1 Introduction

In *Israel J. Math.* 2011, Figiel, Johnson and Pelczyński introduced "Property (k)" for Banach spaces.

Problem 1.1 *does every pre-dual of a σ-finite von Neumann algebra have property (k)?*

In [FJP] it was shown that:

1. if $\mathcal{M}$ is a von Neumann algebra and the pre-dual $\mathcal{M}_*$ is separable, then $\mathcal{M}_*$ has property (k);

2. if μ is a σ-finite measure, then $L_1(\mu)$ has property (k).

This talk is based on joint work with Peter Dodds and Fedor Sukochev.

2 Property (K)

Definition 2.1 *Let $(x_n)_{n=1}^{\infty}$ be a sequence in a (real or complex) vector space X. A sequence $(y_k)_{k=1}^{\infty}$ in X is called a* CCC sequence *of $(x_n)_{n=1}^{\infty}$ if there exists a sequence $1 = N_1 < N_2 < \cdots$ in $\mathbb{N}$ and a sequence $(c_k)_{k=1}^{\infty}$ in $\mathbb{R}^+$ such that*

$$y_k = \sum_{j=N_k}^{N_{k+1}-1} c_j x_j, \quad \sum_{j=N_k}^{N_{k+1}-1} c_j = 1, \quad k = 1, 2, \ldots$$

(CCC=consecutive convex combinations)

Example. $(X, \|\cdot\|)$ a Banach space and $(x_n)_{n=1}^{\infty}$ a sequence in X such that $x_n \to 0$ w.r.t. $\sigma(X, X^*)$. Then there exists a CCC sequence (y_k) of (x_n) such that $\|y_k\| \to 0$.

Indeed: for all $m \in \mathbb{N}$,

$$0 \in \overline{\operatorname{co}\{x_n : n \geq m\}}^{\sigma(X,X^*)} = \overline{\operatorname{co}\{x_n : n \geq m\}}^{\|\cdot\|}.$$

In *Math. Annalen* 1997, Kalton and Pelczyński introduced:

Definition 2.2 *A Banach space X is said to have* property (K) *if every sequence (x_n^*) in X^* satisfying $x_n^* \to 0$ with respect to $\sigma(X^*, X)$ has a CCC sequence (y_k^*) such that $\langle x_k, y_k^* \rangle \to 0$ for every sequence (x_k) in X satisfying $x_k \to 0$ with respect to $\sigma(X, X^*)$.*

Examples.

1. The space c_0 does *not* have property (K).

   Indeed: consider in c_0^* the sequence (e_n^*) of coordinate functionals of the standard basis (e_n) in c_0.

2. Every reflexive space has property (K).

3. Every Grothendieck space has property (K).

   [*Recall*: Banach space X is a *Grothendieck space* if $x_n^* \to 0$ w.r.t. $\sigma(X^*, X)$ implies that $x_n^* \to 0$ w.r.t. $\sigma(X^*, X^{**})$]

   In particular, $X = \ell_\infty$ has property (K); every von Neumann algebra has property (K) (Pfitzner, 1994).

4. Every subspace of a *separable* Banach space with property (K) has property (K).

5. Every *complemented* subspace of a Banach space with property (K) has property (K).

6. If $\Gamma \neq \emptyset$, then $\ell_1(\Gamma)$ has property (K).

   [Indeed, $\ell_1(\Gamma)$ has the Schur property]

Note: "Property (K)"$\Longrightarrow$"Property (k)".

3 Reformulation of Property (K)

Useful observation:

Lemma 3.1 *$(X, \|\cdot\|)$ a Banach space and $(x_n^*)_{n=1}^{\infty} \subseteq X^*$ such that $x_n^* \to 0$ w.r.t. $\sigma(X^*, X)$. Equivalent are:*

(i) for every $(x_n) \subseteq X$ with $x_n \to 0$ w.r.t. $\sigma(X, X^*)$ we have $\langle x_n, x_n^* \rangle \to 0$;

(ii) for every relatively $\sigma(X, X^*)$-compact set $A \subseteq X$ we have

$$\sup_{x \in A} |\langle x, x_n^* \rangle| \to 0, \qquad n \to \infty.$$

(i.e., $x_n^* \to 0$ uniformly on relatively $\sigma(X, X^*)$-compact subsets of X).

