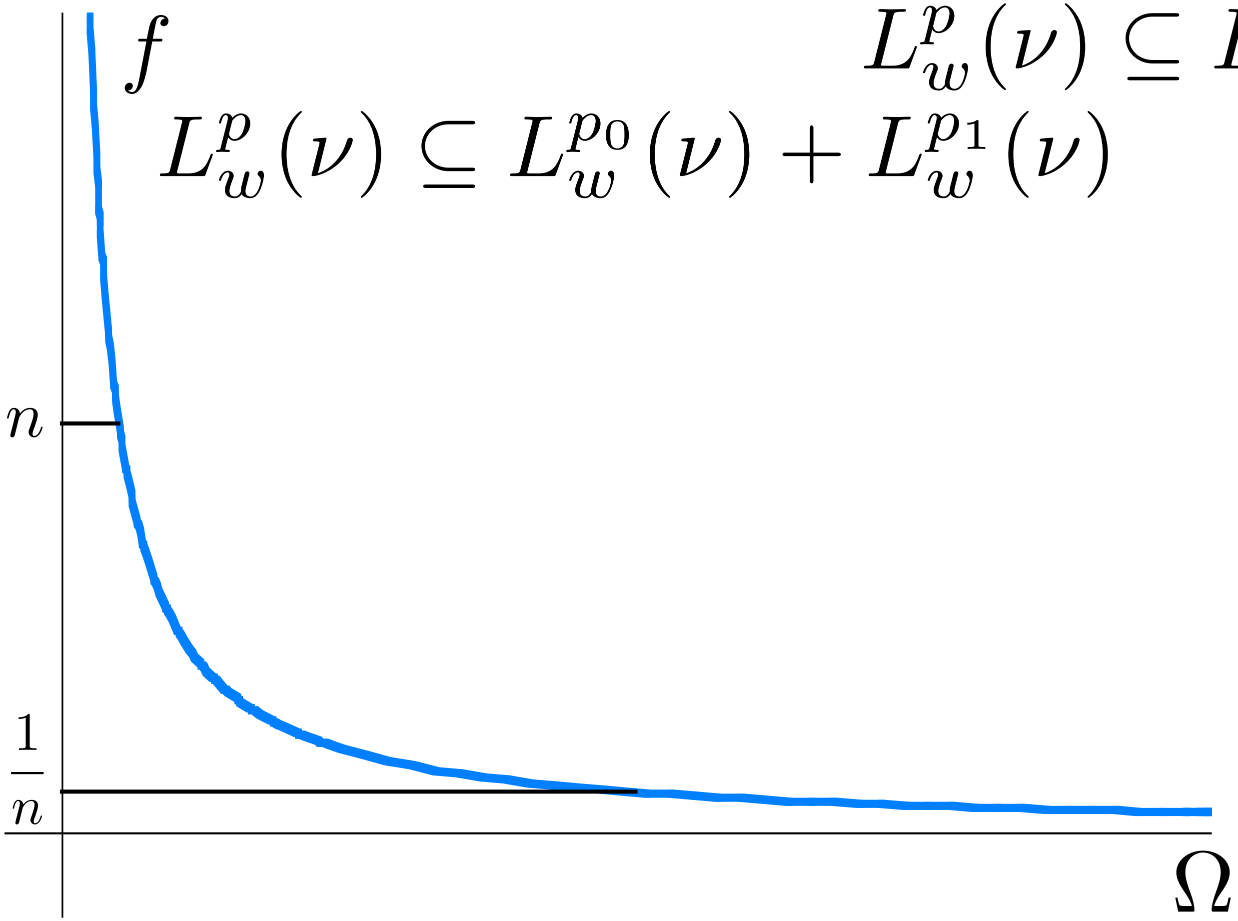

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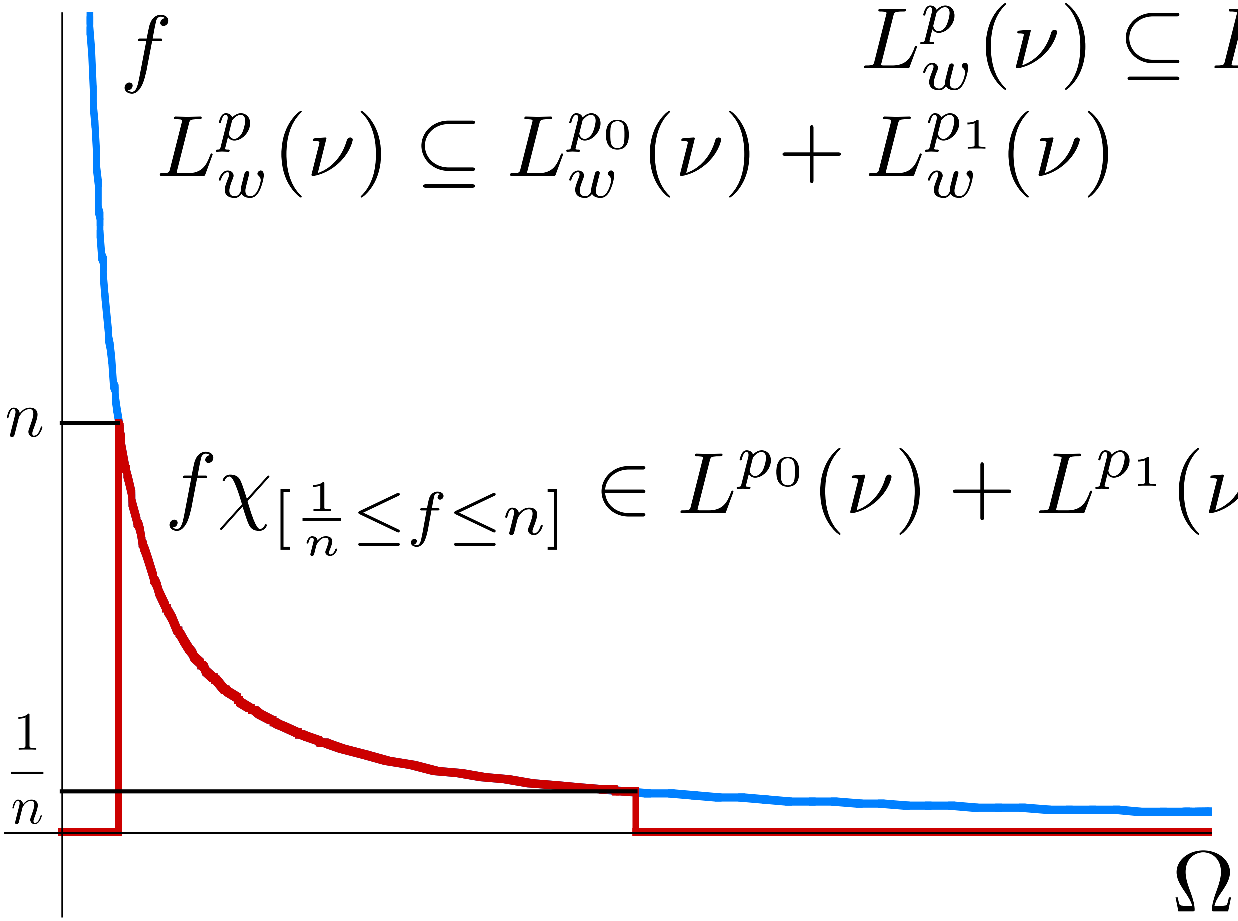

