Vector measures and classical disjointification methods

Enrique A. Sánchez Pérez

I.U.M.P.A.-U. Politécnica de Valencia,
Joint work with **Eduardo Jiménez** (U.P.V.)

Madrid, October 2012

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In this talk we show how the classical disjointification methods (Bessaga-Pelczynski, Kadecs-Pelczynski) can be applied in the setting of the spaces of p-integrable functions with respect to vector measures. These spaces provide in fact a representation of p-convex order continuous Banach lattices with weak unit; the additional tool of the vector valued integral for each function has already shown to be fruitful for the analysis of these spaces. Consequently, our results can be directly extended to a broad class of Banach lattices.

Following this well-known technique, we show that combining Kadecs-Pelczynski Dichotomy, vector measure orthogonality and with disjointness in the range of the integration map, we can determine the structure of the subspaces of our family of Banach function spaces.

These results can already be found in some recent papers and preprints in collaboration with J.M. Calabuig, E. Jiménez, S. Okada, J. Rodríguez and P. Tradacete.

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Disjointification procedures: General Scheme.

Let $m : \Sigma \to E$ be a vector measure. Consider the space $L^p(m)$ of p-integrable functions with respect to m.

1. The first result (Bessaga-Pelczynski) allows to work with orthogonality notions in the range space: **orthogonal integrals**.
2. The second one (Kadec-Pelczynski) provides the tools for analyzing **disjoint functions** in $L^p(m)$.
3. Combining both results: **structure of subspaces** in $L^p(m)$.

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DEFINITIONS: Banach function spaces

- (Ω, Σ, μ) be a *finite* measure space.
- $L^0(\mu)$ space of all (classes of) measurable real functions on Ω.
- A *Banach function space* (briefly B.f.s.) is a Banach space $X \subset L^0(\mu)$ of locally integrable functions with norm $\|\cdot\|_X$ such that if $f \in L^0(\mu)$, $g \in X$ and $|f| \leq |g|$ μ-a.e. then $f \in X$ and $\|f\|_X \leq \|g\|_X$.
- A B.f.s. X has the *Fatou property* if for every sequence $(f_n) \subset X$ such that $0 \leq f_n \uparrow f$ μ-a.e. and $\sup_n \|f_n\|_X < \infty$, it follows that $f \in X$ and $\|f_n\|_X \uparrow \|f\|_X$.
- We will say that X is *order continuous* if for every $f, f_n \in X$ such that $0 \leq f_n \uparrow f$ μ-a.e., we have that $f_n \rightarrow f$ in norm.

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DEFINITIONS: Vector measures and integration

- Let $m : \Sigma \to E$ be a *vector measure*, that is, a countably additive set function, where E is a real Banach space.
- A set $A \in \Sigma$ is *m-null* if $m(B) = 0$ for every $B \in \Sigma$ with $B \subset A$. For each x^* in the topological dual E^* of E, we denote by $|x^* m|$ the variation of the real measure $x^* m$ given by the composition of m with x^*. There exists $x_0^* \in E^*$ such that $|x_0^* m|$ has the same null sets as m. We will call $|x_0^* m|$ a *Rybakov control measure* for m.
- A measurable function $f : \Omega \to \mathbb{R}$ is *integrable with respect to m* if

(i) $\int |f| \, d|x^* m| < \infty$ for all $x^* \in E^*$.

(ii) For each $A \in \Sigma$, there exists $x_A \in E$ such that

$$x^*(x_A) = \int_A f \, dx^* m, \quad \text{for all } x^* \in E.$$

The element x_A will be written as $\int_A f \, dm$.

- A measurable function satisfying only the first requirement is called a *weakly integrable function.*

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DEFINITIONS: Spaces of integrable functions

- Denote by $L^1(m)$ the space of integrable functions with respect to m, where functions which are equal m-a.e. are identified.

- The space $L^1(m)$ is an order continuous Banach function space space endowed with the norm
$$\|f\|_m = \sup_{x^* \in B_{E^*}} \int |f| \, d|x^* m|$$
and the natural order. Note that $L^\infty(|x_0^* m|) \subset L^1(m)$. In particular every measure of the type $|x^* m|$ is finite as $|x^* m|(\Omega) \leq \|x^*\| \cdot \|\chi_\Omega\|_m$.

