Domination of operators in non-commutative setting

Eugeniu Spinu, University of Alberta

joint work with

Timur Oikhberg, University of Illinois in Urbana-Champain

October, 2012

The domination problem

Question

Suppose E and F are ordered Banach spaces, $0 \leqslant T \leqslant S : E \to F$. Suppose S belongs to a certain ideal (such as the ideal of compact, weakly compact, or Dunford-Pettis operators). Does T belong to the same ideal?

Banach lattices: domination problem

Theorem (Fremlin & Dodds; Wickstead)

Suppose E and F are Banach lattices. TFAE:

1. *If $0 \leqslant T \leqslant S : E \to F$, and S is compact, then T is compact.*
2. *One of the three (non-exclusive) statements holds:*
   - (i) *Both $E^\star$ and F are order continuous.*
   - (ii) *F is atomic, and order continuous.*
   - (iii) *$E^\star$ is atomic, and order continuous.*

Theorem (Aliprantis & Burkinshaw)

If E is a Banach lattices, $0 \leqslant T \leqslant S : E \to E$, and S is compact, then T^3 is compact. T^2 need not be compact.

Banach lattices: domination problem

Theorem (Fremlin & Dodds; Wickstead)

Suppose E and F are Banach lattices. TFAE:

1. *If $0 \leqslant T \leqslant S : E \to F$, and S is compact, then T is compact.*
2. *One of the three (non-exclusive) statements holds:*
   - (i) *Both $E^\star$ and F are order continuous.*
   - (ii) *F is atomic, and order continuous.*
   - (iii) *$E^\star$ is atomic, and order continuous.*

Theorem (Aliprantis & Burkinshaw)

If E is a Banach lattices, $0 \leqslant T \leqslant S : E \to E$, and S is compact, then T^3 is compact. T^2 need not be compact.

Ordered spaces: definitions

A real Banach space Z is an ordered Banach space (OBS) if it has a closed proper positive cone Z_+.

Z_+ is generating if $Z_+ - Z_+ = Z$.

Z_+ is normal if its dual $Z^\star$ is generating.

A complex OBS Z is the complexification of its real part $Z_\mathbb{R}$ ($Z = Z_\mathbb{R} + iZ_\mathbb{R}$).

A real Banach space Z is an ordered Banach space (OBS) if it has a closed proper positive cone Z_+.

Z_+ is generating if $Z_+ - Z_+ = Z$.

Z_+ is normal if its dual $Z^\star$ is generating.

A complex OBS Z is the complexification of its real part $Z_\mathbb{R}$ ($Z = Z_\mathbb{R} + iZ_\mathbb{R}$).

Ordered spaces: definitions

A real Banach space Z is an ordered Banach space (OBS) if it has a closed proper positive cone Z_+.

Z_+ is generating if $Z_+ - Z_+ = Z$.

Z_+ is normal if its dual $Z^\star$ is generating.

A complex OBS Z is the complexification of its real part $Z_\mathbb{R}$ ($Z = Z_\mathbb{R} + iZ_\mathbb{R}$).

Ordered spaces: definitions

A real Banach space Z is an ordered Banach space (OBS) if it has a closed proper positive cone Z_+.

Z_+ is generating if $Z_+ - Z_+ = Z$.

Z_+ is normal if its dual $Z^\star$ is generating.

A complex OBS Z is the complexification of its real part $Z_\mathbb{R}$ ($Z = Z_\mathbb{R} + iZ_\mathbb{R}$).

In the following cases, Z is an OBS with a normal and generating cone.

- Z is a Banach lattice.
- Z is a C^*-algebra.
- Z is a non-commutative function space

Suppose $\mathcal{A} \subset B(H)$ is a C^*-algebra. For $x \in B(H)$, define $\mathbf{M}_x : \mathcal{A} \to B(H) : a \mapsto x^*ax$.

Proposition

Suppose x is an element of a C^*-algebra $\mathcal{A}$.