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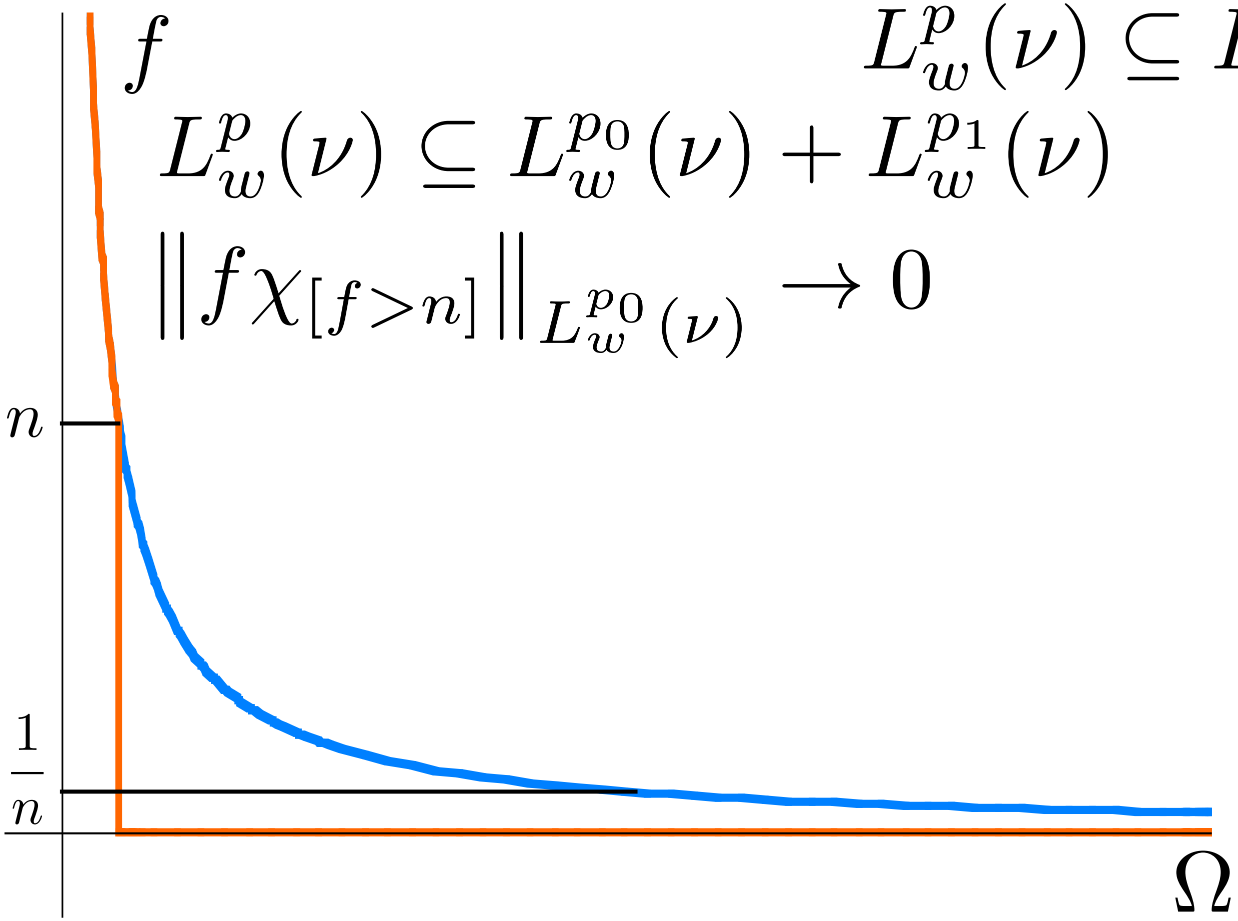