- Given $f \in L^1(m)$, the set function $m_f : \Sigma \to E$ given by $m_f(A) = \int_A f \, dm$ for all $A \in \Sigma$ is a vector measure. Moreover, $g \in L^1(m_f)$ if and only if $gf \in L^1(m)$ and in this case $\int g \, dm_f = \int gf \, dm$.

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$$\|f\| := \|f^p\|_{L^1(m)}^{1/p}.$$

- Generalized Hölder's inequality: $L^p(m) \cdot L^{p'}(m) \subseteq L^1(m)$, and

$$\| \int f \cdot g dm \| \leq \|f\|_{L^p(m)} \cdot \|g\|_{L^{p'}(m)}.$$

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Disjointification procedures:

- **Bessaga-Pelczynski Selection Principle.** If $\{x_n\}_{n=1}^{\infty}$ is a basis of the Banach space X and $\{x_n^*\}_{n=1}^{\infty}$ is the sequence of coefficient functionals, if we take a normalized sequence $\{y_n\}_{n=1}^{\infty}$ that is weakly null, then $\{y_n\}_{n=1}^{\infty}$ admits a basic subsequence that is equivalent to a block basic sequence of $\{x_n\}_{n=1}^{\infty}$

- **Kadec-Pelczynski Disjointification Procedure** / **Dichotomy.**

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THEOREM 4.1. (Figiel-Johnson-Tzafriri, 1975)

Let L be a σ-complete and σ-order continuous Banach lattice. Suppose X is a subspace of L. Either X is isomorphic to a subspace of $L_1(\mu)$ for some measure μ or there is a sequence (x_i) of unit vectors in X and a disjointly supported sequence (e_i) in L with $\|x_i - e_i\| \to 0$. Consequently, every subspace of L contains an unconditional basic sequence.

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- **Bessaga-Pelczynski Selection Principle.** If $\{x_n\}_{n=1}^{\infty}$ is a basis of the Banach space X and $\{x_n^*\}_{n=1}^{\infty}$ is the sequence of coefficient functionals, if we take a normalized sequence $\{y_n\}_{n=1}^{\infty}$ that is weakly null, then $\{y_n\}_{n=1}^{\infty}$ admits a basic subsequence that is equivalent to a block basic sequence of $\{x_n\}_{n=1}^{\infty}$

- **Kadec-Pelczynski Disjointification Procedure / Dichotomy.**

  M. I. Kadec and A. Pelczynski, *Bases, lacunary sequences, and complemented subspaces in the spaces L_p*, Studia Math. (1962)

THEOREM 4.1. (Figiel-Johnson-Tzafriri, 1975)

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Kadec-Pelczynski Disjointification Procedure for sequences.

Let $X(\mu)$ be an order continuous Banach function space over a finite measure μ with a weak unit (this implies $X(\mu) \hookrightarrow L^1(\mu)$). Consider a normalized sequence $\{x_n\}_{n=1}^{\infty}$ in $X(\mu)$. Then

(1) either $\{\|x_n\|_{L^1(\mu)}\}_{n=1}^{\infty}$ is bounded away from zero,

(2) or there exist a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ and a disjoint sequence $\{z_k\}_{k=1}^{\infty}$ in $X(\mu)$ such that $\|z_k - x_{n_k}\| \to_k 0$.

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Applications: Disjointification in spaces $L^p(m)$

- A. Vector measure orthogonality and disjointness.
- B. Positively norming sets in spaces $L^1(m)$ and the structure of their subspaces.
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A. m-orthonormal sequences in $L^2(m)$:

Definition.
A sequence $\{f_i\}_{i=1}^{\infty}$ in $L^2(m)$ is called *m-orthogonal* if $\|\int f_i f_j \, dm\| = \delta_{i,j} k_i$ for positive constants k_i. If $\|f_i\|_{L^2(m)} = 1$ for all $i \in \mathbb{N}$, it is called *m-orthonormal*.