1. *If $\mathbf{M}_x$ is weakly compact, and $0 \leqslant T \leqslant \mathbf{M}_x : \mathcal{A} \to \mathcal{A}$, then T is compact.*
2. *If $0 \leqslant \mathbf{M}_x \leqslant S : \mathcal{A} \to \mathcal{A}$, and S is weakly compact, then $\mathbf{M}_x$ is compact.*

Suppose $\mathcal{A} \subset B(H)$ is a C^*-algebra. For $x \in B(H)$, define $\mathbf{M}_x : \mathcal{A} \to B(H) : a \mapsto x^*ax$.

Proposition

Suppose x is an element of a C^*-algebra $\mathcal{A}$.

1. *If $\mathbf{M}_x$ is weakly compact, and $0 \leqslant T \leqslant \mathbf{M}_x : \mathcal{A} \to \mathcal{A}$, then T is compact.*
2. *If $0 \leqslant \mathbf{M}_x \leqslant S : \mathcal{A} \to \mathcal{A}$, and S is weakly compact, then $\mathbf{M}_x$ is compact.*

Suppose $\mathcal{A} \subset B(H)$ is a C^*-algebra. For $x \in B(H)$, define $\mathbf{M}_x : \mathcal{A} \to B(H) : a \mapsto x^* a x$.

Proposition

Suppose x is an element of a C^*-algebra $\mathcal{A}$.

1. *If $\mathbf{M}_x$ is weakly compact, and $0 \leqslant T \leqslant \mathbf{M}_x : \mathcal{A} \to \mathcal{A}$, then T is compact.*
2. *If $0 \leqslant \mathbf{M}_x \leqslant S : \mathcal{A} \to \mathcal{A}$, and S is weakly compact, then $\mathbf{M}_x$ is compact.*

Compactness of multiplication operators on C^*-algebras: the irreducible case

Proposition

Suppose $\mathcal{A}$ is an irreducible C^*-subalgebra of $B(H)$, and $x \in B(H)$.

1. *If $\mathbf{M}_x : \mathcal{A} \to B(H)$ is compact, and $0 \leqslant T \leqslant \mathbf{M}_x$, then T is compact.*
2. *If $S : \mathcal{A} \to B(H)$ is compact, and $0 \leqslant \mathbf{M}_x \leqslant S$, then $\mathbf{M}_x$ is compact.*

Remark

Irreducibility is essential: there exists an abelian C^*-subalgebra $\mathcal{A} \subset B(H)$, and $x, y \in B(H)$, so that $0 \leqslant \mathbf{M}_x \leqslant \mathbf{M}_y : \mathcal{A} \to B(H)$, $\mathbf{M}_y$ is compact, while $\mathbf{M}_x$ is not.

Compactness of multiplication operators on C^*-algebras: the irreducible case

Proposition

Suppose $\mathcal{A}$ is an irreducible C^*-subalgebra of $B(H)$, and $x \in B(H)$.

1. *If $\mathbf{M}_x : \mathcal{A} \to B(H)$ is compact, and $0 \leqslant T \leqslant \mathbf{M}_x$, then T is compact.*
2. *If $S : \mathcal{A} \to B(H)$ is compact, and $0 \leqslant \mathbf{M}_x \leqslant S$, then $\mathbf{M}_x$ is compact.*

Remark

Irreducibility is essential: there exists an abelian C^*-subalgebra $\mathcal{A} \subset B(H)$, and $x, y \in B(H)$, so that $0 \leqslant \mathbf{M}_x \leqslant \mathbf{M}_y : \mathcal{A} \to B(H)$, $\mathbf{M}_y$ is compact, while $\mathbf{M}_x$ is not.

Compactness of multiplication operators on C^*-algebras: the irreducible case

Proposition

Suppose $\mathcal{A}$ is an irreducible C^*-subalgebra of $B(H)$, and $x \in B(H)$.

1. *If $\mathbf{M}_x : \mathcal{A} \to B(H)$ is compact, and $0 \leqslant T \leqslant \mathbf{M}_x$, then T is compact.*
2. *If $S : \mathcal{A} \to B(H)$ is compact, and $0 \leqslant \mathbf{M}_x \leqslant S$, then $\mathbf{M}_x$ is compact.*

Remark

Irreducibility is essential: there exists an abelian C^*-subalgebra $\mathcal{A} \subset B(H)$, and $x, y \in B(H)$, so that $0 \leqslant \mathbf{M}_x \leqslant \mathbf{M}_y : \mathcal{A} \to B(H)$, $\mathbf{M}_y$ is compact, while $\mathbf{M}_x$ is not.