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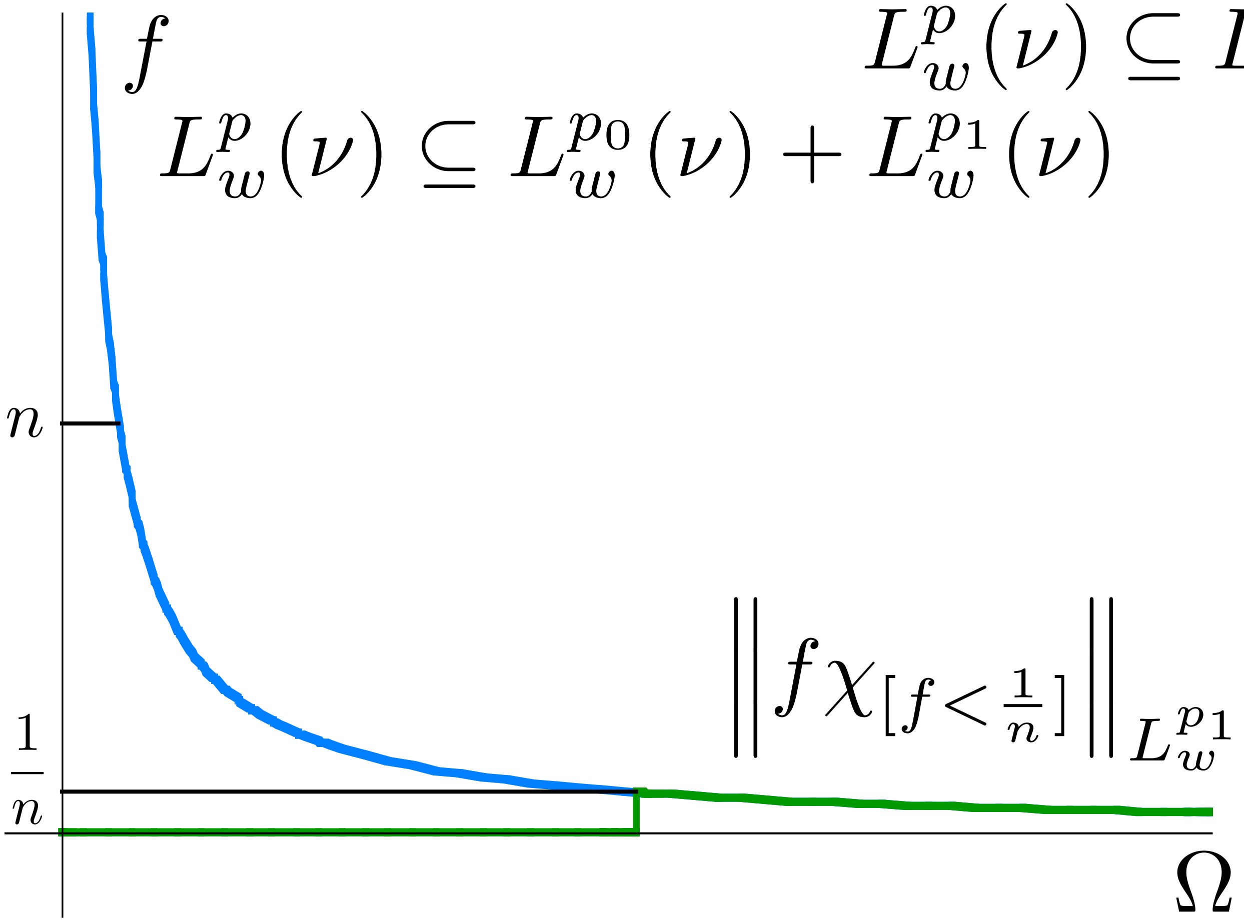