Some notation: Let X be a Banach space.

- For $A \subseteq X$, bounded, define the semi-norm $\rho_A : X^* \to [0, \infty)$ by

$$\rho_A(x^*) = \sup\{|\langle x, x^* \rangle| : x \in A\}.$$

- Let $\mathfrak{S}$ be a collection of bounded sets in X such that span $\bigcup_{A \in \mathfrak{S}} A$ is dense in X. The locally convex topology in X^* generated by

$$\{\rho_A : A \in \mathfrak{S}\}$$

  is denoted by $\tau_{\mathfrak{S}}$: the *topology of uniform convergence on the sets of* $\mathfrak{S}$.

<u>**Recall**</u>:

1. *Mackey topology* $\tau(X^*, X)$: the topology on X^* of uniform convergence on absolutely convex $\sigma(X, X^*)$-compact subsets of X.

2. *Mackey-Arens theorem*: if τ is a locally convex topology on X^*, then the dual of (X^*, τ) equals X if and only if $\tau = \tau_{\mathfrak{S}}$ for some collection $\mathfrak{S}$ of absolutely convex $\sigma(X, X^*)$-compact subsets of X satisfying $\bigcup_{A \in \mathfrak{S}} A = X$.

   Note that then
   $$\sigma(X^*, X) \subseteq \tau \subseteq \tau(X^*, X)$$
   and convex subsets of X^* have the same closure for all such topologies.

3. *Krein-Smulian*: the absolute convex hull of a (relatively) $\sigma(X, X^*)$-compact subset of X is again (relatively) $\sigma(X, X^*)$-compact.

With these observations we find:

Lemma 3.2 *If X is a Banach space, then the following are equivalent:*

(i) X has property (K);

(ii) every sequence (x_n^*) in X^* satisfying $x_n^* \to 0$ w.r.t. $\sigma(X^*, X)$ has a CCC sequence (y_k^*) such that $y_k^* \to 0$ w.r.t. $\tau(X^*, X)$.

Corollary 3.3 *If X is a Banach space such that the Mackey topology $\tau(X^*, X)$ is metrizable on norm bounded subsets of X^*, then X has property (K).*

Proof. If (x_n^*) in X^* satisfies $x_n^* \to 0$ w.r.t. $\sigma(X^*, X)$, then

$$0 \in \overline{\operatorname{co}\{x_n : n \geq m\}}^{\sigma(X, X^*)} = \overline{\operatorname{co}\{x_n : n \geq m\}}^{\tau(X^*, X)}$$

for all $m \in \mathbb{N}$. ■

A general definition. Let X be a Banach space.

Definition 3.4 *Let $\mathfrak{S}$ be a collection of bounded subsets of X. We say that X has property $(K_{\mathfrak{S}})$ if every sequence $(x_n^*) \subseteq X^*$ with $x_n^* \to 0$ w.r.t. $\sigma(X^*, X)$ has a CCC sequence (y_k^*) such that $y_k^* \to 0$ uniformly on the sets in $\mathfrak{S}$, that is,*

$$\sup_{x \in A} |\langle x, y_k^* \rangle| \to 0, \quad k \to \infty,$$

for all $A \in \mathfrak{S}$.

Proposition 3.5 *Suppose that* $\operatorname{span} \bigcup_{A \in \mathfrak{S}} A$ *is dense in A and that $\tau_{\mathfrak{S}} \subseteq \tau(X^*, X)$. If $\tau_{\mathfrak{S}}$ is metrizable on norm bounded subsets of X^*, then X has property $(K_{\mathfrak{S}})$.*

4 Banach lattices

- E a (real) Banach lattice.
- a subset $A \subseteq E$ is order bounded if there exists $0 \leq w \in E$ such that
$$A \subseteq [-w, w],$$
where
$$[-w, w] = \{x \in E : -w \leq x \leq w\}.$$
- $\mathfrak{S}_{ob}$: all order bounded subsets of E.
- $\tau_{ob} = \tau_{\mathfrak{S}_{ob}}$; $(K_{ob}) = (K_{\mathfrak{S}_{ob}})$.