An element a of a Banach algebra $\mathcal{A}$ is multiplication compact if the map $\mathcal{A} \to \mathcal{A} : b \mapsto aba$ is compact.

A Banach algebra is compact if each of its elements is multiplication compact.

A C^*-algebra is compact iff it is C^*-isomorphic to $(\oplus_{i\in I} K(H_i))_{c_0}$.

Proposition

For a C^*-algebra $\mathcal{A}$, TFAE:

1. *$\mathcal{A}$ is compact.*
2. *For any $c \in \mathcal{A}_+$, the order interval $[0, c]$ is compact.*
3. *For any $c \in \mathcal{A}_+$, the order interval $[0, c]$ is weakly compact.*
4. *$\mathcal{A}$ is a hereditary subalgebra of $\mathcal{A}^{**}$.*

Compact C^*-algebras: definitions

An element a of a Banach algebra $\mathcal{A}$ is multiplication compact if the map $\mathcal{A} \to \mathcal{A} : b \mapsto aba$ is compact.

A Banach algebra is compact if each of its elements is multiplication compact.

A C^*-algebra is compact iff it is C^*-isomorphic to $(\oplus_{i \in I} K(H_i))_{c_0}$.

Proposition

For a C^*-algebra $\mathcal{A}$, TFAE:

1. *$\mathcal{A}$ is compact.*
2. *For any $c \in \mathcal{A}_+$, the order interval $[0, c]$ is compact.*
3. *For any $c \in \mathcal{A}_+$, the order interval $[0, c]$ is weakly compact.*
4. *$\mathcal{A}$ is a hereditary subalgebra of $\mathcal{A}^{**}$.*

Compact C^*-algebras: definitions

An element a of a Banach algebra $\mathcal{A}$ is multiplication compact if the map $\mathcal{A} \to \mathcal{A} : b \mapsto aba$ is compact.

A Banach algebra is compact if each of its elements is multiplication compact.

A C^*-algebra is compact iff it is C^*-isomorphic to $(\oplus_{i \in I} K(H_i))_{c_0}$.

Proposition

For a C^*-algebra $\mathcal{A}$, TFAE:

1. *$\mathcal{A}$ is compact.*
2. *For any $c \in \mathcal{A}_+$, the order interval $[0, c]$ is compact.*
3. *For any $c \in \mathcal{A}_+$, the order interval $[0, c]$ is weakly compact.*
4. *$\mathcal{A}$ is a hereditary subalgebra of $\mathcal{A}^{**}$.*

Compact C^*-algebras: definitions

An element a of a Banach algebra $\mathcal{A}$ is multiplication compact if the map $\mathcal{A} \to \mathcal{A} : b \mapsto aba$ is compact.

A Banach algebra is compact if each of its elements is multiplication compact.

A C^*-algebra is compact iff it is C^*-isomorphic to $(\oplus_{i \in I} K(H_i))_{c_0}$.

Proposition

For a C^*-algebra $\mathcal{A}$, TFAE:

1. *$\mathcal{A}$ is compact.*
2. *For any $c \in \mathcal{A}_+$, the order interval $[0, c]$ is compact.*
3. *For any $c \in \mathcal{A}_+$, the order interval $[0, c]$ is weakly compact.*
4. *$\mathcal{A}$ is a hereditary subalgebra of $\mathcal{A}^{**}$.*

Compact maps on compact C^*-algebras

Proposition

Suppose $\mathcal{A}$ is a compact C^*-algebra, E is a generating OBS, and $0 \leqslant T \leqslant S : E \to \mathcal{A}$. If S is compact, then so is T.

Compact maps on scattered C^*-algebras

We say that a C^*-algebra $\mathcal{A}$ is scattered if the spectrum of any self-adjoint element of $\mathcal{A}$ is countable (equivalently, $\mathcal{A}^{\star\star} = (\sum_{i \in I} B(H_i))_\infty$).