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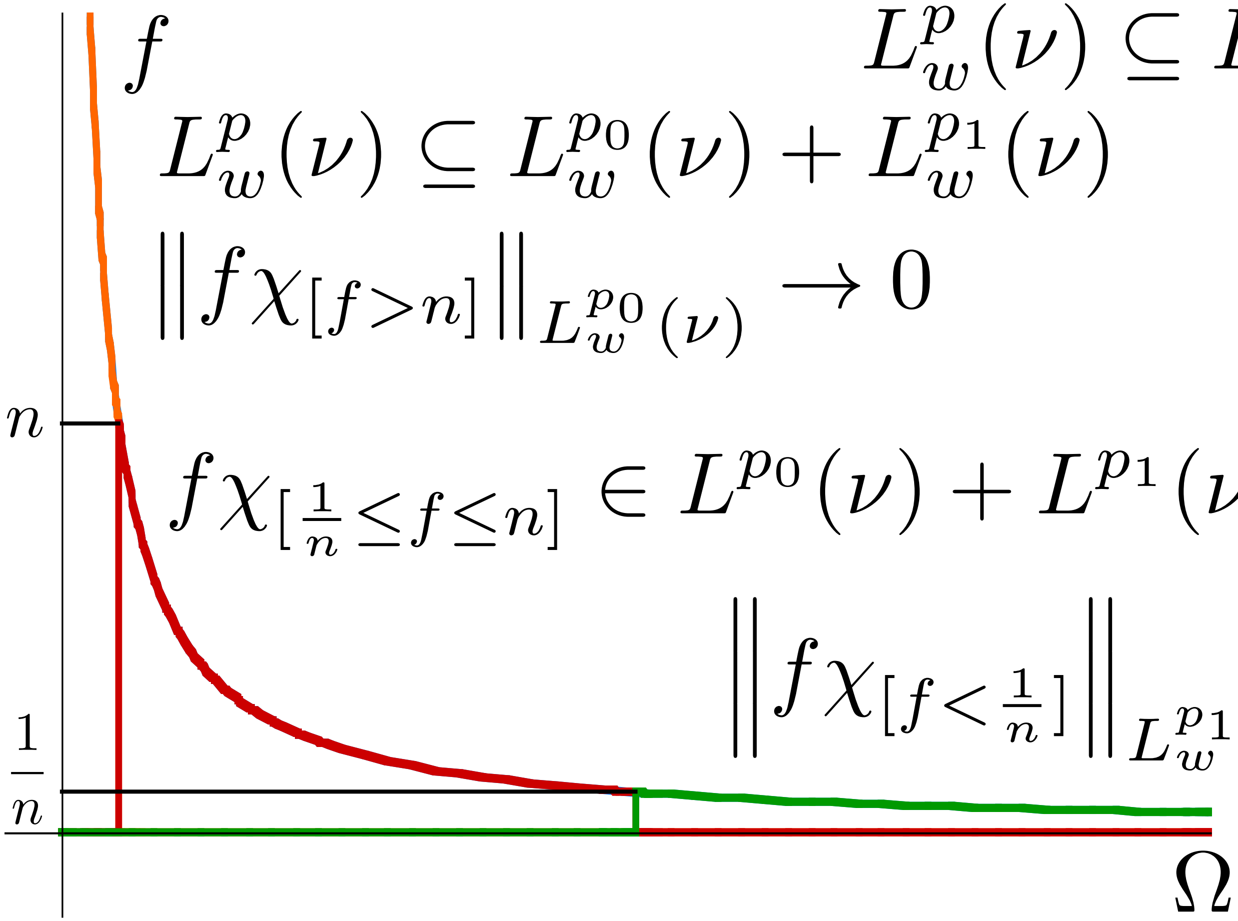