Definition.
Let $m : \Sigma \longrightarrow \ell^2$ be a vector measure. We say that $\{f_i\}_{i=1}^{\infty} \subset L^2(m)$ is a *strongly m-orthogonal sequence* if $\int f_i f_j dm = \delta_{i,j} e_i k_i$ for an orthonormal sequence $\{e_i\}_{i=1}^{\infty}$ in ℓ^2 and for $k_i > 0$. If $k_i = 1$ for every $i \in \mathbb{N}$, we say that it is a *strongly m-orthonormal sequence.*

Definition. A function $f \in L^2(m)$ is *normed by the integral* if $\|f\|_{L^2(m)} = \| \int_\Omega f^2 \, dm\|^{1/2}$.
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Example.

Let $([0,\infty),\Sigma,\mu)$ be Lebesgue measure space. Let $r_k(x) := sign\{\sin(2^{k-1}x)\}$ be the *Rademacher* function of period 2π defined at the interval $E_k = [2(k-1)\pi, 2k\pi]$, $k \in \mathbb{N}$. Consider the vector measure $m : \Sigma \to \ell^2$ given by $m(A) := \sum_{k=1}^{\infty} \frac{-1}{2^k}(\int_{A\cap E_k} r_k d\mu)e_k \in \ell^2$, $A \in \Sigma$.

Note that if $f \in L^2(m)$ then $\int_{[0,\infty)} f dm = (\frac{-1}{2^k}\int_{E_k} f r_k d\mu)_k \in \ell^2$. Consider the sequence of functions

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\begin{aligned}
f_1(x) &= \sin(x)\cdot\chi_{[\pi,2\pi]}(x)\\
f_2(x) &= \sin(2x)\cdot(\chi_{[0,2\pi]}(x)+\chi_{[\frac{7}{2}\pi,4\pi]}(x))\\
f_3(x) &= \sin(4x)\cdot(\chi_{[0,4\pi]}(x)+\chi_{[\frac{23}{4}\pi,6\pi]}(x))\\
&\dots\\
f_n(x) &= \sin(2^{n-1}x)\cdot(\chi_{[0,2(n-1)\pi]}(x)+\chi_{[(2n-\frac{2}{2^n})\pi,2n\pi]}(x)), \quad n \geq 2.
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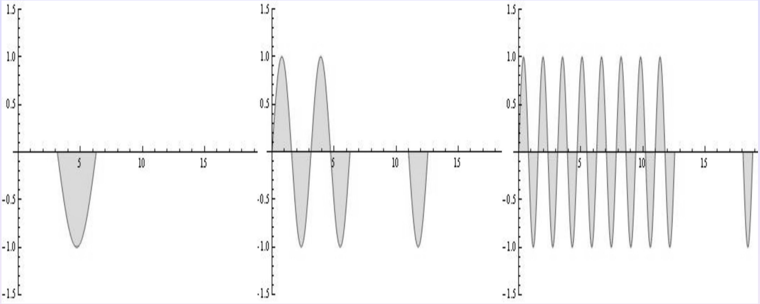

Figura: Functions $f_1(x)$, $f_2(x)$ and $f_3(x)$.

Proposition.

Let $\{g_n\}_{n=1}^{\infty}$ be a normalized sequence in $L^2(m)$. Suppose that there exists a Rybakov measure $\mu = |\langle m, x_0' \rangle|$ for m such that $\{\|g_n\|_{L^1(\mu)}\}_{n=1}^{\infty}$ is not bounded away from zero. Then there are a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$ and an m-orthonormal sequence $\{f_k\}_{k=1}^{\infty}$ such that $\|g_{n_k} - f_k\|_{L^2(m)} \to_k 0$.

Theorem.

Let us consider a vector measure $m : \Sigma \to \ell^2$ and an m-orthonormal sequence $\{f_n\}_{n=1}^{\infty}$ of functions in $L^2(m)$ that are normed by the integrals. Let $\{e_n\}_{n=1}^{\infty}$ be the canonical basis of ℓ^2. If $\lim_n \left\langle \int f_n^2 dm, e_k \right\rangle = 0$ for every $k \in \mathbb{N}$, then there exist a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ and a vector measure $m^* : \Sigma \to \ell^2$ such that $\{f_{n_k}\}_{k=1}^{\infty}$ is strongly m^*-orthonormal.

Moreover, m^* can be chosen to be as $m^* = \phi \circ m$ for some Banach space isomorphism ϕ from ℓ^2 onto ℓ^2, and so $L^2(m) = L^2(m^*)$.

