A non-linear Banach–Stone theorem for lattices of uniformly continuous functions

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Basic problem

Let us focus on the kind of problem we are interested in. Suppose we are given topological spaces X and Y and lattices $L(X)$ and $L(Y)$ consisting in certain real-valued functions on the corresponding spaces. Assume that the lattices $L(Y)$ and $L(X)$ are isomorphic.

What can be said about X and Y?

Lattice isomorphisms are not assumed to be linear. A lattice isomorphism is just a bijection $T : L(Y) \to L(X)$ such that

$$T(f \vee g) = (Tf) \vee (Tg) \qquad (f, g \in L(Y)).$$

Equivalently, T preserves the order in both directions.

Mathematical Archeology I: Continuous functions

Theorem (Kaplansky, 1947) Two compact (Hausdorff) spaces are homeomorphic if (and only if) the corresponding lattices of continuous functions are isomorphic.

Mathematical Archeology II: Bounded uniformly continuous functions

Theorem (Shirota, 1952) Two complete metric spaces are uniformly homeomorphic if (and only if) the lattices of **bounded** uniformly continuos functions are isomorphic.

Mathematical Archeology III: Shirota's claim

Theorem 6. *Let X be a complete metric space. Then X is determined by the lattice of all uniformly continuous real functions on X. Moreover X is determined by the lattice of all bounded uniformly continuous real functions on it.*

Problem

Prove or disprove that two complete metric spaces are uniformly homeomorphic if they have [linearly] isomorphic lattices of uniformly continuous functions.

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Prove or disprove that two complete metric spaces are uniformly homeomorphic if they have [linearly] isomorphic lattices of uniformly continuous functions.

To mention a few people:

★ Garrido&Jaramillo, A Banach-Stone Theorem for Uniformly Continuous Functions, Monatsh. Math. (2000)

★ Hušek&Pulgarín, Banach-Stone-like theorems for lattices of uniformly continuous functions, Quaest. Math. (in press)

★ The referees of F&J Cabello Sánchez, Nonlinear isomorphisms of lattices of Lipschitz functions, Houston J. Math. (2011)

★ The referees of F&J Cabello Sánchez, Lattices of uniformly continuous functions, Top Appl (in press)

... Shirota was right!

Theorem Two complete metric spaces are uniformly homeomorphic if (and only if) the lattices of uniformly continuous functions are isomorphic.

Remarks
Completeness is necessary since any metric space has the same uniformly continuous functions as its completion!

Metrizability is necessary as there are two different uniform structures on the same space having exactly the same uniformly real-valued continuous functions!

Functional representation of isomorphisms

In fact we can say more and give an explicit representation of the order isomorphisms:

Theorem (F&J Cabello-Sánchez)
Every lattice isomorphism $T : U(Y) \to U(X)$ is given by the formula

$$Tf(x) = t(x, f(\tau(x))),$$

where $\tau : X \to Y$ is a uniform homeomorphism and $t : X \times \mathbb{R} \to \mathbb{R}$ is defined by $t(x, c) = (Tc)(x)$.

Remark
If T is linear, then $Tf(x) = w(x)f(\tau(x))$, where $w = T(1)$.

Some ideas of the proof

Step 1 We need to get a point map $\tau : X \to Y$ out from $T : U(Y) \to U(X)$. To this end to each nonnegative $f \in U(Y)$ we attach a regular open set U_f taking the interior of its support:

$$U_f = \overset{\circ}{\overline{\{y \in Y : f(y) > 0\}}} .$$

Every regular open set in Y arises in this way. Then by using ideas of Shirota we show that the condition '$U_f \subset U_g$' can be expressed within the order structure of $U(Y)$. The immediate effect of this is that, assuming $T0 = 0$, we can define an order isomorphism $\mathfrak{T} : RO(Y) \to RO(X)$ between the lattices of regular open sets of the given spaces by the rule

$$\mathfrak{T}(U_f) = U_{Tf}$$

and one checks that given $f, g \in U(Y)$ and $U \in RO(Y)$ one has $f = g$ on U if and only if $Tf = Tg$ on $\mathfrak{T}(U)$.

Some ideas of the proof: the point map

Step 1, continued After that one manages to prove that $\mathfrak{T}$ is implemented by a point map, which is a priori defined only between two dense subsets... More precisely, we have the following.

Lemma Suppose $\mathfrak{T} : RO(Y) \to RO(X)$ is an order isomorphism. Then there are dense subsets $X' \subset X$ and $Y' \subset Y$ and a homeomorphism $\tau : X' \to Y'$ such that if $y = \tau(x)$ one has $y \in U$ if and only if $x \in \mathfrak{T}(U)$.

Some ideas of the proof: the formula

Step 2 Next we must verify that, given $x \in X'$ and $f, g \in U(Y)$ one has $Tf(x) = Tg(x)$ if (and only if) $f(\tau(x)) = g(\tau(x))$. From this the representation

$$Tf(x) = t(x, f(\tau(x))) \qquad (x \in X')$$

follows.

To see this we may assume $f \leq g$ and $f(y) = g(y)$, where $y = \tau(x)$. We construct a uniformly continuous function h between f and g with the following property: every neighbourhood of y contains a regular open set where h agrees with f (and another set where h agrees with g). Therefore, every every neighbourhood of x contains a regular open set where Th agrees with Tf (and another set where Th agrees with Tg) and so $Tf(x) = Tg(x) = Th(x)$.

Some ideas of the proof: uniform continuity

Step 3 Finally we must show that $\tau : X' \to Y'$ is a uniform homeomorphism. This will imply that it extends to a uniform homeomorphism between X and Y. By symmetry, it suffices to prove that τ **is uniformly continuous on** X'. We already know that τ is homeomorphism from X' to Y', and in general if a mapping τ is a homeomorphism but fails to be uniformly continuous, then passing to subsequences without mercy we arrive to the following situation: there are sequences (x_n) and (x'_n) in X' and $\delta > 0$ such that:

- $0 < d(x_n, x'_n) \to 0$;
- $d(x_n, x_m) \geq \delta$ and $d(x'_n, x'_m) \geq \delta$ for $n \neq m$;
- $d(y_n, y'_m) \geq \delta$ for every n and m,

where $y_n = \tau(x_n)$ and $y'_n = \tau(x'_n)$.

Step 3, continued If there is no such a sequence, then passing to a subsequence we may assume the sequence (y_n') uniformly isolated in Y, that is, there is $r > 0$ (independent on n) such that the only point of Y in the ball of radius r centred at y_n' is y_n' itself. This obviously implies that the lattice of restrictions

$$M = \{s \in \mathbb{R}^{\mathbb{N}} : s(n) = f(y_n'), f \in U(Y), f(y_k) = 0 \ \forall k\}$$

is the whole of $\mathbb{R}^{\mathbb{N}}$. But, certainly,

$$L = \{s \in \mathbb{R}^{\mathbb{N}} : s(n) = g(x_n'), g \in U(X), f(x_k) = 0 \ \forall k\}$$

is c_0, the lattice of null sequences. Clearly, T induces a lattice isomorphism between M and L (taking $s \in M$ to the sequence $(Tf)(x_n')$, where f is any uniformly continuous function on Y such that $s(n) = f(y_n')$ and $f(y_n) = 0$ for every n), which is impossible since $\mathbb{R}^{\mathbb{N}}$ and c_0 are not isomorphic.
Indeed, $\mathbb{R}^{\mathbb{N}}$ has the property that every "disjoint family" of nonnegative elements has a supremum, while c_0 lacks it. This is a contradiction again and so τ has to be uniformly continuous.

Thank you!