Proposition

Suppose $\mathcal{A}$ is a scattered C^*-algebra, and E is a generating OBS. If $0 \leqslant T \leqslant S : E \to \mathcal{A}^\star$, and S is compact, then so is T.

Corollary

Suppose $\mathcal{A}$ is a scattered C^*-algebra, and E is a normal OBS. If $0 \leqslant T \leqslant S : \mathcal{A} \to E$, and S is compact, then so is T.

We say that a C^*-algebra $\mathcal{A}$ is scattered if the spectrum of any self-adjoint element of $\mathcal{A}$ is countable (equivalently, $\mathcal{A}^{\star\star} = (\sum_{i \in I} B(H_i))_\infty$).

Proposition

Suppose $\mathcal{A}$ is a scattered C^*-algebra, and E is a generating OBS. If $0 \leqslant T \leqslant S : E \to \mathcal{A}^\star$, and S is compact, then so is T.

Corollary

Suppose $\mathcal{A}$ is a scattered C^*-algebra, and E is a normal OBS. If $0 \leqslant T \leqslant S : \mathcal{A} \to E$, and S is compact, then so is T.

We say that a C^*-algebra $\mathcal{A}$ is scattered if the spectrum of any self-adjoint element of $\mathcal{A}$ is countable (equivalently, $\mathcal{A}^{\star\star} = (\sum_{i \in I} B(H_i))_\infty$).

Proposition

Suppose $\mathcal{A}$ is a scattered C^*-algebra, and E is a generating OBS. If $0 \leqslant T \leqslant S : E \to \mathcal{A}^\star$, and S is compact, then so is T.

Corollary

Suppose $\mathcal{A}$ is a scattered C^*-algebra, and E is a normal OBS. If $0 \leqslant T \leqslant S : \mathcal{A} \to E$, and S is compact, then so is T.

Theorem

Suppose $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras.

1. *Suppose at least one of the two conditions holds:*
   (i) *$\mathcal{A}$ is scattered.*
   (ii) *$\mathcal{B}$ is compact.*

   If $0 \leqslant T \leqslant S : \mathcal{A} \to \mathcal{B}$, and S is compact, then T is compact.
2. *Suppose $\mathcal{A}$ is not scattered, and $\mathcal{B}$ is not compact. Then there exist $0 \leqslant T \leqslant S : \mathcal{A} \to \mathcal{B}$, so that S has rank 1, while T is not compact.*

Compactness of operators on C^*-algebras

Theorem

Suppose $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras.

1. *Suppose at least one of the two conditions holds:*
   (i) *$\mathcal{A}$ is scattered.*
   (ii) *$\mathcal{B}$ is compact.*

   If $0 \leqslant T \leqslant S : \mathcal{A} \to \mathcal{B}$, and S is compact, then T is compact.
2. *Suppose $\mathcal{A}$ is not scattered, and $\mathcal{B}$ is not compact. Then there exist $0 \leqslant T \leqslant S : \mathcal{A} \to \mathcal{B}$, so that S has rank 1, while T is not compact.*

Non-commutative function spaces: definitions

Suppose a von Neumann subalgebra $\mathcal{A} \subseteq B(H)$ is equipped with a normal faithful semi-finite trace τ. $\tilde{\mathcal{A}}$ is the set of closed densely defined operators, affiliated with $\mathcal{A}$. Define the generalized singular value function: for $x \in \tilde{\mathcal{A}}$ and $t \geqslant 0$,
$\mu_x(t) = \inf\{\|xe\| : e = e^* = e^2 \in \mathcal{A}, \tau(1-e) \leqslant t\}$.

Suppose $\mathcal{E}$ is a linear subspace of $\tilde{\mathcal{A}}$ with a complete norm $\|\cdot\|_{\mathcal{E}}$. We say that $\mathcal{E}$ is a non-commutative function space if:

- $L_1(\tau) \cap \mathcal{A} \subset \mathcal{E} \subset L_1(\tau) + \mathcal{A}$.
- For any $x \in \mathcal{E}$ and $a, b \in \mathcal{A}$, we have $axb \in \mathcal{E}$, and $\|axb\|_{\mathcal{E}} \leqslant \|a\|\|x\|_{\mathcal{E}}\|b\|$.