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Proof. (Sketch)

- Take an **m**-orthonormal sequence $\{f_n\}_{n=1}^{\infty}$ in $L^2(\mathbf{m})$ and the sequence of integrals $\left\{\int_\Omega f_n^2 \, d\mathbf{m}\right\}_{n=1}^{\infty}$.
- Disjointification (Bessaga-Pelc.): We get a subsequence $\left\{\int_\Omega f_{n_k}^2 \, d\mathbf{m}\right\}_{k=1}^{\infty}$ that is equivalent to a block basic sequence $\{e'_{n_k}\}_{k=1}^{\infty}$ of the canonical basis of ℓ^2.
- Associated to this sequence there is an isomorphism φ

$$A := \overline{span(e'_{n_k})}^{\ell^2} \xrightarrow{\varphi} B := \overline{span\left(\int_\Omega f_{n_k}^2 \, d\mathbf{m}\right)}^{\ell^2}$$

  such that $\varphi(e'_{n_k}) := \int_\Omega f_{n_k}^2 \, d\mathbf{m}$, $k \in \mathbb{N}$.
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Proof. (Sketch)

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- We can suppose without loss of generality that the elements of the sequence $\{e'_{n_k}\}_{k=1}^{\infty}$ have norm one.

- There is a subspace B^c such that $\ell^2 = B \oplus_2 B^c$ isometrically, where this direct sum space is considered as a Hilbert space (with the adequate Hilbert space norm). We write P_B and P_{B^c} for the corresponding projections.

- Let us consider the linear map $\phi := \varphi^{-1} \oplus Id : B \oplus_2 B^c \xrightarrow{\phi} A \oplus_2 B^c$, where $Id : B^c \to B^c$ is the identity map.

- Let us consider now the vector measure $\mathbf{m}^* := \phi \circ \mathbf{m} : \Sigma \xrightarrow{m} \ell^2 \xrightarrow{\phi} A \oplus_2 B^c$. Then $L^2(\mathbf{m}) = L^2(\phi \circ \mathbf{m}) = L^2(\mathbf{m}^*)$.

- Finally, it is proved that $\{f_{n_k}\}_{k=1}^{\infty}$ is a strongly $\mathbf{m}^*$-orthonormal sequence.

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Corollary.

Let $m : \Sigma \to \ell^2$ be a countably additive vector measure. Let $\{g_n\}_{n=1}^{\infty}$ be a normalized sequence of functions in $L^2(m)$ that are normed by the integrals. Suppose that there exists a Rybakov measure $\mu = |\langle m, x_0' \rangle|$ for m such that $\{\|g_n\|_{L^1(\mu)}\}_{n=1}^{\infty}$ is not bounded away from zero.

If $\lim_n \langle \int g_n^2 dm, e_k \rangle = 0$ for every $k \in \mathbb{N}$, then there is a (disjoint) sequence $\{f_k\}_{k=1}^{\infty}$ such that

(1) $\lim_k \|g_{n_k} - f_k\|_{L^2(m)} = 0$ for a given subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$, and

(2) it is strongly m^*-orthonormal for a certain Hilbert space valued vector measure m^* defined as in the theorem that satisfies that $L^2(m) = L^2(m^*)$.

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(1) $\text{lím}_k \|g_{n_k} - f_k\|_{L^2(m)} = 0$ for a given subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$, and

(2) it is strongly m^*-orthonormal for a certain Hilbert space valued vector measure m^* defined as in the theorem that satisfies that $L^2(m) = L^2(m^*)$.

Corollary.

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Consequences on the structure of $L^1(m)$:

Lemma.
Let $m:\Sigma\to\ell^2$ be a positive vector measure, and suppose that the bounded sequence $\{g_n\}_{n=1}^{\infty}$ in $L^2(m)$ satisfies that $\lim_n\langle\int g_n^2 dm, e_k\rangle=0$ for all $k\in\mathbb{N}$. Then there is a Rybakov measure μ for m such that $\lim_n\|g_n\|_{L^1(\mu)}=0$.