Lemma 4.1 *For a Banach lattice E, the following two conditions are equivalent:*

(i) E has property (K_{ob});

(ii) every sequence (x_n^*) in E^* satisfying $x_n^* \to 0$ with respect to $\sigma(E^*, E)$ has a CCC sequence (y_k^*) such that $|y_k^*| \to 0$ with respect to $\sigma(E^*, E)$.

Proposition 4.2 *If E is a Banach lattice with order continuous norm and weak order unit, then the topology τ_{ob} is metrizable on norm bounded subsets of E^*.*

Proof. Let $0 \leq w \in E$ be a weak order unit.

On the unit ball B_{E^*} the topology τ_{ob} is induced by the semi-norm ρ_w:
$$\rho_w(x^*) = \sup\{|\langle x, x^*\rangle| : x \in E, |x| \leq w\} = \langle w, |x^*|\rangle, \quad x^* \in E^*.$$
■

Theorem 4.3 *If E is a Banach lattice with order continuous norm and weak order unit, then E has property (K_{ob}).*

Corollary 4.4 *Let E be a Banach lattice with order continuous norm and weak order unit. If (x_n^*) is a sequence in E^* satisfying $x_n^* \overset{\sigma(E^*,E)}{\to} 0$, then (x_n^*) has a CCC sequence (y_k^*) such that $|y_k^*| \to 0$ with respect to $\sigma(E^*, E)$.*

The above result is implicit in [FJP], (2011) and improves Sublemma 2.5 in Johnson, 1997.

5 Property (K) in pre-duals of von Neumann algebras

- $\mathcal{M}$ a von Neumann algebra on Hilbert space H.
- $P(\mathcal{M})$ the complete lattice of projections in $\mathcal{M}$.
- $\mathcal{M}$ is called σ-finite if every mutually disjoint system in $P(\mathcal{M})$ is at most countable.
- $\mathcal{M}_*$ the pre-dual of $\mathcal{M}$; $\mathcal{M}_*$ is a bimodule over $\mathcal{M}$.

Definition 5.1 *A subset $A \subseteq \mathcal{M}_*$ is said to be of* uniformly absolutely continuous norm *if $p_\alpha \downarrow_\alpha 0$ in $P(\mathcal{M})$ implies that*

$$\sup_{\varphi \in A} \|p_\alpha x p_\alpha\|_{\mathcal{M}_*} \to_\alpha 0.$$

We use the following ingredients.

Proposition 5.2 (Akemann, 1967) *Every relatively $\sigma(\mathcal{M}_*, \mathcal{M})$-compact subset of $\mathcal{M}_*$ is of uniformly absolutely continuous norm.*

Proposition 5.3 (Raynaud, Xu, 2003) *If$\mathcal{M}$ is σ-finite, then there exists $0 < \varphi_0 \in \mathcal{M}_*$ such that for every $A \subseteq \mathcal{M}$ of uniformly absolutely continuous norm and every $0 < \varepsilon \in \mathbb{R}$ there exists $0 < C_\varepsilon \in \mathbb{R}$ satisfying*

$$A \subseteq C_\varepsilon (\varphi_0 B_{\mathcal{M}} + B_{\mathcal{M}} \varphi_0) + \varepsilon B_{\mathcal{M}_*}.$$

With this we can prove:

Proposition 5.4 *If $\mathcal{M}$ is σ-finite, then the Mackey topology $\tau(\mathcal{M}, \mathcal{M}_*)$ is metrizable on norm bounded subsets of $\mathcal{M}$.*

Proof. Let $0 < \varphi_0 \in \mathcal{M}_*$ be as above and let

$$W = \varphi_0 B_{\mathcal{M}} + B_{\mathcal{M}} \varphi_0.$$

Define the (semi-) norm $\rho_W : \mathcal{M} \to [0, \infty)$ by

$$\rho_W(x) = \sup_{\varphi \in W} |\varphi(x)|, \quad x \in \mathcal{M}.$$

On norm bounded subsets of $\mathcal{M}$ the topology generated by ρ_W and $\tau(\mathcal{M}, \mathcal{M}_*)$ coincide. ■

Remark. In case the underlying Hilbert space H is separable, the result of the above proposition follows from results of Sakai (1965) and Akemann (1967).