$\mathcal{E}$ is called strongly symmetric if, for any $x, y \in \mathcal{E}$, with $y \prec\!\!\prec x$, we have $\|y\|_{\mathcal{E}} \leqslant \|x\|_{\mathcal{E}}$. Here, $\prec\!\!\prec$ refers to the Hardy-Littlewood domination: $y \prec\!\!\prec x$ iff, for any $\alpha > 0$, $\int_0^\alpha \mu_y(t)\,dt \leqslant \int_0^\alpha \mu_x(t)\,dt$.

$\mathcal{E}$ is a normal and generating OBS.

Non-commutative function spaces: definitions

Suppose a von Neumann subalgebra $\mathcal{A} \subseteq B(H)$ is equipped with a normal faithful semi-finite trace τ. $\tilde{\mathcal{A}}$ is the set of closed densely defined operators, affiliated with $\mathcal{A}$. Define the generalized singular value function: for $x \in \tilde{\mathcal{A}}$ and $t \geqslant 0$,

$\mu_x(t) = \inf\{\|xe\| : e = e^* = e^2 \in \mathcal{A}, \tau(1-e) \leqslant t\}$.

Suppose $\mathcal{E}$ is a linear subspace of $\tilde{\mathcal{A}}$ with a complete norm $\|\cdot\|_{\mathcal{E}}$. We say that $\mathcal{E}$ is a non-commutative function space if:

- $L_1(\tau) \cap \mathcal{A} \subset \mathcal{E} \subset L_1(\tau) + \mathcal{A}$.
- For any $x \in \mathcal{E}$ and $a, b \in \mathcal{A}$, we have $axb \in \mathcal{E}$, and $\|axb\|_{\mathcal{E}} \leqslant \|a\|\|x\|_{\mathcal{E}}\|b\|$.

$\mathcal{E}$ is called strongly symmetric if, for any $x, y \in \mathcal{E}$, with $y \prec\prec x$, we have $\|y\|_{\mathcal{E}} \leqslant \|x\|_{\mathcal{E}}$. Here, $\prec\prec$ refers to the Hardy-Littlewood domination: $y \prec\prec x$ iff, for any $\alpha > 0$, $\int_0^\alpha \mu_y(t)\,dt \leqslant \int_0^\alpha \mu_x(t)\,dt$.

$\mathcal{E}$ is a normal and generating OBS.

Non-commutative function spaces: definitions

Suppose a von Neumann subalgebra $\mathcal{A} \subseteq B(H)$ is equipped with a normal faithful semi-finite trace τ. $\tilde{\mathcal{A}}$ is the set of closed densely defined operators, affiliated with $\mathcal{A}$. Define the generalized singular value function: for $x \in \tilde{\mathcal{A}}$ and $t \geqslant 0$,

$\mu_x(t) = \inf\{\|xe\| : e = e^* = e^2 \in \mathcal{A}, \tau(1-e) \leqslant t\}$.

Suppose $\mathcal{E}$ is a linear subspace of $\tilde{\mathcal{A}}$ with a complete norm $\|\cdot\|_{\mathcal{E}}$. We say that $\mathcal{E}$ is a non-commutative function space if:

- $L_1(\tau) \cap \mathcal{A} \subset \mathcal{E} \subset L_1(\tau) + \mathcal{A}$.
- For any $x \in \mathcal{E}$ and $a, b \in \mathcal{A}$, we have $axb \in \mathcal{E}$, and $\|axb\|_{\mathcal{E}} \leqslant \|a\|\|x\|_{\mathcal{E}}\|b\|$.

$\mathcal{E}$ is called strongly symmetric if, for any $x, y \in \mathcal{E}$, with $y \lll x$, we have $\|y\|_{\mathcal{E}} \leqslant \|x\|_{\mathcal{E}}$. Here, $\lll$ refers to the Hardy-Littlewood domination: $y \lll x$ iff, for any $\alpha > 0$, $\int_0^\alpha \mu_y(t)\,dt \leqslant \int_0^\alpha \mu_x(t)\,dt$.