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$$\|f\|_{\theta,q} := \begin{cases} \left[\int_0^\infty \left(t^{-\theta} K(t,f)\right)^q \frac{dt}{t}\right]^{\frac{1}{q}}, & q < \infty \\ \sup\limits_{t>0} t^{-\theta} K(t,f), & q = \infty \end{cases}$$

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$$K(t,f) := \inf\left\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f_0 \in X_0, f_1 \in X_1, f = f_0 + f_1\right\}.$$

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- $L^{p,p}(\mu) = L^p(\mu)$, $1 \leq p < \infty$.
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$$\left(L_w^1(m), L^\infty(m)\right)_{\theta,q} \subsetneq \overline{\mathcal{S}(\Sigma)}^{L_w^{p,q}(m)} \subseteq L_w^{p,q}(m).$$

Distribution and decreasing rearrangement.

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Distribution function. $f : \Omega \longrightarrow \mathbb{R}$ measurable:

$$\|m\|_f : t \in [0,\infty) \longrightarrow \|m\|_f(t) \in [0,\infty)$$

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Decreasing rearrangement function.

$$\begin{aligned}f^*(s) &:= \inf\{t\ge 0 : \|m\|_f(t)\le s\}\\ &= \lambda_{\|m\|_f}(s)\\ &= \sup\{f^*_{x'}(s) : \|x'\|\le 1\}\end{aligned}$$

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For $1 \leq p, q \leq \infty$, the Lorentz space $L^{p,q}(\|m\|)$ consists of all measurable functions $f : \Omega \longrightarrow \mathbb{R}$ such that $\|f\|_{L^{p,q}(\|m\|)} < \infty,$ where

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1) $L^{p,q}(\|m\|)$ is a quasi-Banach lattice.

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$$1 \le p_1 < p_2 < \infty$$

$$L^{p_1,\infty}(\|m\|)$$

$$\cup\!|$$

$$\vdots$$

$$\cup\!|$$

$$L^{p_1,p_1}(\|m\|)$$

$$\cup\!|$$

$$\vdots$$

$$\cup\!|$$

$$L^{p_1,1}(\|m\|)$$

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$$
\begin{array}{ccc}
L^{p_2,\infty}(\|m\|) & & L^{p_1,\infty}(\|m\|) \\
\cup\!| & & \cup\!| \\
\vdots & & \vdots \\
\cup\!| & & \cup\!| \\
L^{p_2,p_2}(\|m\|) & & L^{p_1,p_1}(\|m\|) \\
\cup\!| & & \cup\!| \\
\vdots & & \vdots \\
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\cup\!| & \cup\!| & \cup\!| \\
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c) $$f \in L^1(m) \quad \not\Rightarrow \quad \int_0^t f^*(s)ds < \infty,$$
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The real method. [Mayoral, Naranjo & AF, 2011]

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1) For $0 < \theta < 1 \leq q \leq \infty$,

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2) For $1 \leq p_0 \neq p_1 \leq \infty;\ 0 < \theta < 1 \leq q \leq \infty$,

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[Beauzamy 1978**]** If $0 < \theta < 1 < q < \infty$ and the inclusion $X_0 \cap X_1 \subseteq X_0 + X_1$ is weakly compact, then $(X_0, X_1)_{\theta,q}$ is reflexive.

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Workshop Operators and Banach lattices

Universidad Complutense de Madrid,
Facultad de Ciencias Matemáticas,
25-26 October 2012

Interpolation of spaces of integrable functions with respect to a vector measure

Antonio Fernández (Universidad de Sevilla)