Theorem 5.5 *If $\mathcal{M}$ is σ-finite, then its pre-dual $\mathcal{M}_*$ has property (K).*

Remark. There exist (non σ-finite) measures μ such that $L_1(\mu)$ does *not* have property (K).

6 Non-commutative symmetric spaces

Now we consider the following setting:

- $\mathcal{M}$ a semi-finite von Neumann algebra, $\tau : \mathcal{M}^+ \to [0, \infty]$.

- $S(\tau)$ the $*$-algebra of all τ-measurable operators.

- For $x \in S(\tau)$ define the *generalized singular value function* $\mu(x) : [0, \infty) \to [0, \infty]$ by

  $$\mu(t; x) = \inf\left\{0 \leq s \in \mathbb{R} : \tau\left(e^{|x|}(s, \infty)\right) \leq s\right\}, \quad t \geq 0.$$

  (here, $e^{|x|}$ is the spectral measure of $|x|$)

- If $x, y \in S(\tau)$, then we write $x \prec\prec y$ whenever

  $$\int_0^t \mu(s; x)\, ds \leq \int_0^t \mu(s; y)\, ds, \quad t \geq 0.$$

- $E \subseteq S(\tau)$ linear subspace with norm $\|\cdot\|_E$ such that $(E, \|\cdot\|_E)$ is Banach.

- E is called *symmetric* if $x \in S(\tau)$, $y \in E$ and $\mu(x) = \mu(y)$ imply that $x \in E$ and $\|x\|_E = \|y\|_E$.

- A symmetric space E is called *strongly symmetric* if its norm has the additional property that $x, y \in E$ and $x \prec\prec y$ imply that $\|x\|_E \leq \|y\|_E$.

- The norm on E is called *order continuous* if

$$x_\alpha \downarrow_\alpha 0 \ \text{ in } E \quad \Longrightarrow \quad \|x_\alpha\|_E \downarrow_\alpha 0.$$

  Equivalently:

$$e_n \downarrow 0 \ \text{ in } P(\mathcal{M}) \quad \Longrightarrow \quad \|e_n x e_n\|_E \downarrow 0, \quad x \in E.$$

- If the strongly symmetric space E has order continuous norm, then E is *fully symmetric*: if $x \in S(\tau)$ and $y \in E$, then $x \in E$ (and $\|x\|_E \leq \|y\|_E$).

- If the strongly symmetric space E has order continuous norm, then the Banach dual E^* may be identified with the Köthe dual $E^\times$:

$$E^\times = \{y \in S(\tau) : xy \in L_1(\tau) \ \ \forall \ x \in E\},$$

$$\|y\|_{E^\times} = \sup\{|\tau(xy)| : x \in E, \|x\|_E \leq 1\}, \quad y \in E^\times,$$

  via trace duality

$$\langle x, y \rangle = \tau(xy), \quad x \in E, \quad y \in E^\times.$$

Definition 6.1 *Let $E \subseteq S(\tau)$ be a strongly symmetric space. A subset $A \subseteq E$ is said to be of* uniformly absolutely continuous norm *if*

$$e_n \downarrow 0 \ \ in \ \ P(\mathcal{M}) \quad \Longrightarrow \quad \sup_{x \in A} \|e_n x e_n\|_E \to 0.$$

Note: If E has order continuous norm and if $A \subseteq E$ is of uniformly absolutely continuous norm, then A is relatively $\sigma(E, E^\times)$-compact.

Notation:

- $\mathfrak{S}_{an}$ is the collection of all subsets of E which are of uniformly absolutely continuous norm.
- "Property $(K_{\mathfrak{S}_{an}})$" $\equiv$ "Property (K_{an})".
- $\tau_{\mathfrak{S}_{an}} = \tau_{an}$.