Proposition.
Let $m:\Sigma\to\ell^2$ be a positive (countably additive) vector measure. Let $\{g_n\}_{n=1}^{\infty}$ be a normalized sequence in $L^2(m)$ such that for every $k\in\mathbb{N}$, $\lim_n\langle\int g_n^2 dm, e_k\rangle=0$. Then $L^2(m)$ contains a lattice copy of ℓ^4. In particular, there is a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$ that is equivalent to the unit vector basis of ℓ^4.

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Proposition.
Let $m : \Sigma \to \ell^2$ be a positive (countably additive) vector measure. Let $\{g_n\}_{n=1}^{\infty}$ be a normalized sequence in $L^2(m)$ such that for every $k \in \mathbb{N}$, $\lim_n \left\langle \int g_n^2 dm, e_k \right\rangle = 0$. Then $L^2(m)$ contains a lattice copy of ℓ^4. In particular, there is a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$ that is equivalent to the unit vector basis of ℓ^4.

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Proposition.
Let $m : \Sigma \to \ell^2$ be a positive (countably additive) vector measure. Let $\{g_n\}_{n=1}^{\infty}$ be a normalized sequence in $L^2(m)$ such that for every $k \in \mathbb{N}$, $\text{lím}_n \left\langle \int g_n^2 dm, e_k \right\rangle = 0$. Then $L^2(m)$ contains a lattice copy of ℓ^4. In particular, there is a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ of $\{g_n\}_{n=1}^{\infty}$ that is equivalent to the unit vector basis of ℓ^4.

Theorem.
Let $m : \Sigma \to \ell^2$ be a positive (countably additive) vector measure. Let $\{g_n\}_{n=1}^{\infty}$ be a normalized sequence in $L^2(m)$ such that for every $k \in \mathbb{N}$,

$$\lim_n \left\langle e_k, \int g_n^2 dm \right\rangle = 0$$

for all $k \in \mathbb{N}$. Then there is a subsequence $\{g_{n_k}\}_{k=1}^{\infty}$ such that $\{g_{n_k}^2\}_{k=1}^{\infty}$ generates an isomorphic copy of ℓ^2 in $L^1(m)$ that is preserved by the integration map. Moreover, there is a normalized disjoint sequence $\{f_k\}_{k=1}^{\infty}$ that is equivalent to the previous one and $\{f_k^2\}_{k=1}^{\infty}$ gives a lattice copy of ℓ^2 in $L^1(m)$ that is preserved by I_{m^*}.

Corollary.
Let $m : \Sigma \to \ell^2$ be a positive (countably additive) vector measure. The following assertions are equivalent:

(1) There is a normalized sequence in $L^2(m)$ satisfying that $\text{lím}_n \langle \int_\Omega g_n^2 dm, e_k \rangle = 0$ for all the elements of the canonical basis $\{e_k\}_{k=1}^\infty$ of ℓ^2.

(2) There is an ℓ^2-valued vector measure $m^* = \phi \circ m$ —ϕ an isomorphism— such that $L^2(m) = L^2(m^*)$ and there is a disjoint sequence in $L^2(m)$ that is strongly m^*-orthonormal.

(3) The subspace S that is fixed by the integration map I_m satisfies that there are positive functions $h_n \in S$ such that $\{\int_\Omega h_n dm\}_{n=1}^\infty$ is an orthonormal basis for $I_m(S)$.

(4) There is an ℓ^2-valued vector measure m^* defined as $m^* = \phi \circ m$ —ϕ an isomorphism— such that $L^1(m) = L^1(m^*)$ and a subspace S of $L^1(m)$ such that the restriction of I_{m^*} to S is a lattice isomorphism in ℓ^2.

Theorem.
The following assertions for a positive vector measure $m : \Sigma \rightarrow \ell^2$ are equivalent.

(1) $L^1(m)$ contains a lattice copy of ℓ^2.

(2) $L^1(m)$ has a reflexive infinite dimensional sublattice.

(3) $L^1(m)$ has a relatively weakly compact normalized sequence of disjoint functions.

(4) $L^1(m)$ contains a weakly null normalized sequence.

(5) There is a vector measure m^* defined by $m^* = \phi \circ m$ such that integration map I_{m^*} fixes a copy of ℓ^2.

(6) There is a vector measure m^* defined as $m^* = \phi \circ m$ that is not disjointly strictly singular.

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