$\mathcal{E}$ is a normal and generating OBS.

Non-commutative function spaces: definitions

Suppose a von Neumann subalgebra $\mathcal{A} \subseteq B(H)$ is equipped with a normal faithful semi-finite trace τ. $\tilde{\mathcal{A}}$ is the set of closed densely defined operators, affiliated with $\mathcal{A}$. Define the generalized singular value function: for $x \in \tilde{\mathcal{A}}$ and $t \geqslant 0$,

$\mu_x(t) = \inf\{\|xe\| : e = e^* = e^2 \in \mathcal{A}, \tau(1 - e) \leqslant t\}$.

Suppose $\mathcal{E}$ is a linear subspace of $\tilde{\mathcal{A}}$ with a complete norm $\|\cdot\|_{\mathcal{E}}$. We say that $\mathcal{E}$ is a non-commutative function space if:

- $L_1(\tau) \cap \mathcal{A} \subset \mathcal{E} \subset L_1(\tau) + \mathcal{A}$.
- For any $x \in \mathcal{E}$ and $a, b \in \mathcal{A}$, we have $axb \in \mathcal{E}$, and $\|axb\|_{\mathcal{E}} \leqslant \|a\|\|x\|_{\mathcal{E}}\|b\|$.

$\mathcal{E}$ is called strongly symmetric if, for any $x, y \in \mathcal{E}$, with $y \prec\prec x$, we have $\|y\|_{\mathcal{E}} \leqslant \|x\|_{\mathcal{E}}$. Here, $\prec\prec$ refers to the Hardy-Littlewood domination: $y \prec\prec x$ iff, for any $\alpha > 0$, $\int_0^\alpha \mu_y(t)\,dt \leqslant \int_0^\alpha \mu_x(t)\,dt$.

$\mathcal{E}$ is a normal and generating OBS.

Weakly compact operators on non-commutative function spaces

A non-commutative function space $\mathcal{E}$ is a KB space if any increasing norm bounded sequence in $\mathcal{E}$ is norm-convergent.

TFAE:

1. $\mathcal{E}$ is a KB space.
2. $\mathcal{E}$ is weakly sequentially complete.
3. $\mathcal{E}$ contains no copy of c_0.

Proposition

Suppose $\mathcal{E}$ is a strongly symmetric KB non-commutative function space, X a generating OBS, and $0 \leqslant T \leqslant S : X \to \mathcal{E}$, with S weakly compact. Then T is weakly compact as well.

Weakly compact operators on non-commutative function spaces

A non-commutative function space $\mathcal{E}$ is a KB space if any increasing norm bounded sequence in $\mathcal{E}$ is norm-convergent.
TFAE:

1. $\mathcal{E}$ is a KB space.
2. $\mathcal{E}$ is weakly sequentially complete.
3. $\mathcal{E}$ contains no copy of c_0.

Proposition

Suppose $\mathcal{E}$ is a strongly symmetric KB non-commutative function space, X a generating OBS, and $0 \leqslant T \leqslant S : X \to \mathcal{E}$, with S weakly compact. Then T is weakly compact as well.

Weakly compact operators on non-commutative function spaces

A non-commutative function space $\mathcal{E}$ is a KB space if any increasing norm bounded sequence in $\mathcal{E}$ is norm-convergent.
TFAE:

1. $\mathcal{E}$ is a KB space.
2. $\mathcal{E}$ is weakly sequentially complete.
3. $\mathcal{E}$ contains no copy of c_0.

Proposition

Suppose $\mathcal{E}$ is a strongly symmetric KB non-commutative function space, X a generating OBS, and $0 \leqslant T \leqslant S : X \to \mathcal{E}$, with S weakly compact. Then T is weakly compact as well.

Extra definitions

A Banach lattice, or a non-commutative function space, $\mathcal{E}$, is order continuous if, for any net $x_\alpha \downarrow 0$, we have $\lim \|x_\alpha\| = 0$.

Suppose τ is the canonical trace on $B(H)$, and $x \in B(H)$ has singular values $s_1(x) \geqslant s_2(x) \geqslant \ldots$. Then $\mu_x(t) = s_n(x)$ if $n-1 \leqslant t < n$.