Theorem 6.2 *If $\mathcal{M}$ is σ-finite and $E \subseteq S(\tau)$ is a strongly symmetric space with order continuous norm, then E has property (K_{an}).*

The main ingredients in the proof are:

Proposition 6.3 *Let $E \subseteq S(\tau)$ be a strongly symmetric space with order continuous norm. Suppose that (p_n) is a sequence of projections such that $p_n \uparrow \mathbf{1}$ and $\tau(p_n) < \infty$ for all $n \in \mathbb{N}$.*

If $A \subseteq E$ is of uniformly absolutely continuous norm, then for every $0 < \varepsilon \in \mathbb{R}$ there exists $n = n(\varepsilon) \in \mathbb{N}$ and $0 < C_\varepsilon \in \mathbb{R}$ such that

$$A \subseteq C_\varepsilon (p_n B_{\mathcal{M}} + B_{\mathcal{M}} p_n) + \varepsilon B_E.$$

Proposition 6.4 *If $\mathcal{M}$ is σ-finite and $E \subseteq S(\tau)$ is strongly symmetric space with order continuous norm, then τ_{an} is metrizable on norm bounded subsets of $E^{\times}$.*

Proof. Let (p_n) in $P(\mathcal{M})$ be such that $p_n \uparrow \mathbf{1}$ and $\tau(p_n) < \infty$ for all $n \in \mathbb{N}$. Define

$$W_n = p_n B_{\mathcal{M}} + B_{\mathcal{M}} p_n, \quad n \in \mathbb{N}.$$

On norm bounded subset of $E^{\times}$, the topology τ_{an} coincides with the topology generated by the semi-norms $\left\{\rho_{W_n} : n \in \mathbb{N}\right\}$. ■

<u>**Consequence of the Theorem**</u>:

- Assume that $\tau(\mathbf{1}) < \infty$.
- $E \subseteq S(\tau)$ strongly symmetric with order continuous norm.
- For $x \in E$ let
$$\Omega(x) = \{y \in S(\tau) : y \prec\prec x\}.$$
- Then $\Omega(x) \subseteq E$ and $\Omega(x)$ is of uniformly absolutely continuous norm for all $x \in E$.

This gives:

Proposition 6.5 *If (z_n) is a sequence in $E^{\times}$ such that $z_n \to 0$ w.r.t. $\sigma(E^{\times}, E)$, then there exists a CCC sequence (y_k) of (z_n) such that*

$$\int_0^{\infty} \mu(t; x)\, \mu(t; y_k)\, dt \to 0, \quad k \to \infty,$$

for all $x \in E$.

Proof. Let (y_k) be a CCC sequence of (z_n) such that $y_k \to 0$ uniformly on sets of uniformly absolutely continuous norm. Then use that

$$\int_0^{\infty} \mu(t; x)\, \mu(t; y_k)\, dt = \sup_{y \in \Omega(x)} |\langle y, y_k \rangle|.$$

■

7 Property (k)

Let $(X, \|\cdot\|)$ be a Banach space.

Recall: A subset $A \subseteq L_1[0,1]$ is called order bounded if there exists $0 < w \in L_1[0,1]$ such that

$$A \subseteq [-w, w] + i[-w, w],$$

where

$$[-w, w] = \{f \in L_1[0,1] : -w \leq f \leq w\}.$$

Definition 7.1 *Let $\mathfrak{S}_1$ be the collection of subsets of X which are of the form $T(A)$, where $A \subseteq L_1[0,1]$ is order bounded and $T : L_1[0,1] \to X$ is a bounded linear operator.*

Note: if $A \subseteq L_1[0,1]$ is order bounded, then A is relatively $\sigma(L_1, L_\infty)$-compact. Hence, $T(A) \subseteq X$ is relatively $\sigma(X, X^*)$-compact for every bounded linear operator $T : L_1[0,1] \to X$.