A symmetric sequence space $\mathcal{E}$ gives rise to a Schatten space $\mathcal{S}_\mathcal{E}(H) = \{T \in K(H) : (s_i(T)) \in \mathcal{E}\}$, with $\|T\|_\mathcal{E} = \|(s_i(T))_{i \in \mathbb{N}}\|_\mathcal{E}$. Convention: $\mathcal{S}_p = \mathcal{S}_{\ell_p}$.

For a symmetric sequence space $\mathcal{E}$, TFAE:

- $\mathcal{E}$ is order continuous.
- $\mathcal{E}$ is separable.
- $\mathcal{S}_\mathcal{E}(H)$ is order continuous, for any H.

Extra definitions

A Banach lattice, or a non-commutative function space, $\mathcal{E}$, is order continuous if, for any net $x_\alpha \downarrow 0$, we have $\lim \|x_\alpha\| = 0$.

Suppose τ is the canonical trace on $B(H)$, and $x \in B(H)$ has singular values $s_1(x) \geqslant s_2(x) \geqslant \ldots$. Then $\mu_x(t) = s_n(x)$ if $n - 1 \leqslant t < n$.

A symmetric sequence space $\mathcal{E}$ gives rise to a Schatten space $\mathcal{S}_\mathcal{E}(H) = \{T \in K(H) : (s_i(T)) \in \mathcal{E}\}$, with $\|T\|_\mathcal{E} = \|(s_i(T))_{i \in \mathbb{N}}\|_\mathcal{E}$. Convention: $\mathcal{S}_p = \mathcal{S}_{\ell_p}$.

For a symmetric sequence space $\mathcal{E}$, TFAE:

- $\mathcal{E}$ is order continuous.
- $\mathcal{E}$ is separable.
- $\mathcal{S}_\mathcal{E}(H)$ is order continuous, for any H.

A Banach lattice, or a non-commutative function space, $\mathcal{E}$, is order continuous if, for any net $x_\alpha \downarrow 0$, we have $\lim \|x_\alpha\| = 0$.

Suppose τ is the canonical trace on $B(H)$, and $x \in B(H)$ has singular values $s_1(x) \geqslant s_2(x) \geqslant \ldots$. Then $\mu_x(t) = s_n(x)$ if $n - 1 \leqslant t < n$.

A symmetric sequence space $\mathcal{E}$ gives rise to a Schatten space $\mathcal{S}_\mathcal{E}(H) = \{T \in K(H) : (s_i(T)) \in \mathcal{E}\}$, with $\|T\|_\mathcal{E} = \|(s_i(T))_{i \in \mathbb{N}}\|_\mathcal{E}$. Convention: $\mathcal{S}_p = \mathcal{S}_{\ell_p}$.

For a symmetric sequence space $\mathcal{E}$, TFAE:

- $\mathcal{E}$ is order continuous.
- $\mathcal{E}$ is separable.
- $\mathcal{S}_\mathcal{E}(H)$ is order continuous, for any H.

Extra definitions

A Banach lattice, or a non-commutative function space, $\mathcal{E}$, is order continuous if, for any net $x_\alpha \downarrow 0$, we have $\lim \|x_\alpha\| = 0$.

Suppose τ is the canonical trace on $B(H)$, and $x \in B(H)$ has singular values $s_1(x) \geqslant s_2(x) \geqslant \ldots$. Then $\mu_x(t) = s_n(x)$ if $n - 1 \leqslant t < n$.

A symmetric sequence space $\mathcal{E}$ gives rise to a Schatten space $\mathcal{S}_\mathcal{E}(H) = \{T \in K(H) : (s_i(T)) \in \mathcal{E}\}$, with $\|T\|_\mathcal{E} = \|(s_i(T))_{i\in\mathbb{N}}\|_\mathcal{E}$. Convention: $\mathcal{S}_p = \mathcal{S}_{\ell_p}$.

For a symmetric sequence space $\mathcal{E}$, TFAE:

- $\mathcal{E}$ is order continuous.
- $\mathcal{E}$ is separable.
- $\mathcal{S}_\mathcal{E}(H)$ is order continuous, for any H.