Definition 7.2 *The Banach space X has* property (k) *if X has property $(K_{\mathfrak{S}_1})$.*

Since every set in $\mathfrak{S}_1$ is relatively $\sigma(X, X^*)$-compact it follows that:

$$\text{Property } (K) \quad \Longrightarrow \quad \text{Property } (k).$$

Let $\mathcal{M}$ be a semi-finite von Neumann algebra.

Recall: a strongly symmetric space $E \subseteq S(\tau)$ is called a KB-space if:

1. E has order continuous norm;

2. E has the Fatou property: if $0 \leq x_\alpha \uparrow_\alpha$ in E and $\sup_\alpha \|x_\alpha\|_E = M < \infty$, then there exists $0 \leq x \in E$ such that $x_\alpha \uparrow_\alpha x$ and $\|x\|_E = M$.

Theorem 7.3 *If $\mathcal{M}$ is σ-finite and $E \subseteq S(\tau)$ is a KB-space, then E has property (k).*

The main ingredient in the proof is:

Proposition 7.4 *If $E \subseteq S(\tau)$ is a KB-space and $T : L_1[0,1] \to E$ is a bounded linear operator, then T can be written as*

$$T = (T_1 - T_2) + i(T_3 - T_4),$$

where each $T_j : L_1[0,1] \to E$ is linear and positivity preserving, i.e.,

$$0 \leq f \in L_1[0,1] \quad \Longrightarrow \quad 0 \leq T_j f \in E.$$

Consequently, T maps order bounded sets in $L_1[0,1]$ onto order bounded sets in E.

Fact: if E has order continuous norm, then every order bounded set in E is of uniform absolutely continuous norm.

Corollary 7.5 *If $E \subseteq S(\tau)$ is a KB-space and $T : L_1[0,1] \to E$ is a bounded linear operator, then for each order bounded set $A \subseteq L_1[0,1]$, the image $T(A)$ is a set of uniformly absolutely continuous norm in E.*

The theorem now follows from the fact that E has property (K_{an}).

8 Property (K) in symmetric spaces

Assume:

- $\mathcal{M}$ is a semi-finite and σ-finite von Neumann algebra.
- $E \subseteq S(\tau)$ is a strongly symmetric space with order continuous norm.

We know: then E has property (K_{an}).

Recall the following definition:

Definition 8.1 (Krygin, Sheremet'ev, Sukochev, 1993) *The space E is said to have property (Wm) if for all sequences $(x_n) \subseteq E$ satisfying $x_n \to 0$ both w.r.t. $\sigma(E, E^*)$ and the measure topology, it follows that $\|x_n\|_E \to 0$.*

Proposition 8.2 *If $\tau(\mathbf{1}) < \infty$, then the following statements are equivalent:*

(i) E has property (Wm);

(ii) each relatively $\sigma(E, E^*)$-compact set in E is of uniformly absolutely continuous norm.

Consequently:

Corollary 8.3 *If $\tau(\mathbf{1}) < \infty$ and E has property (Wm), then E has property (K).*

Example

Assume $\tau(\mathbf{1}) = 1$ and let $\phi : [0,1] \to [0,\infty)$ be increasing and concave with $\phi(0+) = \phi(0) = 0$. Define the non-commutative Lorentz space by

$$\Lambda_\phi(\tau) = \left\{ x \in S(\tau) : \|x\|_{\Lambda_\phi} = \int_0^1 \mu(t;x)\,\phi'(t)\,dt < \infty \right\}.$$

The space $\Lambda_\phi(\tau)$ has order continuous norm and has also property (Wm).

Consequently: the space $\Lambda_\phi(\tau)$ has property (K).

The following lifting result may also be of some interest:

Proposition 8.4 *Assume that* $\tau(\mathbf{1}) = 1$. *Suppose that* $E(0,1) \subseteq S(0,1)$ *is a strongly symmetric space with order continuous norm and property* (K). *Then, the corresponding non-commutative space*

$$E(\tau) = \{x \in S(\tau) : \mu(x) \in E(0,1)\},$$

$$\|x\|_{E(\tau)} = \|\mu(x)\|_{E(0,1)},$$

has also order continuous norm and property (K).