Compact operators on Schatten spaces

Theorem

Suppose $\mathcal{E}$ is a separable symmetric sequence space, and H is an inf. dim. Hilbert space.
(1) *If $\mathcal{E}$ does not contain ℓ_1, F is a normal OBS,*
$0 \leqslant T \leqslant S : \mathcal{S}_{\mathcal{E}}(H) \to F$, *and S is compact, then T is compact.*
(2) *Conversely, suppose $\mathcal{E}$ contains ℓ_1, and a Banach lattice F is either not atomic, or not order continuous. Then $\exists$*
$0 \leqslant T \leqslant S : \mathcal{S}_{\mathcal{E}}(\ell_2) \to F$ *so that S is compact, but T is not.*

Weakly compact operators on Schatten spaces

Theorem

Suppose $\mathcal{E}$ is a separable symmetric sequence space, containing ℓ_1, F is a Banach lattice, and H is an inf. dim. Hilbert space. TFAE:

(1) *F is order continuous.*

(2) *If $0 \leqslant T \leqslant S : S_{\mathcal{E}}(H) \to F$, and S is weakly compact, then T is weakly compact as well.*

Weakly compact operators on Schatten spaces

Theorem

Suppose $\mathcal{E}$ is a separable symmetric sequence space, containing ℓ_1, F is a Banach lattice, and H is an inf. dim. Hilbert space. TFAE:
(1) *F is order continuous.*
(2) *If $0 \leqslant T \leqslant S : \mathcal{S}_{\mathcal{E}}(H) \to F$, and S is weakly compact, then T is weakly compact as well.*

Dunford-Pettis Schur multipliers

An operator $T \in B(E, F)$ is Dunford-Pettis if $\lim_n \|Tx_n\| = 0$ whenever $x_n \xrightarrow[weakly]{} 0$. Equivalently, the image of any weakly compact set is compact.

Theorem

Suppose $0 \leqslant \mathbf{S}_\phi \leqslant \mathbf{S}_\psi$ are Schur multipliers from $\mathcal{S}_1$ to $\mathcal{S}_\mathcal{E}$ ($\mathcal{E}$ is a symmetric sequence space). If $\mathbf{S}_\psi$ is Dunford-Pettis, then the same is true for $\mathbf{S}_\phi$.

There exist examples of Banach lattices E, F, and $0 \leqslant T \leqslant S : E \to F$, so that S is Dunford-Pettis, but T is not.

Dunford-Pettis Schur multipliers

An operator $T \in B(E, F)$ is Dunford-Pettis if $\lim_n \|Tx_n\| = 0$ whenever $x_n \xrightarrow[weakly]{} 0$. Equivalently, the image of any weakly compact set is compact.

Theorem

Suppose $0 \leqslant \mathbf{S}_\phi \leqslant \mathbf{S}_\psi$ are Schur multipliers from $\mathcal{S}_1$ to $\mathcal{S}_\mathcal{E}$ ($\mathcal{E}$ is a symmetric sequence space). If $\mathbf{S}_\psi$ is Dunford-Pettis, then the same is true for $\mathbf{S}_\phi$.

There exist examples of Banach lattices E, F, and $0 \leqslant T \leqslant S : E \to F$, so that S is Dunford-Pettis, but T is not.

Dunford-Pettis Schur multipliers

An operator $T \in B(E, F)$ is Dunford-Pettis if $\lim_n \|Tx_n\| = 0$ whenever $x_n \xrightarrow[weakly]{} 0$. Equivalently, the image of any weakly compact set is compact.

Theorem

Suppose $0 \leqslant \mathbf{S}_\phi \leqslant \mathbf{S}_\psi$ are Schur multipliers from $\mathcal{S}_1$ to $\mathcal{S}_\mathcal{E}$ ($\mathcal{E}$ is a symmetric sequence space). If $\mathbf{S}_\psi$ is Dunford-Pettis, then the same is true for $\mathbf{S}_\phi$.

There exist examples of Banach lattices E, F, and $0 \leqslant T \leqslant S : E \to F$, so that S is Dunford-Pettis, but T is not.